On the normal structure of the automorphism group and the ideal structure of the endomorphism ring of abelian p-groups

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Introduction

This paper consists of an application of the results of the earlier paper [5] to abelian p-groups. The results are part of the author's PhD dissertation completed in Spring 1964 at New Mexico State University. The author expresses his gratitude for the help and suggestions of Prof. J. M. IRWIN and Prof. L. FUCHS. The research was supported by the National Science Foundation under Research Grant GP 377. The notation will be that of [5] (Cf. Section 0 this paper).

In non-abelian group theory one object of investigation is the normal structure of groups. Thereby chains of normal subgroups and the corresponding quotient groups are of particular interest. Analogously one may study the ideal structure of rings. We already know that, for a characteristic subgroup H of an abelian group G, $\mathrm{Aut}_H G$ is a normal subgroup of $\mathrm{Aut}\, G$. For a fully invariant subgroup H we know that $\mathrm{End}_H\, G$ is a two-sided ideal of $\mathrm{End}\, G$. It is thus the obvious thing to consider natural chains of characteristic and fully invariant subgroups and the corresponding chains of normal subgroups of the automorphism group and of two-sided ideals of the endomorphism ring. The chains $\{\mathrm{Aut}_{p^iG} G\}$ and $\{\mathrm{Aut}_{G[p^i]} G\}$ have been studied by SHODA ([6]) for finite abelian groups, FREEDMAN ([1]) for reduced countable abelian p-groups, and by FUCHS ([3]) for arbitrary abelian p-groups. We resume the study of these chains in the general case and obtain an explicit representation of $\mathrm{Aut}_{pG}\, G$ as a semidirect product of three groups [Cf. Def. 1. 23] isomorphic to $\mathrm{Hom}\,(G/pG, pG)$, $\mathrm{Hom}\,(G/B_1 \oplus pG, pG)$ and $\mathrm{Aut}\, B_1$, where $B = \bigoplus_{i=1}^{\infty} B_i$ is a basic subgroup of G. It is then easy, for example, to determine the center of $\mathrm{Aut}_{pG}\, G$. Since $\mathrm{Aut}_{p^{i+1}G}\, G/\mathrm{Aut}_{p^iG}\, G$ is isomorphic with $\mathrm{Aut}_{p^{i+1}G}\, p^iG$ if either i is an integer, or if G is countable and i is an arbitrary ordinal, the structure of this factor group can be described in a satisfactory way in these important cases. Our results improve the results of the previous papers. We also study the corresponding chain $\{\mathrm{End}_{p^iG}\, G\}$ of two-sided ideals in $\mathrm{End}\, G$ and determine the quotient rings in the cases named above. It is easy to see that $\mathrm{Aut}_{G[p^i]}\, G/\mathrm{Aut}_{G[p^{i+1}]}\, G$ is an elementary abelian p-group for $i \geqq 1$, but $\mathrm{Aut}\, G/\mathrm{Aut}_{G[p]}\, G$ is considerably more complicated. Therefore we attack this problem in Section 2 only in a special case.

If G is divisible then $\mathrm{Aut}\, G/\mathrm{Aut}_{G[p]}\, G$ is isomorphic to the general linear group $GL(r(G), p)$ of dimension $r(G)$ ($=$rank of G) over the prime field of characteristic p. The results for the chain $\{\mathrm{End}_{G[p^i]}\, G\}$ of two-sided ideals of $\mathrm{End}\, G$ correspond to those for the automorphism group.

Section O. Preliminaries

In this section we state for convenience some definitions and theorems of [5] which are needed in this paper.

[5] 1. 4 Definition. For $H < G$, let $\mathrm{Aut}_H\, G = \{\alpha \in \mathrm{Aut}\, G : \alpha | H = 1\}$.

[5] 1. 6 **Proposition.** *Let $H < G$ and let $\varphi: G \to G/H$ be the natural homomorphism. Then the sequence*

$$0 \to \mathrm{Hom}\,(G/H, H) \xrightarrow{U} \mathrm{Aut}_H\, G \xrightarrow{T} \mathrm{Aut}\, G/H$$

is exact, where

U: $\mathrm{Hom}\ (G/H,\ H) \to \mathrm{Aut}_H\ G: \xi U = 1 + \varphi\xi$, and T: $\mathrm{Aut}_H\ G \to \mathrm{Aut}\ G/H$: $\varphi(\alpha T) = \alpha\varphi$.

[5] 1. 9 **Lemma.** *Let H be a characteristic subgroup of G. Then the sequence*

$$1 \to \mathrm{Aut}_H\, G \xrightarrow{I} \mathrm{Aut}\, G \xrightarrow{R} \mathrm{Aut}\, H$$

is exact, where I is the injection, and R is the restriction map.

[5] 1. 11 **Lemma.** *If $H_1 < H_2 < G$ and H_1, H_2 are characteristic, then*

$$1 \to \mathrm{Aut}_{H_2}\, G \xrightarrow{I} \mathrm{Aut}_{H_1}\, G \xrightarrow{R} \mathrm{Aut}_{H_1}\, H_2$$

is exact, where I is the injection and R the restriction map.

[5] 1. 12 Definition. For $H < G$, let $\mathrm{End}_H\, G = \{\delta \in \mathrm{End}\, G : H\delta = 0\}$.

[5] 1. 16 **Proposition.** *Given*

$$0 \to H \xrightarrow{\iota} G \xrightarrow{\varphi} G/H \to 0\ (\mathrm{ex}),$$

ι injection map, φ natural homomorphism. Then the sequence

$$0 \to \{\delta \in \mathrm{End}_H\, G : G\delta < H\} \xrightarrow{I} \mathrm{End}_H\, G \xrightarrow{T} \mathrm{End}\, G/H \to$$
$$\to \mathrm{Ext}\,(G/H, H) \to \mathrm{Ext}\,(G/\ H, G) \to \mathrm{Ext}\,(G/H, G/H) \to 0$$

is exact, where I is the injection map and T is given by $\varphi(\delta T) = \delta\varphi$. I and T are ring homomorphisms.

[5] 1. 19 **Proposition.** *Given* $0 \to H \xrightarrow{\iota} G \xrightarrow{\varphi} G/H \to 0$(ex), ι *injection map,* φ *natural homomorphism, and given that H is fully invariant in G, then the sequence*

$$0 \to \mathrm{End}_H\, G \xrightarrow{I} \mathrm{End}\, G \xrightarrow{R} \mathrm{End}\, H \to \mathrm{Ext}\,(G/H, G)$$

is exact, where I is the injection and R the restriction map. I and R are ring homomorphisms.

[5] 1. 22 **Lemma.** *If* $H_1 < H_2 < G$, H_1, H_2 *fully invariant, then*

$$0 \to \mathrm{End}_{H_2}\, G \xrightarrow{I} \mathrm{End}_{H_1}\, G \xrightarrow{R} \mathrm{End}_{H_1}\, H_2$$

is exact, where I is the injection and R the restriction map. I and R are ring homomorphisms.

[5] 1. 23 **Proposition.** *If H is a fully invariant subgroup of G and* $1+\mathrm{End}_H G \subset \subset \mathrm{Aut}\, G$, *then*

$$(\mathrm{Aut}\, G)\, R = \textit{group of units of } (\mathrm{End}\, G)\, R,$$

where R is the restriction map to H.

Section 1. The chains $\{\mathrm{Aut}_{p^n G} G\}$ and $\{\mathrm{End}_{p^n G} G\}$.

In this section G denotes an abelian p-group. The groups $p^n G$ are fully invariant subgroups of G. By [5] 1. 9 and [5] 1. 19 the sequences

(1. 1) $$1 \to \mathrm{Aut}_{p^n G}\, G \xrightarrow{I} \mathrm{Aut}\, G \xrightarrow{R} \mathrm{Aut}\, p^n\, G$$

and

(1. 2) $$0 \to \mathrm{End}_{p^n G}\, G \xrightarrow{I} \mathrm{End}\, G \xrightarrow{R} \mathrm{End}\, p^n\, G \to \mathrm{Ext}(G/p^n\, G, G)$$

are exact, where I is the injection, R the restriction map, in particular, $\mathrm{Aut}_{p^n G}\, G$ is normal in Aut G and $\mathrm{End}_{p^n G}\, G$ is a two-sided ideal of End G.

What can be said about im R in 1. 1 and 1. 2? For a first result we only need to restate Lemma 1 in [3].

