On limitings and topogenous orders

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In Chapter 7 of his book [2] Á. CSÁSZÁR has shown how topological spaces, uniform spaces and proximity spaces are capable of a unified treatment within the framework of the theory of syntopogenous structures. In this little note we wish to carry through similar investigations with respect to so-called limitings as introduced by H.-J. KOWALSKY and independently by H. R. FISCHER. (Earlier than both authors mentioned, G. CHOQUET has introduced „structures pseudo-topologiques" which are closely related to limitings, and „structures pré-topologiques" which are in fact the same as principal limitings. See [1], [3] and [4]. We shall adopt the terminology and notations of [3].)

§ 1. Preliminaries

Let us begin by recapitulating for the reader's convenience the basic definitions we shall need in the sequel. Let E be a nonvoid set, $\Phi(E)$ the set of all filters defined on E with its natural partial ordering, and $\mathfrak{P}(\Phi(E))$ the power-set of the set $\Phi(E)$.

We start with the following

Definition 1. A mapping τ of E into $\mathfrak{P}(\Phi(E))$ is a limiting on E, if for any $x\in E$ it satisfies the following conditions:

$$(\mathrm{L}_1)\qquad \mathscr{F},\mathscr{G}\in\tau(x)\Rightarrow\mathscr{F}\wedge\mathscr{G}\in\tau(x),$$

and

$$\left.\begin{matrix}\mathscr{F}\in\tau(x)\\ \mathscr{G}\supseteqq\mathscr{F}\end{matrix}\right\}\Rightarrow\mathscr{G}\in\tau(x).$$

$$(\mathrm{L}_2)\qquad \dot{x}=\{H|x\in H\subseteqq E\}\in\tau(x).$$

The set $\boldsymbol{T}=\boldsymbol{T}(E)$ of all limitings on E can be endowed with a partial order:

Definition 2. Let $\tau,\tau'\in\boldsymbol{T}$. Then $\tau\subseteqq\tau'$ means that $\tau(x)\subseteqq\tau'(x)$ holds [1]) for any $x\in E$.

An important subset of the set of all limitings on a given space is characterized by

Definition 3. A mapping τ of E into $\mathfrak{P}(\Phi(E))$ is a principal limiting on E, if for any $x\in E$ it satisfies the following conditions:

$$(\mathrm{L}_2)\qquad \dot{x}\in\tau(x).$$

[1]) KOWALSKY and FISCHER adopt the reverse of our convention: they write $\tau_1\leqq\tau_2$ iff $\tau_1\supseteqq\tau_2$.

(L_3) There exists a filter $\mathfrak{B}(x)$ on E such that

$$\tau(x)=\{\mathcal{F}\,|\,\mathcal{F}\geqq\mathfrak{B}(x)\}.$$

Condition (L_3) clearly implies (L_1), so principal limitings are limitings in the sense of Definition 1.

We still need the following

Definition 4. A relation $<$ between subsets of the space E is a perfect topogenous order on E, if it satisfies the following conditions:

(O_1) $\quad \emptyset<\emptyset,\ E<E.$

(O_2) $\quad A<B\Rightarrow A\subseteqq B.$

(O_3) $\quad A\subseteqq A'<B'\subseteqq B\Rightarrow A<B.$

(Q') $\quad \left.\begin{matrix}A<B\\ A'<B'\end{matrix}\right\}\Rightarrow A\cap A'<B\cap B'.$

(P') $\quad A_i<B\,(i\in I)\Rightarrow\bigcup\{A_i\,|\,i\in I\}<B.$

for any index-set I.

The set of perfect topogenous orders on E possesses a natural partial order, namely the one induced by set-theoretical inclusion in $\mathfrak{P}(E)\times\mathfrak{P}(E)$:

$$<_1\subseteqq<_2 \quad\text{iff}\quad A<_1B\Rightarrow A<_2B.$$

§ 2. On principal limitings

Our first aim is to show that the concept of principal limiting is equivalent to that of perfect topogenous order. This equivalence is established by the following

Theorem 1.

(1) *If $<$ is a perfect topogenous order on E then for any $x\in E$*

$$\mathfrak{B}(x)=\{V\,|\,x<V\}$$

is a filter, and the mapping

$$\tau_<(x)=\{\mathcal{F}\,|\,\mathcal{F}\geqq\mathfrak{B}(x)\}\qquad(x\in E)$$

is a principal limiting on E.

