The idempotent separating congruences of a bisimple inverse semigroup with identity *)

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A congruence ϱ on a semigroup is called idempotent separating if each congruence class contains at most one idempotent. These congruences have been studied on regular semigroups by HOWIE [3] and MUNN [4] The main purpose of this note is to determine the idempotent separating congruences of a bisimple inverse semigroup with identity S in terms of certain normal subgrouphs of the group of units of S. We show that these congruences are uniquely determined by the congruence class containing the identity. We also give a class of examples of bisimple inverse semigroups with identity on which $\mathscr{H}$ (Green's relation) is not a congruence, i. e. $\mathscr{H}$ properly includes each idempotent separating congruence of S.

We will use the terminology and definitions of [2]. Let $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$ and $\mathscr{D}$ be Green's relations [2]. Bisimple inverse Semigroups with identity have been investigated by CLIFFORD [1] and WARNE [7]—[9].

CLIFFORD [1] determined the structure of bisimple inverse semigroups with identity S in terms of their right unit subsemigroups P. Let us briefly review his construction. If $a, b \in P$, there exists $c \in P$ such that $Pa \cap Pb = Pc$. Choose a representative element from each $\mathscr{L}$-class of P, and let $a \vee b$ be the representative of the $\mathscr{L}$-class containing c. Define a binary operation $*$ on P by

(1) $$(a * b)b = a \vee b$$

Then, $S \cong P \times P$ under the following definitions of equality and multiplication.

(2) $(a, b) = (c, d)$ if $a = uc$, $b = ud$ where $u \in U$, the group of units of P.

(3) $$(a, b)(c, d) = ((c * b)a, (b * c)d).$$

Conversely, if P is a right cancellative semigroup with identity such that the intersection of two principal left ideals is a principal left ideal, $S = P \times P$ under the above equality and multiplication is a bisimple inverse semigroup with identity with right unit subsemigroup P. The following are also established:

(4) $$a * a \in U$$

(5) $$E_S = ((a, a) : a \in P),$$

*) A summary of a portion of this article appeared as a research announcement in the *Bull. Amer. Math. Soc.* **72** (1966), 679—682.

where E_s is the set of idempotents of S.

(6) (b, a) is the inverse of (a, b) (This is actually established in [7]).

(7) $Pa = Pb$ if and only if $a = ub$ for some $u \in U$.

(8) $((1, u): u \in U) \cong U$ is the group of units of S.

(9) $1 * a = u \in U$ and $a * 1 = ua$.

The following is given in [8]

(10) $(a, b)\mathcal{R}(c, d)$ if $Pa = Pc$

$(a, b)\mathcal{L}(c, d)$ if $Pb = Pd$.

Let us now review the work of Preston [5] on the congruences of an arbitrary inverse semigroup. Let S be an inverse semigroup and let E_S be divided into disjoint sets E_α, where α ranges over some set J, such that $E = \bigcup_{\alpha \in J} E_\alpha$. Suppose further that to each pair $\alpha, \beta \in J$, there corresponds $\nu \in J$, such that $E_\alpha E_\beta \subseteq E_\nu$ and that to each $a \in S$ and $\alpha \in J$, there corresponds $\beta \in J$, such that $aE_\alpha a^{-1} \subseteq E_\beta$. For each $\alpha \in J$, let $N(E_\alpha)$ denote an inverse subsemigroup of S with E_α as its set of idempotents. If the $N(E_\alpha)$ satisfy the further conditions:

(A) $aa^{-1}, bb^{-1} \in E_\alpha, a, ab^{-1} \in N(E_\alpha)$, together imply that $b \in N(E_\alpha)$;

(B) $aa^{-1}, bb^{-1} \in E_\alpha, ab^{-1} \in N(E_\alpha)$, together imply that

$aN(E_\beta)b^{-1} \subseteq N(E_\gamma)$, where $aE_\beta a^{-1} \subseteq E_\gamma$,

then the inverse subsemigroup $N = \bigcup_{\alpha \in J} N(E_\alpha)$, will be said to be a normal subsemigroup of S, with components $N(E_\alpha)$.

If S is an inverse semigroup and ϱ is a congruence on S, the *kernel* of ϱ is the inverse image of $E_{S/\varrho}$ under the canonical homomorphism.

Theorem 1. (PRESTON) *Let N, with components $N(E_\alpha)$, be a normal subsemigroup of the inverse semigroup S, and define the relation ϱ_N over S, by $a\varrho_N b$, if and only if, for some α, $aa^{-1}, bb^{-1} \in E_\alpha$ and $ab^{-1} \in N(E_\alpha)$. Then ϱ_N is a congruence relation over S, with kernel N.*

Conversely, every congruence relation ϱ over S, has a kernel N which is a normal subsemigroup of S, such that ϱ_N is ϱ. S/ϱ is itself an inverse semigroup with the components of N as its idempotents. A congruence ϱ_N is idempotent separating if and only if $N = U(N_e : e \in E)$ where each N_e is a group and $N_e N_f \subseteq N_{ef}$ and $aN_f a^{-1} \subseteq N_g$ where $g = afa^{-1}$.