1. 3 Lemma. *If n is a positive integer and* $\delta \in \mathrm{End}\, p^n G$, *then* δ *can be extended to a* $\bar\delta \in \mathrm{End}\, G$ *in such a way that* $\bar\delta$ *is onto if* δ *is onto, and* $\bar\delta$ *is one-to-one if* δ *is one-to-one, in particular, if* $\delta \in \mathrm{Aut}\, p^n\, G$, *then* $\bar\delta \in \mathrm{Aut}\, G$.

The next lemma which answers our question for countable groups and endomorphisms is more or less Exercise 39 in [4].

1. 4 Lemma. *If G is countable, then every* $\delta \in \mathrm{End}\, p^n G$, *n arbitrary, can be extended to* $\bar\delta \in \mathrm{End}\, G$.

The corresponding lemma for automorphisms is due to ZIPPIN (Cf. [3]) and is contained in FREEDMAN [1], FUCHS [3] and more or less in KAPLANSKY [4] (Exercise 38).

1. 5 Lemma *(Zippin). If G is countable, then every $\alpha \in \operatorname{Aut} p^n G$, n arbitrary, can be extended to $\bar{\alpha} \in \operatorname{Aut} G$.*

Immediate consequences of 1. 1, 1. 2, 1. 3, 1. 4, and 1. 5 are the following theorems.

1. 6 Theorem. *The sequence*

$$1 \to \operatorname{Aut}_{p^n G} G \xrightarrow{I} \operatorname{Aut} G \xrightarrow{R} \operatorname{Aut} p^n G \to 1 \tag{1. 7}$$

is exact either for positive integers n and arbitrary p-groups G, or for arbitrary ordinals n and countable p-groups G.

1. 8 Theorem. *The sequence*

$$0 \to \operatorname{End}_{p^n G} G \xrightarrow{I} \operatorname{End} G \xrightarrow{R} \operatorname{End} p^n G \to 0 \tag{1. 9}$$

is exact either for positive integers n and arbitrary p-groups G, or for arbitrary ordinals n and countable p-groups G.

Remark. In his dissertation (An Extension of Ulm's Theorem, New Mexico State University, May 1964) Roger W. Mitchell constructed a number of interesting and surprising examples of abelian p-groups with elements of infinite height (Lemma 2. 1). The following is one of his examples. Let $B = \bigoplus_{i=1}^{\infty} \langle x_i \rangle$, $o(x_i) = p^i$, and let $\bar{B} = \left(\bigoplus_{i=1}^{\infty} {}^* \langle x_i \rangle \right)_t$. Then $G = \bar{B}/B[p]$ is a p-group with countable basic $B/B[p] \cong B$, such that $G^1 = p^\omega G = \bar{B}[p]/B[p]$ is an elementary abelian p-group of order $2^{\aleph_0}$. This group shows that 1. 6 is not true for arbitrary abelian p-groups and arbitrary ordinals n, since by [7], Thm 2, $|\operatorname{Aut} G| = 2^{|B|} = 2^{\aleph_0}$ while $|\operatorname{Aut}(G^1)| = 2^{2^{\aleph_0}}$ and thus not every automorphism of G^1 can be extended to an automorphism of G.

We now turn our attention to the sequence

$$0 \to \operatorname{Hom}(G/pG, pG) \xrightarrow{U} \operatorname{Aut}_{pG} G \xrightarrow{T} \operatorname{Aut}(G/pG) \text{ (ex)} \tag{1. 10}$$

which follows from [5], 1. 6. First $\operatorname{im} T$ will be determined. Let

$$\varphi: G \to G/pG$$

be the natural homomorphism. Let B be a basic subgroup of $G, B = \bigoplus_{i=1}^{\infty} B_i$, $B_1 = \bigoplus_{\lambda \in \Lambda} \langle a_\lambda \rangle$, $\bigoplus_{i \geqq 2} B_i = \bigoplus_{\mu \in M} \langle b_\mu \rangle$. Thus $G = B_1 \oplus G_1$. Then

$$G/pG = (\bigoplus_{\lambda} \langle a_\lambda \varphi \rangle) \oplus (\bigoplus_{\mu} \langle b_\mu \varphi \rangle) = B_1 \varphi \oplus G_1 \varphi.$$

1. 11 Definition. If, for any abelian group G, $G = H \oplus K$, let $[\pi H]: G \to H$ and $[\pi K]: G \to K$ be the projections.

Note that

$$[\pi B_1]\varphi = \varphi[\pi B_1 \varphi] \quad \text{and} \quad [\pi G_1]\varphi = \varphi[\pi G_1 \varphi]. \tag{1. 12}$$

There are certain obvious subgroups of Aut $G\varphi$, namely

$$A'_2 = \{1+[\pi G_1\varphi]\xi : \xi \in \operatorname{Hom}(G_1\varphi, B_1\varphi)\} < \operatorname{Aut} G\varphi$$

and

$$A'_3 = \{[\pi B_1\varphi]\alpha + [\pi G_1\varphi] : \alpha \in \operatorname{Aut} B_1\varphi\} < \operatorname{Aut} G\varphi.$$

It is easy to see that $A'_2\cdot A'_3 = A'_3\cdot A'_2$, in fact $A'_2 <| A'_2 A'_3$, and $A'_2 \cap A'_3 = \langle 1\rangle$, therefore we have

1. 13 Lemma. *$A'_2 \cap A'_3 = \langle 1\rangle$, $A'_2\cdot A'_3 < \operatorname{Aut} G$, and $A'_2 <| A'_3\cdot A'_3$.*

There are subgroups of $\operatorname{Aut}_{pG} G$ corresponding to A'_2 and A'_3. Let

$$\psi : G \to G/(B_1 \oplus pG)$$

be the natural homomorphism. Then

$$A_2 = \{1+\psi\xi : \xi \in \operatorname{Hom}\big(G/(B_1\oplus pG), B_1\big) < \operatorname{Aut}_{pG} G,$$

and

$$A_3 = \{[\pi B_1]\alpha + [\pi G_1] : \alpha \in \operatorname{Aut} B_1\} < \operatorname{Aut}_{pG} G.$$

Further we have analogous to 1. 13

1. 14 Lemma. *$A_2 \cap A_3 = \langle 1\rangle$, $A_2\cdot A_3 < \operatorname{Aut}_{pG} G$, and $A_2 <| A_2\cdot A_3$.*

1. 15 Lemma. *The map T in 1. 10 maps $A_2\cdot A_3$ isomorphically onto $A'_2\cdot A'_3$.*

Proof. a) Note that

$$\varphi|B_1 : B_1 \to B_1\varphi \tag{1. 16}$$

is an isomorphism, and that

$$\sigma : G\psi = G/(B_1\oplus pG) \to G_1/pG_1 = G_1\varphi : (g\psi)\sigma = g[\pi G_1]\varphi = g\varphi[\pi G_1\varphi] \tag{1. 17}$$

is an isomorphism. Therefore

$$F : \operatorname{Aut} B_1 \to \operatorname{Aut} B_1\varphi : \alpha F = (\varphi|B_1)^{-1}\alpha\varphi \tag{1.18}$$

is an isomorphism and

$$S : \operatorname{Hom}(G\psi, B_1) \to \operatorname{Hom}(G_1\varphi, B_1\varphi) : (g\psi\sigma)(\xi S) = g\psi\xi\varphi \tag{1. 19}$$

is an isomorphism.

b) Let $(1+\psi\xi)([\pi B_1]\alpha + [\pi G_1])$ be an arbitrary element in $A_2\cdot A_3$. Then

$$\varphi[(1+\psi\xi)([\pi B_1]\alpha+[\pi G_1])]T = (1+\psi\xi)([\pi B_1]\alpha+[\pi G_1])\varphi =$$
$$= (1+\psi\xi)([\pi B_1]\varphi(\varphi|B_1)^{-1}\alpha\varphi + \varphi[\pi G_1\varphi]) = (1+\psi\xi)\varphi([\pi B_1\varphi](\alpha F) + [\pi G_1\varphi]) =$$
$$= [\varphi + \psi\sigma(\xi S)][[\pi B_1\varphi](\alpha F)+[\pi G_1\varphi]] = \varphi[1+[\pi G_1\varphi](\xi S)][[\pi B_1\varphi](\alpha F)+[\pi G_1\varphi]],$$

hence

$$[(1+\psi\xi)([\pi B_1]\alpha+[\pi G_1])]T = [1+[\pi G_1\varphi](\xi S)][[\pi B_1\varphi](\alpha F)+[\pi G_1\varphi]], \tag{1. 20}$$

and it is now obvious that T maps $A_2\cdot A_3$ isomorphically onto $A'_2\cdot A'_3$.