(2) *If τ is a principal limiting on E with*

$$\tau(x)=\{\mathcal{F}\,|\,\mathcal{F}\geqq\mathfrak{B}(x)\}$$

for any $x\in E$ then the relation $<_\tau$ defined by

$$A<_\tau B \quad\text{iff}\quad x\in A\Rightarrow B\in\mathfrak{B}(x)$$

is a perfect topogenous order on E.

(3) *The mappings $<\,\to\tau_<$ and $\tau\to\,<_\tau$ are one-to-one correspondences, inverse to each other, between the sets of all perfect topogenous orders and all principal limitings on E which invert the respective partial orders.*

PROOF.

(1) The proof is straightforward.

(2) (O_1): $\emptyset <_\tau \emptyset$, $E <_\tau E$ is clear.

(O_2): Conditions (L_2) and (L_3) together imply $\dot{x} \geqq \mathfrak{B}(x)$, i. e. $B \in \mathfrak{B}(x) \Rightarrow \Rightarrow x \in B$. Thus, by the definition of $<_\tau$, we have $A <_\tau B \Rightarrow A \subseteqq B$.

(O_3): Let $A \subseteqq A' <_\tau B' \subseteqq B$. Then $x \in A \Rightarrow x \in A' \Rightarrow B' \in \mathfrak{B}(x) \Rightarrow B \in \mathfrak{B}(x)$.

(Q'): Let $A <_\tau B$ and $A' <_\tau B'$. Then

$$x \in A \cap A' \left\} \begin{matrix} \Rightarrow x \in A \Rightarrow B \in \mathfrak{B}(x) \\ \Rightarrow x \in A' \Rightarrow B' \in \mathfrak{B}(x) \end{matrix} \right\} \Rightarrow B \cap B' \in \mathfrak{B}(x).$$

(P'): Let $A_i <_\tau B$ for $i \in I$. Then $x \in \bigcup_{i \in I} A_i \Rightarrow (x \in A_{i_0}$ for some $i_0 \in I) \Rightarrow B \in \mathfrak{B}(x)$.

(3) Let $< \to \tau_<$; $\tau \to <_\tau$.

If $\tau = \tau_<$ then $<_\tau = <$. As a matter of fact, $A <_\tau B$ iff $x \in A \Rightarrow B \in \mathfrak{B}(x)$. Now, if $\tau = \tau_<$ then $\mathfrak{B}(x) = \{V | x < V\}$, and so $A <_\tau B$ iff $x \in A \Rightarrow x < B$, and this is true iff $A < B$, the second „iff" being a consequence of the fact that the relation $<$ satisfies condition (P').

Again, let $\tau \to <_\tau$; $< \to \tau_<$.

If $< = <_\tau$ then $\tau_< = \tau$. As a matter of fact, for $< = <_\tau$ we get

$$\mathfrak{B}_{\tau_<}(x) = \{V | x <_\tau V\} = \{V | V \in \mathfrak{B}(x)\} = \mathfrak{B}(x).$$

Thus $\tau_< = \tau$.

It is not hard to see that $<_1 \subseteqq <_2$ iff $\tau_{<_1} \supseteqq \tau_{<_2}$.

Indeed, each of the following statements is equivalent to the next one:

$$<_1 \subseteqq <_2,$$

$$\mathfrak{B}_1(x) = \{V | x <_1 V\} \subseteqq \{V | x <_2 V\} = \mathfrak{B}_2(x)$$

for any $x \in E$, [2])

$$\tau_{<_1}(x) = \{\mathscr{F} | \mathscr{F} \geqq \mathfrak{B}_1(x)\} \supseteqq \{\mathscr{F} | \mathscr{F} \geqq \mathfrak{B}_2(x)\} = \tau_{<_2}(x)$$

for any $x \in E$.

$$\tau_{<_1} \supseteqq \tau_{<_2}.$$

This shows that the one-to-one correspondence considered inverts the respective partial orders, completing thus the proof of Theorem 1.

One easily sees that part (1) of the theorem just proved remains valid if we drop condition (P') from the definition of the relation $<$, i. e. if we start with what might be called semi-topogenous $\cap$-orders. Then, however, part (2) of the theorem will not reproduce the relation $<$ originally given; it will lead to the smallest perfect topogenous order containing $<$, i. e. to $<^p$ in the terminology of [2]. Thus the correspondence between semi-topogenous $\cap$-orders and principal limitings on

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[2]) In order to establish

$$\mathfrak{B}_1(x) \subseteqq \mathfrak{B}_2(x) (x \in E) \Rightarrow <_1 \subseteqq <_2$$

one has to make use of condition (P').

a given space is many-to-one. A one-to-one correspondence between equivalence classes of semi-topogenous $\cap$-orders and principal limitings results, if we define

$$<_1 \sim <_2 \quad \text{iff} \quad <_1^p = <_2^p.$$

Of course, Theorem 1. states just this equivalence, with the equivalence classes replaced by their perfect representatives.