Let P be a right cancellative semigroup with identity and group of units U. A subgroup V is called a right normal divisor of P if and only if $aV \subseteq Va$ for all $a \in P$.

The following lemma is established in [8].

Lemma. *If S is a bisimple inverse semigroup with identity with right unit subsemigroup P, U, the group of units of P, is a right normal divisor if and only if $\mathcal{H}$ is a congruence on S.*

Theorem 2. *Let S be a bisimple inverse semigroup with identity and right unit subsemigroup P. There exists a 1—1 correspondence between the congruences on S such that the congruence class containing the identity is a group and the right normal divisors of P. In this case, each congruence is uniquely determined by the congruence class containing the identity. These congruences are precisely the idempotent separating congruences of S. If ϱ^V is the congruence corresponding to the right normal divisor V, $\varrho^V_{(a,b)} = ((a, vb): v \in V)$. If V_1, V_2 are right normal divisors of P, $V_1 \subseteqq V_2$ if and only if $\varrho^{V_1} \subseteqq \varrho^{V_2}$. Each idempotent separating congruence of S is contained in $\mathscr{H}$. If $M = \{g \in U / xg \in Ux$ for all $x \in P\}$, ϱ^M is the maximal idempotent separating congruence of S.*

PROOF. Let V be a right normal divisor of P and let $N_{(a,a)} = ((a, va): v \in V)$. We will show that $N = U(N_{(a,a)}: a \in P)$ is a normal subsemigroup corresponding to the idempotent separating congruence ϱ_N. If $u, v \in V$, by (2), (3), and (4), $(a, ua)(a, va) = (a, uva)$ and it easily follows that $N_{(a,a)}$ is a group. If $u, v_1 \in V$, a, $b \in P$, there exists $s, t \in V$ such that $(b * a)u^{-1} = s(b * a)$ and $(a * b)v_1 = t(a * b)$ since V is a right normal divisor of P. Thus, by (1), (2), (3) and (4), $(a, ua)(b, v_1 b) = (u^{-1}a, a)(b, v_1 b) = ((b * a)u^{-1}a, (a * b)v_1 b) = (s(b * a)a, t(a * b)b) = (avb, s^{-1}t(avb))$ and $N_{(a,a)}N_{(b,b)} \subseteqq N_{(avb, avb)}$. Next, let $(a, b) \in S$ and by (5) $(c, c) \in E_S$. Thus, by (3) and (1), $(a, b)(c, c)(b, a) = ((c * b)a, bvc)(b, a) = ((c * b)a, (c * b)a)$. Noting that if $v' \in V$ there exists $w \in V$ such that $(b * c)v' = w(b * c)$, similarly we have $(a, b)(c, v'c)(b, a) = ((c * b)a, (b * c)v'c)(b, a) = ((c * b)a, w(bvc))(b, a) = (w^{-1}(c * b)a, bvc)(b, a) = (w^{-1}(c * b)a, (c * b)a) = ((c * b)a, w(c * b)a)$.

Hence, $(a, b)N_{(c,c)}(b, a) \subseteqq N_{((c*b)a(c*b)a)}$ and the desired result follows from (6) and theorem 1. Conversely, suppose that ϱ is any congruence such that the congruence class containing the identity is a group. Thus, by theorem 1, $\varrho = \varrho_N$ where N is a normal subsemigroup of S. Denote the congruence class containing the identity by N_1. Suppose that (a, a) and (b, b) are in the same component of N. Thus, by (4) and (2), $(1, a)(a, a)(a, 1) = (1, 1)$ and $(1, a)(b, b)(a, 1) = (1, a)(b, 1)(1, b) \cdot (a, 1) = (b * a, a * b)(a * b, b * a) = (b * a, b * a)$. Thus by theorem 1, $(b * a, b * a) \in N_1$. Thus, $b * a \in U$ by (2). Interchanging a and b, $a * b \in U$. Hence, since $(b * a)a = avb = (a * b)b$ by (1), $(a, a) = (b, b)$ by (2) and ϱ is idempotent separating. By (8), $N_1 = ((1, v): v \in V)$ where V is a subgroup of U. By (9) and theorem 1 if $a \in P$ and $s \in V$, there exists $t \in U$ and $v \in V$ such that $(1, a)(1, s)(a, 1) = (1 * a, (a * 1)s) \cdot (a, 1) = (t, tas)(a, 1) = (1, as)(a, 1) = (a * as, as * a) = (1, v)$. Hence, by (2), there exists $w \in U$ such that $a * as = w$ and $as * a = wv$. Thus, by (1), $was = wva$ and $as = va$, i. e. $aV \subseteqq Va$ and V is a right normal divisor of P. Let $N_{(a,a)}$ denote the component of N containing (a, a). By theorem 1, (4) and (6), $(1, a)N_{(a,a)}(a, 1) \subseteqq N_{(1,1)}$. Since $N_{(a,a)}$ is a group. $N_{(a,a)} \subseteqq H_{(a,a)}$, the $\mathscr{H}$-class containing (a, a). Thus, it follows from (10), (2), and (3) that $N_{(a,a)} = ((a, wa): w \in W)$ where W is a subgroup of U. Now $(1, a)(a, wa)(a, 1) = (1, wa)(a, 1) = (w^{-1}, 1) = (1, w)$ and $w \in V$. Thus, $N_{(a,a)} = \{(a, va): v \in V\}$. Hence, we have the desired correspondence. Since $(a, ua) = (a, 1)(1, u)(1, a)$ for $u \in U$ by (4) and (2), each component is uniquely determined by the congruence class containing the identity. Next let us show that $\varrho^V \subseteqq \mathscr{H}$.