1. 21 Proposition. *In 1. 10 im $T = A'_2\cdot A'_3$.*

PROOF. It is left to show that $\alpha T \in A_2' \cdot A_3'$ for all $\alpha \in \mathrm{Aut}_{pG} G$.

a) For all $\alpha \in \mathrm{Aut}_{pG} G$, $(\alpha T)|B_1\varphi \in \mathrm{Aut}\, B_1\varphi$. Proof: Let $\alpha \in \mathrm{Aut}_{pG} G$. Recall $B_1 = \bigoplus_{\lambda \in \Lambda} \langle a_\lambda \rangle$. For all a_λ, $o(a_\lambda \alpha) = p$, hence $a_\lambda \alpha = \sum_{\mu \in \Lambda} k_\mu a_\mu + pg$, $g \in G$. Now $(a_\lambda \varphi)(\alpha T) = a_\lambda \alpha \varphi = \sum_{\mu \in \Lambda} k_\mu (a_\mu \varphi) \in B_1 \varphi$, hence $(B_1\varphi)(\alpha T) < B_1\varphi$ for all αT. Thus in particular $(B_1\varphi)(\alpha^{-1}T) < B_1\varphi$, i. e. $B_1\varphi < (B_1\varphi)(\alpha T)$ and therefore $(B_1\varphi)(\alpha T) = B_1\varphi$, $\alpha T | B_1\varphi \in \mathrm{Aut}\, B_1\varphi$.

b) Let $\alpha \in \mathrm{Aut}_{pG} G$. By a) $(\alpha T)|B_1\varphi \in \mathrm{Aut}\, B_1\varphi$, say $(\alpha T)|B_1\varphi = \beta^{-1}$. By 1. 20 with $\xi = 0$ $\left[\alpha\left([\pi B_1](\beta F^{-1}) + [\pi G_1]\right)\right] T = (\alpha T)\left[[\pi B_1 \varphi]\beta + [\pi G_1 \varphi]\right]$, and thus $\alpha T \cdot \left[[\pi B_1\varphi]\beta + [\pi G_1 \varphi]\right] \in \mathrm{im}\, T \cap \mathrm{Aut}_{B_1\varphi} G\varphi$.

c) Let $\gamma \in \mathrm{Aut}_{pG} G$ be such that $\gamma T \in \mathrm{Aut}_{B_1\varphi} G\varphi$. Recall $\bigoplus_{i \geq 2} B_i = \bigoplus_{\mu \in M} \langle b_\mu \rangle \cdot$ $p(b_\mu \gamma - b_\mu) = pb_\mu \gamma - pb_\mu = pb_\mu - pb_\mu = 0$, hence $b_\mu \gamma - b_\mu = \sum_\lambda k_\lambda a_\lambda + pg$, $g \in G$. Then $(b_\mu \varphi)(\gamma T - 1) = b_\mu \varphi(\gamma T) - b_\mu \varphi = b_\mu \gamma \varphi - b_\mu \varphi = (b_\mu \gamma - b_\mu)\varphi = \sum_\lambda k_\lambda (a_\lambda \varphi) \in B\varphi_1$. $(\gamma T - 1)|B_1 \varphi = 0$. Hence $\gamma T = 1 + (\gamma T - 1) \in A_2'$.

d) For arbitrary $\alpha \in \mathrm{Aut}_{pG} G$ we know by b) and c) $\alpha T([\pi B_1 \varphi]\beta + [\pi G_1 \varphi]) \in \mathrm{im}\, T \cap \mathrm{Aut}_{B_1\varphi} G\varphi < A_2'$, where $\beta^{-1} = (\alpha T)|B_1\varphi$, hence $\alpha T \in A_2' \cdot A_3'$ which was to be proved.

The structure of $\mathrm{Aut}_{pG} G$ now follows immediately from 1.15 and 1.21. In order to formulate the result with ease we introduce the following notion.

1. 22 Definition. Let A be an arbitrary multiplicative group. A will be called the semi-direct product of its subgroups A_1, A_2, A_3 if $A_1 <| A$, $A_1 A_2 <| A$, $A = A_1 A_2 A_3$, $A_1 \cap A_2 = \langle 1 \rangle$ and $A_1 A_2 \cap A_3 = \langle 1 \rangle$.

We now have the basic theorem

1. 23 Theorem. *Let G be an abelian p-group. For some basic $B = \bigoplus_{i=1}^{\infty} B_i$, write $G = B_1 \oplus G_1$, and let $[\pi B_1]$ and $[\pi G_1]$ be the corresponding projections from G onto B_1 respectively G_1. Let $\varphi\colon G \to G/pG$ and $\psi\colon G \to G/(B_1 \oplus pG)$ be the natural homomorphisms. Then*

$$A_1 = \{1 + \varphi\xi\colon \xi \in \mathrm{Hom}\,(G/pG, pG)\} < \mathrm{Aut}_{pG}\, G,$$

$$A_2 = \{1 + \psi\eta\colon \eta \in \mathrm{Hom}\,(G/(B_1 \oplus pG), B_1)\} < \mathrm{Aut}_{pG}\, G,$$

$$A_3 = \{[\pi B_1]\alpha + [\pi G_1]\colon \alpha \in \mathrm{Aut}\, B_1\} < \mathrm{Aut}_{pG}\, G,$$

and $\mathrm{Aut}_{pG}\, G$ is the semi-direct product of A_1, A_2 and A_3.

Remark. The groups A_1, A_2 and A_3 are well-known groups. In fact

$$A_1 \cong \mathrm{Hom}\,(G/pG, pG) \cong \bigoplus_{r(G/pG)}^{*} (pG)[p],$$

$$A_2 \cong \mathrm{Hom}\,(G/(B_1 \oplus pG), B_1) \cong \bigoplus_{r(pG/p^2G)}^{*} \bigoplus_{f_G(0)} Z(p),$$

and

$$A_3 \cong \mathrm{Aut}\, B_1 \cong GL(f_G(0), p).$$

Here $f_G(n)$ is the n-th Ulm-invariant of G.

Before we apply 1. 23 we want to state some of the properties of $\mathrm{Aut}_{pG}\, G$, in particular, we want to find the center $Z(\mathrm{Aut}_{pG}\, G)$ of $\mathrm{Aut}_{pG}\, G$. First note that $\mathrm{Aut}_{pG}\, G$ contains a subgroup A isomorphic to $\mathrm{Hom}\,(G/(B_1 \oplus pG), B_1 \oplus pG)$, namely let ($\psi$ as before)

$$A = \{1+\psi\xi\colon \xi \in \mathrm{Hom}\,(G/(B_1 \oplus pG), B_1 \oplus pG)\}. \tag{1. 24}$$

Since $\mathrm{Hom}\,(G\psi, B_1 \oplus pG) \cong \mathrm{Hom}\,(G\psi, B_1) \oplus \mathrm{Hom}\,(G\psi, pG)$, A is a direct product, in fact, if

$$C = \{1+\psi\xi\colon \xi \in \mathrm{Hom}\,(G/(B_1 \oplus pG), pG)\} \tag{1. 25}$$

then

$$A = C \times A_2. \tag{1. 26}$$

Furthermore $\mathrm{Hom}\,(G/pG, pG) = \mathrm{Hom}\,(G\varphi, pG) = \mathrm{Hom}\,(B_1\varphi \oplus G_1\varphi, pG) = [\pi B_1\varphi]\,\mathrm{Hom}\,(B_1\varphi, pG) \oplus [\pi G_1\varphi]\,\mathrm{Hom}\,(G_1\varphi, pG)$. The isomorphism σ defined in 1. 17 yields the isomorphism

$$S\colon \mathrm{Hom}\,(G_1\varphi, pG) \to \mathrm{Hom}\,(G\psi, pG)\colon (g\psi)(\eta S) = g\psi\sigma\eta = g\varphi[\pi G_1\varphi]\eta \tag{1. 27}$$

and therefore $C = \{1+\psi\xi\colon \xi \in \mathrm{Hom}\,(G\psi, pG)\} = \{1+\varphi[\pi G_1\varphi]\eta\colon \eta \in \mathrm{Hom}\,(G_1\varphi, pG)\}$. If $D = \{1+\psi[\pi B_1\varphi]\xi\colon \xi \in \mathrm{Hom}\,(B_1\varphi, pG)\}$, then

$$A_1 = D \times C. \tag{1. 28}$$

Since A and A_1 are abelian groups and $C < A \cap A_1$, every element of C commutes with every element of A and A_1. Let $[\pi B_1]\alpha + [\pi G_1] \in A_3$ and $1+\psi\xi \in C$. Then the equations

$$([\pi B_1]\alpha + [\pi G_1])(1+\psi\xi) = [\pi B_1]\alpha + [\pi G_1] + [\pi G_1]\psi\xi = [\pi B_1]\alpha + [\pi G_1] + \psi\xi$$

and

$$(1+\psi\xi)([\pi B_1]\alpha + [\pi G_1]) = [\pi B_1]\alpha + [\pi G_1] + \psi\xi[\pi G_1] = [\pi B_1]\alpha + [\pi G_1] + \psi\xi$$

show that

$$C < Z(A_1 A_2 A_3) = Z(\mathrm{Aut}_{pG}\, G).$$

1. 29 Proposition. *Let G be as in* 1. 23, $G = B_1 \oplus G_1$.