§ 3. The general case of limitings

So far, we have characterized principal limitings as being equivalent in a natural way to perfect topogenous orders. The question naturally arises, whether a similar characterization could not be found also for (general) limitings in the sense of Definition 1. In order to obtain such a characterization, we shall have recourse to filters of perfect topogenous orders. Let us give for completeness' sake a formal

Definition 5. A nonvoid set $\mathscr{H}$ of perfect topogenous orders on a set E is a filter of perfect topogenous orders on E if it satisfies the filter-axiom:

(F)
$$<_1, <_2 \in \mathscr{H} \Rightarrow <_1 \cap <_2 \in \mathscr{H},$$
$$\left.\begin{array}{l} <_1 \in \mathscr{H} \\ <_1 \subseteqq <_2 \end{array}\right\} \Rightarrow <_2 \in \mathscr{H}.$$

We are able to prove at once the following

Proposition 1. *Let $\mathscr{H}$ be a filter of perfect topogenous orders on the set E. Then for any $x \in E$ and $< \in \mathscr{H}$*

$$\mathscr{F}_<(x) = \{V | x < V\}$$

is a filter on E, and

$$\tau_{\mathscr{H}}(x) = \{\mathscr{G} | \mathscr{G} \geqq \mathscr{F}_<(x) \quad \textit{for some} \quad < \in \mathscr{H}\} (x \in E)$$

is a limiting on E.

Proof. $\mathscr{F}_<(x)$ is clearly a filter for any $x \in E$ and $< \in \mathscr{H}$. Let $\mathscr{G}_1, \mathscr{G}_2 \in \tau_{\mathscr{H}}(x)$, i. e. $\mathscr{G}_1 \geqq \mathscr{F}_{<_1}(x)$ and $\mathscr{G}_2 \geqq \mathscr{F}_{<_2}(x)$ for some $<_1, <_2 \in \mathscr{H}$. Then $\mathscr{G}_1 \wedge \mathscr{G}_2 \geqq \mathscr{F}_{<_1}(x) \wedge \wedge \mathscr{F}_{<_2}(x) = \mathscr{F}_{<_1 \cap <_2}(x)$, with $<_1 \cap <_2 \in \mathscr{H}$. The remaining part of (L_1) is clear, and so is (L_2).

Two filters of perfect topogenous orders on a set E will be said to be τ-equivalent if they give raise to the same limiting in the sense of Proposition 1.:

$$\mathscr{H}_1 \sim_\tau \mathscr{H}_2 \quad \text{iff} \quad \tau_{\mathscr{H}_1} = \tau_{\mathscr{H}_2}.$$

A filter of perfect topogenous orders will be said to be τ-maximal, if it contains every filter τ-equivalent to it. In other words, $\mathscr{H}_M$ is τ-maximal iff

$$\mathscr{H} \sim_\tau \mathscr{H}_M \quad \text{implies} \quad \mathscr{H} \subseteqq \mathscr{H}_M.$$

Of course, we have still to prove the existence of τ-maximal filters. The first step in this direction will be made by establishing the following

Proposition 2. *Let τ be a limiting on E. To each element $\{\mathscr{F}_x(\in\tau(x))|x\in E\}$ of the cartesian product $\mathbf{X}\{\tau(x)|x\in E\}$ we make correspond the relation $<$ defined by*

$$A<B \quad \textit{iff} \quad x\in A\Rightarrow x\in B\in\mathscr{F}_x.$$

The set $\mathscr{H}_\tau$ of the relations $<$ so defined is a filter of perfect topogenous orders on E, and the product of the mappings $\tau\to\mathscr{H}_\tau$ and $\mathscr{H}\to\tau_{\mathscr{H}}$ is the identity mapping on $\boldsymbol{T}(E)$.

Proof. Each relation $<$ is a perfect topogenous order. The conditions laid down in Definition 4. can be checked in much the same way as in the proof of Theorem 1. (2). Moreover, the set $\mathscr{H}_\tau$ is a filter of perfect topogenous orders on E:

Let $<_1, <_2\in\mathscr{H}_\tau$. Then

$$A<_1B \quad \text{iff} \quad x\in A\Rightarrow x\in B\in\mathscr{F}_x,$$

$$A<_2B \quad \text{iff} \quad x\in A\Rightarrow x\in B\in\mathscr{G}_x,$$

and so, for $< = <_1\cap<_2$, $A<B$ iff $x\in A\Rightarrow x\in B\in\mathscr{F}_x\wedge\mathscr{G}_x$.