If $(a, b)\varrho^V(c, d)$, $(a, a) = (c, c)$ by (6) and Theorem 1, and $Pa = Pc$ by (2) and (7). Now, again by Theorem 1, $(a, b)(d, c) = ((d * b)a, (b * d)c)$, $(d * b)a = ua$, and $(b * d)c = vc$ for some $u, v \in U$. Thus, $d * b$, $b * d \in U$ since P is right cancellative. Hence $(d * b)b = bvd = (b * d)d$ and $Pb = Pd$ by (2) and (7). Hence, $(a, b) \mathscr{H} (c, d)$

by (10). We now show that $\varrho^V_{(c,d)}=((c, ud): u\in V)$. Since $\varrho^V\subseteqq\mathscr{H}$, $(a, b)\in\varrho^V_{(c,d)}$ implies that $a=sc$ and $b=td$ where $s, t\in U$ by (7) and (10). Thus, by (4), (2), and theorem 1 there exists $v\in V$ such that $(a, b)(d, c)=(a, b)(t^{-1}b, c)=(a, b)(b, tc)= =(a, tc)=(sc, tc)=(c, s^{-1}tc)=(c, vc)$. Hence, since P is right cancellative, using (2), we obtain $s^{-1}t=v\in V$, i. e. $(a, b)=(sc, td)=(c, s^{-1}td)$ and $\varrho^V_{(c,d)})\subseteqq((c, vd): v\in V)$. If $v\in V$, by (4) and (2), $(c, vd)(vd, c)=(c, c), (c, d)(d, c)=(c, c), (c, vd)(d, c)= =(v^{-1}c, c)=(c, vc)$ and $(c, vd)\in\varrho^V(c, d)$ by theorem 1. Thus, $\varrho^V_{(c,d)}=((c, vd): v\in V)$. We next show that $V_1\subseteqq V_2$ if and only if $\varrho^{V_1}\subseteqq\varrho^{V_2}$. Suppose that $\varrho^{V_1}\subseteqq\varrho^{V_2}$. If $v\in V_1$, $(1, v)\varrho^{V_2}$ $(1,1)$ and $v\in V_2$, ie $V_1\subseteqq V_2$. Clearly, if $V_1\subseteqq V_2$, $\varrho^{V_1}\subseteqq\varrho^{V_2}$. Since $M=\{g\in U/xg\in Ux$ for all $x\in P\}$ is the greatest right normal divisor of P[6, lemma 2.11]. ϱ^M is the maximal idempotent separating congruence of S, q.e.d. By virtue of theorem 2,

if $\mathscr{H}$ is a congruence on S, then $\mathscr{H}$ is the maximal idempotent separating congruence on S. This situation occurs in many cases [8].

However let us now give a class of examples of a bisimple inverse semigroup S with identity on which $\mathscr{H}$ is not a congruence. By the lemma, it is only necessary to construct a right cancellative semigroup P with identity for which U is not a right normal divisor and for which the intersection of two principal left ideals is a principal left ideal and then apply Clifford's structure theorem described above. Let F be the positive part of any ordered field and let $F^*=F\setminus 0$. The required example is given by $P=F^*xF$ under the multiplication

$$(a, b)(c, d)=(ac, bc+d).$$

It is seen by straightforward calculations that P has the required properties.

Remark. The fact (theorem 2) that each idempotent separating congruence is contained in $\mathscr{H}$ also follows from [4], p. 389, Theorem 2.

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(Received September 27, 1965.)