A) If $G_1 = 0$, *then* $\mathrm{Aut}_{pG}\, G = \mathrm{Aut}\, B_1 \cong GL(f_G(o), p)$ *and the center is known.*
B) If $G = Z(2) \oplus 2G$, *then* $\mathrm{Aut}_{2G}\, G = \{1+\varphi\xi\colon \xi \in \mathrm{Hom}\,(G/2G, 2G)\}$ *is abelian.*
C) *In all other cases*

$$Z(\mathrm{Aut}_{pG}\, G) = \{1+\psi\xi\colon \xi \in \mathrm{Hom}\,(G/(B_1 \oplus pG), pG)\}.$$

Proof. The cases A) and B) are clear. Therefore assume $G_1 \neq 0$, i. e. $pG \neq 0$, and $G \neq Z(2) \oplus 2G$. We have $\mathrm{Aut}_{pG}\, G = D \cdot C \cdot A_2 \cdot A_3$. We show in a) that, for $B_1 \neq \langle 0 \rangle$, $\alpha_2 \in A_2$, $\alpha_3 \in A_3$ and $\delta\alpha_2\alpha_3 = \alpha_2\alpha_3\delta$ for all $\delta \in D$ implies $\alpha_2\alpha_3 = 1$. We show in b) that, for $B_1 \neq \langle 0 \rangle$, $\delta \in D$ and $\delta\alpha_2\alpha_3 = \alpha_2\alpha_3\delta$ for all $\alpha_2 \in A_3$ and for all $\alpha_3 \in A_3$ implies $\delta = 1$.

a) Assume $\delta\alpha_2\alpha_3 = \alpha_2\alpha_3\delta$ for $\alpha_2 \in A_2$, $\alpha_3 \in A_3$ and for all $\delta \in D$. Let

$$\alpha_2 = 1+\psi\eta,\ \eta \in \mathrm{Hom}\,(G/(B_1 \oplus pG), B_1),\ \alpha_3 = [\pi B_1]\alpha + [\pi G_1],\ \alpha \in \mathrm{Aut}\, B_1,$$

and

$$\delta = 1+\varphi[\pi B_1\varphi]\xi,\ \xi \in \mathrm{Hom}\,(B_1\varphi, pG).$$

Then

$$\delta\alpha_2\alpha_3 = [\pi B_1]\alpha + [\pi G_1] + \psi\eta\alpha + \varphi[\pi B_1\varphi]\xi \tag{1.30}$$

and

$$\alpha_2\alpha_3\delta = [\pi B_1]\alpha + [\pi G_1] + \psi\eta\alpha + [\pi B_1]\alpha\varphi\xi + \psi\eta\alpha\varphi\xi. \tag{1.31}$$

It follows that

$$\varphi\xi = \alpha\varphi\xi \quad \text{on} \quad B_1 \quad \text{for all} \quad \xi \in \operatorname{Hom}(B_1\varphi, pG).$$

If $B_1 \neq 0$, then it follows easily that $\alpha = 1$. With $\alpha = 1$ we have

$$\psi\eta\varphi\xi = 0 \quad \text{for all} \quad \xi \in \operatorname{Hom}(B_1\varphi, pG).$$

Since ψ is onto, clearly $\eta = 0$.

b) Assume that $\delta\alpha_2\alpha_3 = \alpha_2\alpha_3\delta$ for $\delta \in D$, for all $\alpha_2 \in A_2$, and for all $\alpha_3 \in A_3$. Using the notation of a) it follows using 1. 30 and 1. 31 that (choosing $\alpha = 1$)

$$\psi\eta\varphi\xi = 0 \quad \text{for all} \quad \eta \in \operatorname{Hom}\big(G/(B_1 \oplus pG),\, B_1\big).$$

If $G \neq B_1 \oplus pG$ and $B_1 \neq \langle 0 \rangle$, then necessarily $\xi = 0$. If $G = B_1 \oplus pG$, then $\eta = 0$ and

$$\alpha\varphi\xi = \varphi\xi \quad \text{on } B_1 \text{ for all} \quad \alpha \in \operatorname{Aut} B_1.$$

If $\operatorname{Aut} B_1 \neq \langle 1 \rangle$, then $\xi = 0$. If $\operatorname{Aut} B_1 = \langle 1 \rangle$, but $B_1 \neq \langle 0 \rangle$, then $B_1 = Z(2)$ and $G = Z(2) \oplus 2G$.

c) Assume $\alpha_0\alpha_1\alpha_2\alpha_3$ is a central element of $\operatorname{Aut}_{pG} G$ where $\alpha_0 \in D$, $\alpha_1 \in C$, $\alpha_2 \in A_2$ and $\alpha_3 \in A_3$. Then for all $\delta \in D$

$$\delta\alpha_0\alpha_1\alpha_2\alpha_3 = \alpha_0\alpha_1\alpha_2\alpha_3\delta,$$

hence

$$\delta\alpha_2\alpha_3 = \alpha_2\alpha_3\delta \text{ for all } \delta \in D.$$

By a), if $B_1 \neq \langle 0 \rangle$, $\alpha_2\alpha_3 = 1$. If $\alpha_0\alpha_1$ is central then $\alpha_0\alpha_1\alpha_2\alpha_3 = \alpha_2\alpha_3\alpha_0\alpha_1$ for all $\alpha_2 \in A_2$ and for all $\alpha_3 \in A_3$, hence

$$\alpha_0\alpha_2\alpha_3 = \alpha_2\alpha_3\alpha_0$$

for all $\alpha_2 \in A_2$ and for all $\alpha_3 \in A_3$. By b), if $B_1 \neq \langle 0 \rangle$, $\alpha_0 = 1$. This proves C) if $B_1 \neq \langle 0 \rangle$. But if $B_1 = \langle 0 \rangle$, then $\operatorname{Aut}_{pG} G = C$ and C) is also true.

Now consider the sequence

$$1 \to \operatorname{Aut}_{p^n G} G \xrightarrow{I} \operatorname{Aut}_{p^{n+1} G} G \xrightarrow{R} \operatorname{Aut}_{p^{n+1} G} p^n G \text{ (ex)}$$

which follows from [5], 1. 10. Applying 1. 6, then 1. 23 to $\operatorname{Aut}_{p^{n+1}G} p^n G = \operatorname{Aut}_{p(p^n G)} p^n G$ we obtain

1. 32 Theorem. *The sequence*

$$1 \to \operatorname{Aut}_{p^n G} G \xrightarrow{I} \operatorname{Aut}_{p^{n+1} G} G \xrightarrow{R} \operatorname{Aut}_{p^{n+1} G} p^n G \to 1 \tag{1.33}$$

is exact either for positive integers n and arbitrary p-group G, or for arbitrary ordinals n and countable p-groups G.