Thus $<\in\mathscr{H}_\tau$, since $\tau(x)$ contains with $\mathscr{F}_x$ and $\mathscr{G}_x$ their intersection $\mathscr{F}_x\wedge\mathscr{G}_x$.

Again, let $<\in\mathscr{H}_\tau$, i. e. $A<B$ iff $x\in A\Rightarrow x\in B\in\mathscr{F}_x$, and let $<\subseteqq<_1$, with $<_1$ a perfect topogenous order on E. Then we have $A<_1B$ iff $x\in A\Rightarrow x<_1B$, and if we put $\mathfrak{V}_1(x)=\{V|x<_1V\}$ then $A<_1B$ iff $x\in A\Rightarrow B\in\mathfrak{V}_1(x)$. Now $\mathfrak{V}_1(x)\geqq\mathscr{F}_x\wedge\wedge\dot{x}\in\tau(x)$, and so $\mathfrak{V}_1(x)\in\tau(x)$. This proves $<_1\in\mathscr{H}_\tau$.

Now, let $\tau\to\mathscr{H}_\tau$; $\mathscr{H}\to\tau_{\mathscr{H}}$.

If $\mathscr{H}=\mathscr{H}_\tau$ then $\tau_{\mathscr{H}}=\tau$. As a matter of fact, for $\mathscr{H}=\mathscr{H}_\tau$ we have

$$\tau_{\mathscr{H}}(x)=\{\mathscr{G}|\mathscr{G}\geqq\mathscr{F}_<(x) \text{ for some } <\in\mathscr{H}_\tau\}.$$

Now, $<\in\mathscr{H}_\tau$ means that $A<B$ iff $x\in A\Rightarrow x\in B\in\mathscr{F}_x\in\tau(x)$. Thus one has

$$\mathscr{F}_<(x)=\{V|x<V\}=\{V|x\in V\in\mathscr{F}_x\}=\mathscr{F}\wedge\dot{x}.$$

This yields $\mathscr{F}_<(x)\subseteqq\mathscr{F}_x$, and so $\mathscr{F}_x\in\tau_{\mathscr{H}}(x)$. Thus we get $\tau(x)\subseteqq\tau_{\mathscr{H}}(x)$. (If $<$ runs through $\mathscr{H}_\tau$, $\mathscr{F}_x$ in the above reasoning will run through $\tau(x)$.)

In order to establish the reverse inclusion $\tau_{\mathscr{H}}(x)\subseteqq\tau(x)$, it suffices to show that $\mathscr{F}_<(x)\in\tau(x)$ for any $<\in\mathscr{H}_\tau$. This, however, is true since as we have just seen $\mathscr{F}_<(x)=\mathscr{F}_x\wedge\dot{x}$, and $\dot{x}\in\tau(x)$. — This completes the proof of Proposition 2.

Next we have

Proposition 3. *If $\mathscr{H}$ is a filter of perfect topogenous orders and $\tau=\tau_{\mathscr{H}}$ then $\mathscr{H}_\tau$ is τ-equivalent to $\mathscr{H}$ and $\mathscr{H}\subseteqq\mathscr{H}_\tau$.*

Proof. By the proposition just proved, the mapping $\mathscr{H}\to\tau_{\mathscr{H}}$ transforms $\mathscr{H}_\tau$ into τ. On the other hand, the filter $\mathscr{H}$ we start with has this same image $\tau_{\mathscr{H}}=\tau$ (this is, in fact, the definition of τ), and so $\mathscr{H}$ and $\mathscr{H}_\tau$ are τ-equivalent. Moreover, for $<\in\mathscr{H}$ we have $A<B$ iff $x\in A\Rightarrow B\in\mathscr{F}_<(x)$, and in view of $\mathscr{F}_<(x)\in\tau_{\mathscr{H}}(x)$ there results $<\in\mathscr{H}_\tau$. Thus we have $\mathscr{H}\subseteqq\mathscr{H}_\tau$.

Propositions 2. and 3. together yield the following

Theorem 2. *Any filter of perfect topogenous orders on a set E is contained in a τ-equivalent τ-maximal filter, or equivalently, each τ-equivalence class contains a τ-maximal member.*

PROOF. To an arbitrary filter $\mathscr{H}$ of perfect topogenous orders there corresponds by Proposition 2. a limiting $\tau_{\mathscr{H}}$, and to $\tau=\tau_{\mathscr{H}}$ there corresponds by Proposition 3. a filter $\mathscr{H}_\tau$ such that $\mathscr{H}\sim_\tau\mathscr{H}_\tau$ and $\mathscr{H}\subseteq\mathscr{H}_\tau$. Since $\mathscr{H}_\tau$ depends only on τ and not on $\mathscr{H}$ itself, it is τ-maximal.