If $B^{(n)}=\bigoplus_{i=1}^{\infty} B_i^{(n)}$ *is a basic of* p^nG, $p^nG=B_1^{(n)}\oplus G_1^{(n)}$, *then* $\mathrm{Aut}_{p^{n+1}G}p^nG$ *is the semidirect product of groups* $A_1^{(n)}$, $A_2^{(n)}$ *and* $A_3^{(n)}$ *where*

$$A_1^{(n)} \cong \mathrm{Hom}\,(p^n G/p^{n+1} G, p^{n+1} G), \quad A_2^{(n)} \cong \mathrm{Hom}\,(p^n G/(B_1^{(n)}\oplus p^{n+1} G), B_1^{(n)}),$$

and

$$A_3^{(n)} \cong \mathrm{Aut}\, B_1^{(n)}.$$

Remark. Note that

$$A_1^{(n)} \cong \oplus^*_{r(p^n G/p^{n+1} G)}(p^{n+1} G)[p], \quad A_2^{(n)} \cong \oplus^*_{r(p^{n+1}G/p^{n+2} G)}\oplus_{f_G(n)}Z(p),$$

and

$$A_3^{(n)} \cong GL(f_G(n), p).$$

Remark. Theorem 1.32 contains an explicit determination of the factor groups $\mathrm{Aut}_{p^{n+1}G}G/\mathrm{Aut}_{p^nG}G$ if n is a positive integer and for arbitrary n if G is countable. We shall derive Fuchs' results in [3] from ours. $\mathrm{Aut}_{pG}G$ contains the normal subgroup A (1. 24) and $A\cdot D$ (1. 28). Hence there are two normal subgroups A_n^* and A_n^{**} of $\mathrm{Aut}_{p^{n+1}G}G$ such that

$$\mathrm{Aut}_{p^nG}G<|A_n^*<|A_n^{**}<|\mathrm{Aut}_{p^{n+1}G}G$$

and

$$A_n^*/\mathrm{Aut}_{p^n G}G \cong \oplus^*_{r(p^{n+1} G/p^{n+2} G)}(p^n G)[p], \quad A_n^{**}/A_n^* \cong \oplus^*_{f_G(n)}(p^{n+1} G)[p],$$

and

$$\mathrm{Aut}_{p^{n+1}G}G/A_n^{**} \cong GL(f_G(n), p).$$

It is easy to check that

$$A_n^* = \mathrm{Aut}_{B_1^{(n)}\oplus p^{n+1} G}G = \mathrm{Aut}_{(p^n G)[p]+p^{n+1} G}G$$

and that

$$A_n^*=\{\alpha\in\mathrm{Aut}_{p^{n+1}G}G\colon \text{ for all } g\in(p^nG)[p],\ g\alpha=g \bmod (p^{n+1}G)[p]\}.$$

The above factor groups were determined by Fuchs ([3], Theorems 2, 3, 4). Thus his results follow from our more precise results.

Freedman ([1]) uses a different approach for reduced countable p-groups and obtains the following somewhat abstract result on the structure of the quotient groups $\mathrm{Aut}_{p^{n+1}G}G/\mathrm{Aut}_{p^nG}G$ (Cor. 5. 4): "The factor group $\mathrm{Aut}_{p^{n+1}G}G/\mathrm{Aut}_{p^nG}G$ is the product of a p-subgroup $\overline{K}_{n+1}$ and a subgroup isomorphic to $GL(f_G(n), p)$. $\overline{K}_{n+1}$ is generated by two abelian p-elementary subgroups whose intersection is the common center of $\overline{K}_{n+1}$ and $\mathrm{Aut}_{p^{n+1}G}G/\mathrm{Aut}_{p^nG}G$, except when p^nG is of exponent 1."

$\overline{K}_{n+1}$ corresponds to $A_1^{(n)}\cdot A_2^{(n)}$ in our notation. For a comparison it suffices to consider A_1A_2 (cf. 1. 23). For $1+\varphi\xi\in A_1$ and $1+\psi\eta\in A_2$ it follows by induction that

$$[(1+\varphi\xi)(1+\psi\eta)]^n = 1+\varphi n\xi+\psi n\eta+\psi 1/2(n-1)n\eta\varphi\xi$$

and hence $[(1+\varphi\xi)(1+\psi\eta)]^p=1$ if $p\neq 2$, and if $p=2|[(1+\psi\xi)(1+\psi\eta)]^4=1$. Thus $A_1^{(n)}A_2^{(n)}$ is a p-group, and we have obtained all of Freedman's results concerning the quotient group $\mathrm{Aut}_{p^{n+1}G}G/\mathrm{Aut}_{p^nG}G$ from our more explicit results.

We shall now carry through the analogous program for the ideal structure of the endomorphism ring. Consider the exact sequence

$$(1.34)\qquad \begin{aligned} &0\to\{\varepsilon\in\mathrm{End}_{pG}G : \mathrm{im}\,\varepsilon<pG\}\xrightarrow{I}\mathrm{End}_{pG}G\xrightarrow{T}\mathrm{End}\,G/pG\to\\ &\to\mathrm{Ext}\,(G/pG, pG)\to\mathrm{Ext}\,(G/pG, G)\to\mathrm{Ext}\,(G/pG, G/pG)\to 0\end{aligned}$$

which follows from [5] 1. 17. We continue to use the notation introduced at the beginning of the section. Note that

$$(1.35)\qquad \{\delta\in \mathrm{End}_{pG}G\colon \mathrm{im}\,\delta < pG\} = \varphi\mathrm{Hom}\,(G/pG, pG),$$

and we assume in the following (also in analogous cases) that φ Hom$(G/pG, pG)$ is a ring with the obvious compositions. Also

(1. 36)

$$\mathrm{End}\,G = [\pi B_1]\,\mathrm{End}\,B_1 \oplus [\pi B_1]\,\mathrm{Hom}\,(B_1, G_1) \oplus [\pi G_1]\,\mathrm{Hom}\,(G_1, B_1) \oplus [\pi G_1]\,\mathrm{End}\,G_1.$$

Obviously $[\pi B_1]\,\mathrm{End}\,B_1 < \mathrm{End}_{pG}\,G$. Furthermore $[\pi G_1]\,\mathrm{Hom}\,(G_1, B_1) < \mathrm{End}_{pG}\,G$, since, for all $\xi\in\mathrm{Hom}\,(G_1, B_1)$ and for all $g\in G$, $(pg)[\pi G_1]\xi = p[g[\pi G_1]\xi] = 0$. We have similarly

$$(1.37)\qquad \mathrm{End}\,G\varphi = [\pi B_1\varphi]\,\mathrm{End}\,B_1\varphi \oplus [\pi B_1\varphi]\,\mathrm{Hom}\,(B_1\varphi, G_1\varphi) \oplus [\pi G_1\varphi]$$
$$\mathrm{Hom}\,(G_1\varphi, B_1\varphi) \oplus [\pi G_1\varphi]\,\mathrm{End}\,G_1\varphi.$$

1. 38 Lemma. *T: $\mathrm{End}_{pG}G \to \mathrm{End}\,G\varphi$ maps $[\pi B_1]\,\mathrm{End}\,B_1 \oplus [\pi G_1]\,\mathrm{Hom}\,(G_1, B_1)$ isomorphically onto $[\pi B_1\varphi]\,\mathrm{End}\,B_1\varphi \oplus [\pi G_1\varphi]\,\mathrm{Hom}\,(G_1\varphi, B_1\varphi)$.*

Proof. Note that

$$(1.39)\qquad S\colon \mathrm{Hom}\,(G_1, B_1) \to \mathrm{Hom}\,(G_1\varphi, B_1\varphi)\colon (g\varphi)(\xi S) = g\xi\varphi,$$

$g\in G_1$, $\xi\in\mathrm{Hom}\,(G_1, B_1)$, and that

$$(1.40)\qquad F\colon \mathrm{End}\,B_1 \to \mathrm{End}\,B_1\varphi\colon \delta F = (\varphi|B_1)^{-1}\delta\varphi,\ \delta\in\mathrm{End}\,B_1,$$

are isomorphisms. Let $\xi\in\mathrm{Hom}\,(G_1, B_1)$ and $\delta\in\mathrm{End}\,B_1$. Then, using 1. 12, $\varphi([\pi B_1]\delta + [\pi G_1]\xi)T = ([\pi B_1]\delta + [\pi G_1]\xi)\varphi = [\pi B_1]\varphi(\varphi|B_1)^{-1}\delta\varphi + [\pi G_1]\xi\varphi =$
$= \varphi[[\pi B_1\varphi](\delta F) + [\pi G_1\varphi](\xi S)]$, thus $([\pi B_1]\delta + [\pi G_1]\xi)T = [\pi B_1\varphi](\delta F) + [\pi G_1\varphi](\xi S)$ which clearly implies the assertion.

1. 41 Proposition. *In* 1. 34 $\mathrm{im}\,T = [\pi B_1\varphi]\,\mathrm{End}\,B_1\varphi \oplus [\pi G_1\varphi]\,\mathrm{Hom}\,(G_1\varphi, B_1\varphi)$.