The results so far obtained enable us to characterize limitings as being equivalent to τ-maximal filters of perfect topogenous orders:

Theorem 3.

(1) *Let $\mathscr{H}$ be a τ-maximal filter of perfect topogenous orders on the set E. Then for any $x\in E$ and $<\in\mathscr{H}$*

$$\mathscr{F}_<(x)=\{V|x<V\}$$

is a filter on E, and

$$\tau_{\mathscr{H}}(x)=\{\mathscr{G}|\mathscr{G}\supseteq\mathscr{F}_<(x)\quad \text{for some}\ <\in\mathscr{H}\}\ (x\in E)$$

is a limiting on E.

(2) *Let τ be a limiting on E. To each element $\{\mathscr{F}_x(\in\tau(x))|x\in E\}$ of the cartesian product $\mathbf{X}\{\tau(x)|x\in E\}$ we make correspond the relation $<$ defined by*

$$A<B \quad \text{iff} \quad x\in A\Rightarrow x\in B\in\mathscr{F}_x.$$

The set $\mathscr{H}_\tau$ of the relations $<$ so defined is a τ-maximal filter of perfect topogenous orders on E.

(3) *The mappings $\mathscr{H}\to\tau_{\mathscr{H}}$ and $\tau\to\mathscr{H}_\tau$ are one-to-one correspondences, inverse to each other, between the sets of all τ-maximal filters of perfect topogenous orders and all limitings on E, which preserve the respective partial orders.* [3])

PROOF.

(1) This is a special case of Proposition 1.

(2) $\mathscr{H}_\tau$ is a filter by Proposition 2., and it is τ-maximal by Theorem 2.

(3) Let $\mathscr{H}\to\tau_{\mathscr{H}}$; $\tau\to\mathscr{H}_\tau$. If $\tau=\tau_{\mathscr{H}}$ then $\mathscr{H}_\tau=\mathscr{H}$. As a matter of fact, $\tau=\tau_{\mathscr{H}}$ implies $\mathscr{H}\sim_\tau\mathscr{H}_\tau$ and by the τ-maximality of both $\mathscr{H}$ and $\mathscr{H}_\tau$ equality follows.

Again, let $\tau\to\mathscr{H}_\tau$; $\mathscr{H}\to\tau_{\mathscr{H}}$. If $\mathscr{H}=\mathscr{H}_\tau$ then $\tau_{\mathscr{H}}=\tau$. This is the concluding part of the statement of Proposition 2. and was proved there.

Finally, the relation $\mathscr{H}_{\tau_1}\subseteq\mathscr{H}_{\tau_2}$ iff $\tau_1\subseteq\tau_2$ is an easy consequence of the way we defined the $\mathscr{H}$'s with the help of the τ's and the τ's with the help of the $\mathscr{H}$'s.

Of course, with respect to the one-to-one correspondence just established, a situation analogous to that described in connection with Theorem 1. prevails: τ-maximal filters of perfect topogenous orders can be replaced by τ-equivalence classes.

§ 4. Direct characterization of τ-equivalence and τ-maximality

In the previous section τ-equivalence of two filters of perfect topogenous orders has been defined to mean equality of the corresponding limitings. Here we are going to characterize τ-equivalence directly,[4]) without reference to limitings, and this direct characterization will throw additional light also upon τ-maximality.

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[3]) The partial order of the $\mathscr{H}$'s is of course the one defined by set-theoretical inclusion for sets of perfect topogenous orders.

[4]) The fundamental ideas underlying this section have been suggested to the author by Professor Á. CSÁSZÁR.

Let us adopt first of all the following convention: If $<_1$ and $<_2$ are two perfect topogenous orders on E then for $x \in E$ we say that $<_2$ *is smaller at* x *than* $<_1$ (written $<_2 \subseteq_x <_1$) iff $x <_2 V \Rightarrow x <_1 V$ for any $V \subseteq E$. — Making use of condition (P′) in Definition 4. we immediately get the following

Proposition 4. *If* $<_1$ *and* $<_2$ *are perfect topogenous orders on a set* E *then* $<_2 \subseteq <_1$ *if and only if* $<_2 \subseteq_x <_1$ *holds for any* $x \in E$.