Proof. Let $\delta\in\mathrm{End}_{pG}G$. Then, for all $g\in G$, $p(g\delta) = (pg)\delta = 0$, hence $g\delta = \sum_\lambda k_\lambda a_\lambda + pg'$, and thus $(g\varphi)(\delta T) = (g\delta)\varphi = \sum_\lambda k_\lambda(a_\lambda\varphi)$. Therefore $\mathrm{im}\,T < [\pi B_1\varphi]\cdot\mathrm{End}\,B_1\varphi \oplus [\pi G_1\varphi]\,\mathrm{Hom}\,(G_1\varphi, B_1\varphi)$ and 1. 38 concludes the proof.

1. 42 Theorem. *Let G be an abelian p-group. For some basic $B = \bigoplus_{i=1}^{\infty} B_i$, write $G = B_1 \oplus G_1$, and let $[\pi B_1]$ and $[\pi G_1]$ be the corresponding projections from G onto B_1 respectively G_1. Let $\varphi\colon G \to G/pG$ be the natural homomorphism. Then*

$$(1.43)\qquad \mathrm{End}_{pG}G = [\pi B_1]\,\mathrm{End}\,B_1 \oplus \varphi\,\mathrm{Hom}\,(G/pG, pG) \oplus [\pi G_1]\,\mathrm{Hom}\,(G_1, B_1).$$

$\varphi\,\mathrm{Hom}\,(G/pG, pG)$ is a two-sided ideal of $\mathrm{End}_{pG}G$ with trivial multiplication, $\varphi\,\mathrm{Hom}\,(G/pG, pG) \oplus [\pi G_1]\,\mathrm{Hom}\,(G_1, B_1)$ is a two-sided ideal of $\mathrm{End}_{pG}G$, and $[\pi B_1]\,\mathrm{End}\,B_1$ is a subring of $\mathrm{End}_{pG}G$.

PROOF. 1. 43 is an immediate consequence of 1. 38 and 1. 41. By 1. 35 $\{\delta\in \mathrm{End}_{pG}G\colon\ \mathrm{im}\,\delta<pG\}=\varphi\,\mathrm{Hom}\,(G/pG,\,pG)$. That $\varphi\,\mathrm{Hom}\,(G/pG,\,pG)\oplus[\pi G_1]$ $\mathrm{Hom}\,(G_1,\,B_1)$ is a two-sided ideal of $\mathrm{End}_{pG}G$ is easy to see.

As in the case of $\mathrm{Aut}_{pG}G$ it is easy to find the center $Z(\mathrm{End}_{pG}G)$ of $\mathrm{End}_{pG}G$.

1. 44 Proposition. *Let G be as in* 1. 42, $G=B_1\oplus G_1$. *If* $G_1=0$, *then* $\mathrm{End}_{pG}G=\mathrm{End}\,G=\mathrm{End}\,B_1$. *If* $G_1\neq 0$, *then* $Z(\mathrm{End}_{pG}G)=[\pi G_1]\varphi\,\mathrm{Hom}\,(G_1/pG,\,pG)$, *and the multiplication in* $Z(\mathrm{End}_{pG}G)$ *is trivial.*

PROOF. a) The case $G_1=0$ is clear.

b) Assume now $G_1\neq 0$, i. e. $pG\neq 0$. Further assume $B_1\neq 0$ and $G_1\varphi\neq 0$. $(Z(\mathrm{End}_{pG}G))T<Z(\mathrm{im}\,T)$. Let $\gamma,\,\delta\in\mathrm{End}\,B_1\varphi,\ \xi,\,\eta\in\mathrm{Hom}\,(G_1\varphi,\,B_1\varphi)$. Then $([\pi B_1\varphi]\gamma+[\pi G_1\varphi]\xi)([\pi B_1\varphi]\delta+[\pi G_1\varphi]\eta)=[\pi B_1\varphi]\gamma\delta+[\pi G_1\varphi]\xi\delta$, and $([\pi B_1\varphi]\delta+[\pi G_1\varphi]\eta)([\pi B_1\varphi]\gamma+[\pi G_1\varphi]\xi)\ =\ [\pi B_1\varphi]\delta\gamma+[\pi G_1\varphi]\eta\gamma$. If $[\pi B_1\varphi]\gamma+[\pi G_1\varphi]\xi\in Z(\mathrm{im}\,T)$, then $[\pi B_1\varphi]\gamma\delta+[\pi G_1\varphi]\xi\delta\ =\ [\pi B_1\varphi]\delta\gamma+[\pi G_1\varphi]\eta\gamma$ which implies (multiplication by $[\pi G_1\varphi]$ from the left) $\xi\delta=\eta\gamma$ for all $\delta\in\mathrm{End}\,B_1\varphi$ and all $\eta\in\mathrm{Hom}\,(G_1\varphi,\,B_1\varphi)$. Let $\delta=0$; then $\eta\gamma=0$ for all $\eta\in\mathrm{Hom}\,(G_1\varphi,\,B_1\varphi)$ and hence $\gamma=0$. Let $\eta=0$; then $\xi\delta=0$ for all $\delta\in\mathrm{End}\,B_1\varphi$, hence $\xi=0$. Thus under the stated hypotheses $Z(\mathrm{im}\,T)=0$, $Z(\mathrm{End}_{pG}G)<\varphi\,\mathrm{Hom}\,(G/pG,\,pG)$.

c) Assume $B_1\neq 0$. $\varphi\,\mathrm{Hom}\,(G\varphi,\,pG)=[\pi B_1]\varphi\,\mathrm{Hom}\,(G\varphi,\,pG)\oplus[\pi G_1]\,\varphi\,\mathrm{Hom}\,(G\varphi,pG)=[\pi B_1]\varphi\,\mathrm{Hom}\,(B_1\varphi,pG)\oplus[\pi G_1]\varphi\,\mathrm{Hom}\,(G_1\varphi,pG)$. Let $\xi\in\mathrm{Hom}\,(B_1\varphi,pG)$, $\eta\in\mathrm{Hom}\,(G_1\varphi,pG)$, and $\gamma\in\mathrm{End}\,B_1$. Then $([\pi B_1]\gamma)([\pi B_1]\varphi\xi+[\pi G_1]\varphi\eta)=[\pi B_1]\gamma\varphi\xi$ and $([\pi B_1]\varphi\xi+[\pi G_1]\varphi\eta)([\pi B_1]\gamma)=0$. Hence if $[\pi B_1]\varphi\xi+[\pi G_1]\varphi\eta\in Z(\mathrm{End}_{pG}G)$, then $\gamma\varphi\xi=0$ for all $\gamma\in\mathrm{End}\,B_1$, i.e. $\xi=0$.

d) Let $\xi,\,\eta\in\mathrm{Hom}\,(G\varphi,pG)$ and $\zeta\in\mathrm{Hom}\,(G_1,\,B_1)$. Then $([\pi G_1]\varphi\xi)(\varphi\eta)=0$ and $(\varphi\eta)([\pi G_1]\varphi\xi)=0$, $([\pi G_1]\varphi\xi)([\pi G_1]\zeta)=0$ and $([\pi G_1]\zeta)([\pi G_1]\varphi\xi)=0$, since $[\pi G_1]\,\mathrm{Hom}\,(G_1,\,B_1)=[\pi G_1]\varphi\,\mathrm{Hom}\,(G_1/pG,\,B_1)$.

e) If $B_1\neq 0$ and $G_1\varphi\neq 0$, then by b) $Z(\mathrm{End}_{pG}G)<\varphi\mathrm{Hom}\,(G\varphi,\,pG)$; by c) $Z(\mathrm{End}_{pG}G)<[\pi G_1]\varphi$ Hom $(G_1\varphi,$ $pG)$, and by c) and d) $Z(\mathrm{End}_{pG}G)=[\pi G_1]\varphi\,\mathrm{Hom}\,(G_1\varphi,\,pG)$. Clearly the multiplication in $[\pi G_1]\varphi\,\mathrm{Hom}\,(G_1\varphi,\,pG)$ is trivial. If $B_1=0$, then $\mathrm{End}_{pG}G=\varphi\,\mathrm{Hom}\,(G/pG,\,pG)=[\pi G_1]\varphi\,\mathrm{Hom}\,(G_1\varphi,\,pG)$ which has trivial multiplication. If $B_1\neq 0$, $G_1\varphi=0$, i. e. G_1 is divisible, then $\mathrm{End}_{pG}G=[\pi B_1]\,\mathrm{End}\,B_1\oplus[\pi B_1]\varphi\,\mathrm{Hom}\,(B_1\varphi,\,pG)$ and it follows easily that $Z(\mathrm{End}_{pG}G)=\langle 0\rangle=[\pi G_1]\varphi\,\mathrm{Hom}\,(G_1\varphi,\,B_1)$.