It is also clear that Proposition 1. implies

Proposition 5. *If* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *are filters of perfect topogenous orders on the set E then* $\tau_{\mathscr{H}_1} \subseteq \tau_{\mathscr{H}_2}$ *if and only if for any* $x \in E$ *and any* $<_1 \in \mathscr{H}_1$ *there exists a* $<_2 \in \mathscr{H}_2$ *satisfying* $<_2 \subseteq_x <_1$.

If the condition just described holds (i. e. if $\tau_{\mathscr{H}_1} \subseteq \tau_{\mathscr{H}_2}$) then we say that $\mathscr{H}_2$ is τ-coarser than $\mathscr{H}_1$. We clearly have the

Proposition 6. *Two filters of perfect topogenous orders on a set* E *are* τ*-equivalent iff each of them is* τ*-coarser than the other.*

This is the direct characterization of τ-equivalence hinted at previously. Its usefulness will be shown by the following considerations.

Let $\mathscr{H}_0$, $\mathscr{H}_1$ and $\mathscr{H}_2$ be filters of perfect topogenous orders over the given set E. If we put

$$\{\mathscr{H}_1, \mathscr{H}_2\} = \{< \mid <_1 \cap <_2 \subseteq < \quad \text{for some} \quad <_1 \in \mathscr{H}_1 \quad \text{and} \quad <_2 \in \mathscr{H}_2\}$$

then $\{\mathscr{H}_1, \mathscr{H}_2\}$ is clearly a filter of perfect topogenous orders on E (in fact the smallest filter containing both $\mathscr{H}_1$ and $\mathscr{H}_2$), and we have the implication expressed by

Proposition 7.

$$\left.\begin{matrix}\mathscr{H}_1 \sim_\tau \mathscr{H}_0 \\ \mathscr{H}_2 \sim_\tau \mathscr{H}_0\end{matrix}\right\} \Rightarrow \{\mathscr{H}_1, \mathscr{H}_2\} \sim_\tau \mathscr{H}_0.$$

Proof. For $< \in \mathscr{H}_0$ and $x \in E$ given, there exist perfect topogenous orders $<_1 \in \mathscr{H}_1$ and $<_2 \in \mathscr{H}_2$ such that $<_1 \subseteq_x <$ and $<_2 \subseteq_x <$, and so $<_1 \cap <_2 \subseteq_x <$ follows with $<_1 \cap <_2 \in \{\mathscr{H}_1, \mathscr{H}_2\}$.

On the other hand, let $< \in \{\mathscr{H}_1, \mathscr{H}_2\}$, i. e. let $<_1 \cap <_2 \subseteq <$ for some $<_1 \in \mathscr{H}_1$ and $<_2 \in \mathscr{H}_2$. Then for $x \in E$ given there exist $<_\alpha, <_\beta \in \mathscr{H}_0$! such that $<_\alpha \subseteq_x <_1$ and $<_\beta \subseteq_x <_2$, and consequently $<_\alpha \cap <_\beta \subseteq_x <_1 \cap <_2 \subseteq <$, whence $<_\alpha \cap$ $\cap <_\beta \subseteq_x <$ results with $<_\alpha \cap <_\beta \in \mathscr{H}_0$. This completes the proof of the proposition.

The existence of a τ-maximal filter in each τ-equivalence class now follows by

Proposition 8. *If* $\mathscr{H}_0$ *is a filter of perfect topogenous orders on a set* E *then the set-theoretical union*

$$\mathscr{U} = \cup \{\mathscr{H} \mid \mathscr{H} \sim_\tau \mathscr{H}_0\}$$

is a filter of perfect topogenous orders on E, τ*-equivalent to* $\mathscr{H}_0$.

Proof. $\mathscr{U}$ is nonvoid because $\mathscr{H}_0$ is. Moreover, $\mathscr{U}$ satisfies condition (F) of Definition 5. Indeed,

$$<_1, <_2 \in \mathscr{U} \Rightarrow <_1 \cap <_2 \in \mathscr{U}$$

is a direct consequence of Proposition 7. The remaining part of condition (F) is clear, and so is the equivalence $\mathscr{U} \sim_\tau \mathscr{H}_0$.

The proposition just proved clearly implies Theorem 2. As a matter of fact, here we have established slightly more, namely the following

Theorem 2a. *Any filter of perfect topogenous orders on a set E is contained in a τ-equivalent τ-maximal filter, which is the set-theoretical union of all the filters τ-equivalent to the given one, or equivalently, each τ-equivalence class contains a τ-maximal member, namely the set-theoretical union of the filters contained in that class.*

§ 5. Some further remarks

To any limiting τ on a set E there corresponds in a natural way a principal limiting $\psi\tau$ on E. (See [3], p. 273.) Let indeed be $\mathfrak{B}(x) = \wedge \{\mathscr{F} | \mathscr{F} \in \tau(x)\}$, and

$$\psi\tau(x) = \{\mathscr{G} | \mathscr{G} \geqq \mathfrak{B}(x)\}.$$

The connection existing between the τ-maximal filter $\mathscr{H}_\tau$ and $<_{\psi\tau}$ is described by the following simple

Proposition 4.

$$<_{\psi\tau} = \cap \{< | < \in \mathscr{H}_\tau\}.$$

PROOF.