Now [5] 1. 22, 1. 8 and 1. 41 yield the following theorem analogous to 1. 32.

1. 44 Theorem. *The sequence*

$$0\to\mathrm{End}_{p^nG}G\xrightarrow{I}\mathrm{End}_{p^{n+1}G}G\xrightarrow{R}\mathrm{End}_{p^{n+1}G}\,p^nG\to 0, \tag{1. 45}$$

I the injection, R the restriction map, is exact either for positive integers n and arbitrary p-groups G, or for arbitrary ordinals n and countable p-groups G.

Let $B^{(n)}=\bigoplus_{i=1}^{\infty}B_i^{(n)}$ *be a basic of* p^nG, $p^nG=B_1^{(n)}\oplus G_1^{(n)}$, *further let* $[\pi B_1^{(n)}]$ *and* $[\pi G_1^{(n)}]$ *be the corresponding projections from* p^nG *onto* $B_1^{(n)}$ *respectively* $G_1^{(n)}$, *and finally let* $\varphi_n\colon p^nG\to(p^nG)/(p^{n+1}G)$ *be the natural homomorphism. Then*

$$\mathrm{End}_{p^{n+1}G}p^nG=[\pi B_1^{(n)}]\,\mathrm{End}\,B_1^{(n)}\oplus\varphi_n\ \mathrm{Hom}\,((p^nG)/(p^{n+1}G),\,p^{n+1}G)\oplus$$
$$\oplus[\pi G_1^{(n)}]\,\mathrm{Hom}\,(G_1^{(n)},\,B_1^{(n)}).$$

φ_n Hom $((p^nG)/(p^{n+1}G), p^{n+1}G)$ *is a two-sided ideal of* $\mathrm{End}_{p^{n+1}G}p^nG$ *with trivial multiplication, and* φ_n Hom $((p^nG)/(p^{n+1}G), p^{n+1}G)\oplus[\pi G_1^{(n)}]$ Hom $(G_1^{(n)}, B_1^{(n)})$ *is a two-sided ideal of* $\mathrm{End}_{p^{n+1}G}p^nG$.

Section 2. The chains $\{\mathrm{Aut}_{G[p^i]}G\}$ and $\{\mathrm{End}_{G[p^i]}G\}$

The groups $G[p^i]$, $i=1, 2, \ldots$, form a countable sequence of fully invariant subgroups of G. By [5] 1. 9 and [5] 1. 19 the sequences

$$1\to \mathrm{Aut}_{G[p^i]}G \xrightarrow{I} \mathrm{Aut}\, G \xrightarrow{R} \mathrm{Aut}\, G[p^i] \tag{2.1}$$

and

$$0\to \mathrm{End}_{G[p^i]}G \xrightarrow{I} \mathrm{End}\, G \xrightarrow{R} \mathrm{End}\, G[p^i] \to \mathrm{Ext}\,(G/G[p^i], G) \tag{2.2}$$

are exact, in particular $\mathrm{Aut}_{G[p^i]}G$ is normal in Aut G and $\mathrm{End}_{G[p^i]}G$ is a two-sided ideal of End G. Hence we have a chain

$$\mathrm{Aut}\, G > \mathrm{Aut}_{G[p]}G > \mathrm{Aut}_{G[p^2]}G > \ldots > \mathrm{Aut}_{G[p^i]}G > \ldots \tag{2.3}$$

of normal subgroups of Aut G, and $\bigcap_i \mathrm{Aut}_{G[p^i]}G=\langle 1\rangle$. Similarly

$$\mathrm{End}\, G > \mathrm{End}_{G[p]}G > \mathrm{End}_{G[p^2]}G > \ldots > \mathrm{End}_{G[p^i]}G > \ldots \tag{2.4}$$

is a chain of two-sided ideals of End G, and obviously $\bigcap_i \mathrm{End}_{G[p^i]}G=\langle 0\rangle$.

Except for a special case we are unable to determine im R in 2. 1 or 2. 2. However with the help of [5] 1. 23 we obtain a relationship between im R in 2. 1 and im R in 2. 2. The following lemma is needed.

2. 5 Lemma. *Let G be an abelian p-group. Then, for all* $\delta\in\mathrm{End}_{G[p^i]}G$, $1-\delta\in\mathrm{Aut}\, G$, *in fact* $1-\delta\in\mathrm{Aut}_{G[p^i]}G$.

PROOF. a) W: $\mathrm{End}_{G[p^i]}G\to\mathrm{Hom}\,(p^iG, G)$: $(p^ig)(\delta W)=g\delta$, $g\in G$, $\delta\in\mathrm{End}_{G[p^i]}G$, is an isomorphism: δW is well-defined since ker $\delta > G[p^i]$, δW is homomorphic from p^iG to G, W is obviously homomorphic and one-to-one. Let $\xi\in\mathrm{Hom}(p^iG, G)$. Then $p^i\xi\in\mathrm{End}_{G[p^i]}G$ and $(p^ig)(p^i\xi W)=gp^i\xi$, hence $p^i\xi W=\xi$. b) By a) every $\delta\in\mathrm{End}_{G[p^i]}G$ may be written in the form $\delta=p^i\xi$ where $\xi=\delta W\in\mathrm{Hom}\,(p^iG, G)$. Now $1+\delta+\delta^2+\ldots=1+(p^i\xi)+(p^i\xi)^2+\ldots\in\mathrm{End}\, G$ since G is a p-group. If $o(g)=p^n$, then $g(1+\delta+\delta^2+\ldots)=g\big(1+p^i\xi+(p^i\xi)^2+\ldots+(p^i\xi)^{n-1}\big)$ and $g(1+\delta+\delta^2+\ldots)(1-\delta)=g(1-\delta)(1+\delta+\delta^2+\ldots)=g\big(1+p^i\xi+\ldots+(p^i\xi)^{n-1}-(p^i\xi)-\ldots-(p^i\xi)^n\big)=g\big(1-(p^i\xi)^n\big)=g$. Hence $1-\delta$ has an inverse, $1-\delta\in\mathrm{Aut}\, G$.

The next proposition is now an immediate consequence of 2. 5 and [5] 1. 23.

2. 6 Proposition. *Let G be an abelian p-group, i a positive integer. Let* R: End $G\to$ End $G[p^i]$ *be the restriction map. Then*

$$[\mathrm{Aut}\, G]R=\{\textit{units of } [\mathrm{End}\, G]R\}.$$

[5] 1. 11 yields for each $i\geqq 0$ the exact sequence

$$1\to \mathrm{Aut}_{G[p^{i+1}]}G \xrightarrow{I} \mathrm{Aut}_{G[p^i]}G \xrightarrow{R} \mathrm{Aut}_{G[p^i]}G[p^{i+1}]. \tag{2.7}$$

Furthermore it follows from [5] 1. 6 that

$$(2.8)\qquad 0 \to \mathrm{Hom}\,(G/G[p],\, G[p]) \xrightarrow{U} \mathrm{Aut}_{G[p]}\, G \xrightarrow{T} \mathrm{Aut}\, G/G[p]$$

and for $i=1, 2, \ldots$

$$(2.9)\qquad 0 \to \mathrm{Hom}\,(G[p^{i+1}]/G[p^i],\, G[p^i]) \xrightarrow{U} \mathrm{Aut}_{G[p^i]}\, G[p^{i+1}] \xrightarrow{T} \mathrm{Aut}\,(G[p^{i+1}]/G[p^i])$$

are exact.