We have $A <_{\psi\tau} B$ iff $x \in A \Rightarrow B \in \mathfrak{B}(x)$, i. e. iff $x \in A \Rightarrow B \in \mathscr{F}$ for any $\mathscr{F} \in \tau(x)$.

On the other hand, putting $<_D = \cap \{< | < \in \mathscr{H}_\tau\}$, we obtain $A <_D B$ iff $A < B$ for any $< \in \mathscr{H}_\tau$, i. e. iff $x \in A \Rightarrow B \in \mathscr{F}$ for any $\mathscr{F} \in \tau(x)$, the same condition as before. (If $<$ runs through $\mathscr{H}_\tau$, $\mathscr{F}$ will run through $\tau(x)$. Of course, $x \in B$ is implied by $\dot{x} \in \tau(x)$.)

In [2] a central role is played by syntopogenous structures. (See [2], Chapter 7.) One might ask what is the connection between syntopogenous structures and limitings. It is not hard to see that the natural link between these two concepts is formed by the topologies (in the usual sense of the word).

One indeed has the following

Proposition 5. *If $\mathscr{S}$ is a syntopogenous structure on the set E, then for $x \in E$*

$$\mathfrak{B}(x) = \{V | x < V \quad \text{for some} \quad < \in \mathscr{S}\}$$

is a filter, and the filters $\mathfrak{B}(x)$ $(x \in E)$ are the neighborhood filters of a topology on E.

Any topology on E as defined by its neighborhood filters $\mathfrak{B}(x)$ $(x \in E)$ can be derived in this manner from the corresponding simple and perfect syntopogenous structure.

The PROOF of this Proposition is implicitely contained in [2]. (See [2], (15.10) and (7. 20).)

Thus any syntopogenous structure on a set E gives raise in a natural way to a principal limiting on E, and a principal limiting corresponds in this way to a syntopogenous structure if and only if its generating filters $\mathfrak{B}(x)(x \in E)$ are the neighborhood filters of a topology on E. (These principal limitings are simply called topologies in [3].) In order to get a one-to-one correspondence between equivalence classes of syntopogenous structures and those principal limitings which are topologies, one has to put

$$\mathcal{S}_1 \sim \mathcal{S}_2 \quad \text{iff} \quad \mathcal{S}_1^{tp} = \mathcal{S}_2^{tp}.$$

(See [2], (8. 54).)

§ 6. Pseudo-topologies and prae-topologies in the sense of G. Choquet

As we mentioned right at the outset, prior to the introduction of limitings by H.-J. Kowalsky and H. R. Fisher some closely related notions have been considered by G. Choquet [5]) in his paper [1].

Choquet's "structures pseudo-topologiques" (see [1], pp. 79—80) can be defined in our terminology as follows:

Definition 6. A mapping ϱ of E into $\mathfrak{P}(\Phi(E))$ is a pseudo-topology on E, if for any $x \in E$ it satisfies the following conditions:

$$(\mathrm{F}_1) \qquad \left.\begin{matrix} \mathcal{F} \in \varrho(x) \\ \mathcal{F} \leqq \mathcal{F}' \end{matrix}\right\} \Rightarrow \mathcal{F}' \in \varrho(x).$$

$$(\mathrm{F}_2) \qquad \mathcal{F} \notin \varrho(x) \Rightarrow (\exists \mathcal{F}')\{(\mathcal{F} \leqq \mathcal{F}') \& [\mathcal{F}' \leqq \mathcal{F}'' \rightarrow \mathcal{F}'' \notin \varrho(x)]\}.$$

$$(\mathrm{F}_3) \qquad \dot{x} \in \varrho(x).$$

One sees that condition (F_2) can be given the following equivalent formulation:

(F_{2a}) If $\mathcal{F} \notin \varrho(x)$ then $\mathcal{U} \notin \varrho(x)$ for some ultrafilter $\mathcal{U}$ containing $\mathcal{F}$.