2. 10 Proposition. *If G is an abelian p-group, then for $i=1, 2, \ldots$*

$$0 \to \mathrm{Hom}\,(G[p^{i+1}]/G[p^i],\, G[p^i]) \xrightarrow{U} \mathrm{Aut}_{G[p^i]}\, G[p^{i+1}] \to 1$$

is exact, thus

$$\mathrm{Aut}_{G[p^i]}\, G[p^{i+1}] \cong \oplus^*_{r(G[p^{i+1}]/G[p^i])}\, G[p].$$

Proof. Let $\varphi\colon G[p^{i+1}] \to G[p^{i+1}]/G[p^i]$ be the natural homomorphism. For $\alpha \in \mathrm{Aut}_{G[p^i]} G[p^{i+1}]$, $x \in G[p^{i+1}]$, we have $px \in G[p^i]$, hence $p(x\alpha) = (px)\alpha = px$, $x\alpha - x \in G[p] < G[p^i]$. Thus $0 = (x\alpha - x)\varphi = x\alpha\varphi - x\varphi = (x\varphi)(\alpha T) - x\varphi$, i. e. $\alpha T = 1$.

$$\mathrm{Aut}_{G[p^i]} G[p^{i+1}] \cong \mathrm{Hom}\,(G[p^{i+1}]/G[p^i],\, G[p^i]) \cong \oplus^*_{r(G[p^{i+1}]/G[p^i])} G[p].$$

2. 11 Corollary. *For $i=1, 2, 3, \ldots$ the factor groups $\mathrm{Aut}_{G[p^i]}G/\mathrm{Aut}_{G[p^{i+1}]}G$ are elementary abelian p-groups.*

The corresponding results for the endomorphism ring are the following. Let $\varphi\colon G \to G/G[p]$ and, for $i=1, 2, \ldots$, $\varphi_i\colon G[p^{i+1}] \to G[p^{i+1}]/G[p^i]$ be the natural homomorphisms. Then we get from [5] 1. 16 and [5] 1. 22 the exact sequence

$$(2.12)\qquad \begin{aligned} &0 \to \varphi\, \mathrm{Hom}\,(G/G[p],\, G[p]) \xrightarrow{I} \mathrm{End}_{G[p]}\, G \xrightarrow{T} \mathrm{End}\, G/G[p] \to \\ &\to \mathrm{Ext}\,(G/G[p],\, G[p]) \to \mathrm{Ext}\,(G/G[p],\, G) \to \mathrm{Ext}\,(G/G[p],\, G/G[p]) \to 0, \end{aligned}$$

furthermore for $i=1, 2, \ldots$

$$(2.13)\qquad \begin{aligned} &0 \to \varphi_i\, \mathrm{Hom}\,\big(G[p^{i+1}]/G[p^i],\, G[p^i]\big) \xrightarrow{I} \mathrm{End}_{G[p^i]}\, G[p^{i+1}] \to \\ &\xrightarrow{T} \mathrm{End}\, G[p^{i+1}]/G[p^i] \to \mathrm{Ext}\,(G[p^{i+1}/G[p^i],\, G[p^i]) \to \\ &\to \mathrm{Ext}\,(G[p^{i+1}]/G[p^i],\, G[p^{i+1}]) \to \mathrm{Ext}\,(G[p^{i+1}]/G[p^i],\, G[p^{i+1}]/G[p^i]) \to 0, \end{aligned}$$

and finally for $i=1, 2, \ldots$

$$(2.14)\qquad 0 \to \mathrm{End}_{G[p^{i+1}]}\, G \xrightarrow{I} \mathrm{End}_{G[p^i]}\, G \xrightarrow{R} \mathrm{End}_{G[p^i]}\, G[p^{i+1}].$$

2.15 Proposition. *If G is an abelian p-group, then,* for $i=1, 2, \ldots,$

$$0 \to \varphi_i\, \mathrm{Hom}\, G[p^{i+1}]/G[p^i],\, G[p^i]) \xrightarrow{I} \mathrm{End}_{G[p^i]}\, G[p^{i+1}] \to 0$$

is exact, thus

$$\mathrm{End}_{G[p^i]}\, G[p^{i+1}] \cong \oplus^*_{r(G[p^{i+1}]/G[p^i])}\, G[p]$$

and has trivial multiplication.

PROOF. $x \in G[p^{i+1}]$ implies $px \in G[p^i]$ and thus $(px)\delta = 0$ for all $\delta \in \mathrm{End}_{G[p^i]}G[p^{i+1}]$. Therefore, for all $\delta \in \mathrm{End}_{G[p^i]}G[p^{i+1}]$, $p(x\delta) = 0$, $x\delta \in G[p] < G[p^i]$, hence $x\delta\varphi_i = 0$, i. e. $(x\varphi_i)(\delta T) = x\delta\varphi_i = 0$, $\delta T = 0$.

2. 16 Corollary. *For $i = 1, 2, 3, \ldots$ the quotient rings $\mathrm{End}_{G[p^i]}G/\mathrm{End}_{G[p^{i+1}]}G$ are as abelian groups elementary p-groups and have trivial multiplication.*

The case of a divisible group is easy to handle.

2. 17 Theorem. *If G is a divisible p-group then, for $i = 1, 2, \ldots$, the following sequences are exact:*

(2. 18) $$0 \to \mathrm{End}_{G[p^i]} G \xrightarrow{I} \mathrm{End}\, G \xrightarrow{R} \mathrm{End}\, G[p^i] \to 0,$$

(2. 19) $$0 \to \mathrm{End}_{G[p^{i+1}]} G \xrightarrow{I} \mathrm{End}_{G[p^i]} G \xrightarrow{R} \mathrm{End}_{G[p^i]} G[p^{i+1}] \to 0.$$

Therefore there is a ring isomorphism

$$\mathrm{End}\, G/\mathrm{End}_{G[p]}G \cong \mathrm{End}\, G[p],$$

and for $i = 1, 2, \ldots$ there is a group isomorphism

$$\mathrm{End}_{G[p^i]}G/\mathrm{End}_{G[p^{i+1}]}G \cong \overset{*}{\bigoplus_{r(G)}} \bigoplus_{r(G)} Z(p),$$

and $\mathrm{End}_{G[p^i]}G/\mathrm{End}_{G[p^{i+1}]}G$ has trivial multiplication.

PROOF. If G is divisible, $\mathrm{Ext}(G/G[p^i], G) = \langle 0 \rangle$, and therefore 2. 18 follows from 2. 2. Then 2. 19 follows in turn from 2. 18. The first isomorphism is just a restatement of 2. 18 for $i = 1$, the second isomorphism follows from 2. 19 using 2. 15 and the fact that for a divisible p-group $r(G[p]) = r(G)$ and $r(G[p^{i+1}]/ /G[p^i]) = r(G)$.

We may use 2. 6 to derive from 2. 17 information on the automorphism group of a divisible p-group.

2. 20 Theorem. *If G is a divisible p-group, then, for $i = 1, 2, \ldots$, the following sequences are exact:*

(2. 21) $$1 \to \mathrm{Aut}_{G[p^i]} G \xrightarrow{I} \mathrm{Aut}\, G \xrightarrow{R} \mathrm{Aut}\, G[p^i] \to 1,$$

(2. 22) $$1 \to \mathrm{Aut}_{G[p^{i+1}]} G \xrightarrow{I} \mathrm{Aut}_{G[p^i]} G \xrightarrow{R} \mathrm{Aut}_{G[p^i]} G[p^{i+1}] \to 1.$$

Therefore

$$\mathrm{Aut}\, G/\mathrm{Aut}_{G[p]}G \cong GL(r(G), p),$$

and for $i = 1, 2, \ldots$

$$\mathrm{Aut}_{G[p^i]}G/\mathrm{Aut}_{G[p^{i+1}]}G \cong \overset{*}{\bigoplus_{r(G)}} \bigoplus_{r(G)} Z(p).$$

PROOF. Since $\mathrm{Aut}\, G[p^i] = \{\text{units of } \mathrm{End}\, G[p^i]\}$, 2. 21 follows immediately from 2. 18 using 2. 6. Then 2. 22 follows from 2. 21. $G[p]$ can be taken to be a vector space over the prime field of characteristic p of dimension $r(G)$, hence $\mathrm{Aut}\ G[p] \cong \cong GL(r(G), p)$ and the first isomorphism is immediate from 2. 21. 2. 22 together with 2. 10 implies the second isomorphism using the relations between the ranks stated in the proof of 2. 19.

We conclude the case of a divisible group with two remarks: Firstly

$$\text{End}_{G[p^i]}G = p^i\text{End } G \tag{2. 23}$$

in this case. Obviously $p^i\text{End } G < \text{End}_{G[p^i]}G$. Let $\delta \in \text{End}_{G[p^i]}G$. Since G is divisible, $p^iG = G$ and every element of G may be written in the form of p^ig. Let δ' be defined by $(p^ig)\delta' = g\delta$. Then δ' is well defined since $(G[p^i])\delta = 0$, and clearly δ' is homomorphic. Now $g(p^i\delta') = (p^ig)\delta' = g\delta$, hence $\delta = p^i\delta' \in p^i\text{End } G$ which proves $\text{End}_{G[p^i]}G = p^i\text{End}G$.

Secondly, for a divisible p-group G, End G is the ring of row-finite $r(G) \times r(G)$ matrices over the p-adic integers (Cf. [2], 212—13). This fact constitutes another powerful approach to the structure of End G.

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