A partial clarification of the relation in which pseudo-topologies stand to limitings, is achieved by the following

Proposition 6. *Every pseudo-topology on a set E is a limiting on E.*

Proof. We have only to show that $\mathcal{F}, \mathcal{G} \in \varrho(x) \Rightarrow \mathcal{F} \wedge \mathcal{G} \in \varrho(x)$. Suppose $\mathcal{F}, \mathcal{G} \in \varrho(x)$ and $\mathcal{F} \wedge \mathcal{G} \notin \varrho(x)$. Then, by (F_{2a}) there exists an ultrafilter $\mathcal{U}$ such that $\mathcal{F} \wedge \mathcal{G} \leqq \mathcal{U}$ and $\mathcal{U} \notin \varrho(x)$. From this we infer

$$(*) \qquad \mathcal{F} \leqq \mathcal{U} \quad \text{and/or} \quad \mathcal{G} \leqq \mathcal{U}.$$

As a matter of fact, $\mathcal{F} \not\leqq \mathcal{U}$ and $\mathfrak{G} \not\leqq \mathcal{U}$ together imply $E - F \in \mathcal{U}$ and $E - G \in \mathcal{U}$ for some $F \in \mathcal{F}$ and some $G \in \mathcal{G}$. Then, however

$$(E - F) \cap (E - G) = E - (F \cup G) \in \mathcal{U}$$

follows, and this contradicts $F \cup G \in \mathcal{F} \wedge \mathcal{G} \leqq \mathcal{U}$.

Thus the hypothesis we started with implies $(*)$, and $(*)$ in turn implies by (F_1) $\mathcal{U} \in \varrho(x)$, a contradiction. The contradiction obtained proves $\mathcal{F} \wedge \mathcal{G} \in \varrho(x)$ and establishes thereby the proposition. [6])

[5]) I am indebted to Professor Á. Császár for having kindly directed my attention to G. Choquet's paper [1].

[6]) The statement to the contrary in [3], p. 280., bottom, seems to require correction.

It might be conjectured that the converse of the proposition just proved is not true, i. e. that not every limiting on a set is a pseudo-topology.

Among pseudo-topologies so called prae-topologies ("structures pré-topologiques", see [1], p. 83) merit particular attention:

Definition 7. A pseudo-topology on a set E is a prae-topology on E, if for any $x \in E$ it satisfies the additional requirement

$$(U_2') \qquad \wedge\{\mathscr{F} \mid \mathscr{F} \in \varrho(x)\} \in \varrho(x).$$

A characterization of prae-topologies is given by the following

Proposition 7. *The concept of prae-topology and that of principal limiting coincide: Every prae-topology on a set E is a principal limiting on E, and conversely.*

PROOF. If a pseudo-topology ϱ satisfies also condition (U_2') then it is not only a limiting but a principal limiting, (L_3) being satisfied with

$$\mathfrak{B}(x) = \wedge\{\mathscr{F} \mid \mathscr{F} \in \varrho(x)\} \ (x \in E).$$

On the other hand, a principal limiting τ always satisfies (F_{2a}). As a matter of fact, let $\mathscr{F} \notin \tau(x)$. Then $\mathscr{U} \notin \tau(x)$ for some ultrafilter $\mathscr{U}$ satysfying $\mathscr{F} \leqq \mathscr{U}$, since otherwise we would have

$$\mathscr{F} = \wedge\{\mathscr{U} \mid \mathscr{F} \leqq \mathscr{U}\} \in \tau(x),$$

a contradiction. [7])

References

[1] G. Choquet, Convergences. *Ann. Univ. Grenoble,* **23** (1947/48), 57—112.
[2] Á. Császár, Foundations of General Topology. *Oxford—London—New York—Paris, 1963.*
[3] H. R. Fischer, Limesräume. *Math. Ann.* **137** (1959), 269—303.
[4] H. -J. Kowalsky, Limesräume und Komplettierung. *Math. Nachr.* **12** (1954), 301—340.

(Received August 28, 1965.)

[7]) Here we made use of the following known fact, which we prove for completeness' sake: *Any filter on a set E is equal to the intersection of the ultrafilters containing it.*

PROOF. It will be sufficient to prove the following statement:
If $H \notin \mathscr{F}$ then there exists an ultrafilter $\mathscr{U}$ such that $\mathscr{F} \leqq \mathscr{U}$ and $CH \in \mathscr{U}$.

Now, $H \notin \mathscr{F}$ implies $CH \cap F \neq \emptyset$ for any $F \in \mathscr{F}$, since $CH \cap F = \emptyset \Rightarrow F \subseteqq H \Rightarrow H \in \mathscr{F}$. Thus CH and $\mathscr{F}$ generate a filter, and any ultrafilter containing this filter has the required properties.