Publ. Math. Debrecen
48 / 3-4 (1996), 357–366

Reduction theorems of certain Landsberg spaces to Berwald spaces

By S. BÁCSÓ (Debrecen) and M. MATSUMOTO (Kyoto)

Abstract. We shall show three theorems according to which, under certain conditions a Landsberg space reduces to a Berwald space. These conditions concern the Douglas tensor, the T-tensor and the quartic metric.

0. Introduction

We have several interesting theorems such that if a Finsler space F^n is a Landsberg space ($C_{hij|k}y^k = 0$) and satisfies some additional conditions, then F^n becomes a Berwald space ($C_{hij|k} = 0$). Such theorems suggest us to consider the existence of essentially Landsberg spaces which are not Berwald spaces.

In a recent paper [3][1], which is a joint work of the first author and his colleagues, the additional condition of the above mentioned reduction ([3], Theorem 1) is that the Douglas tensor of F^n vanishes. This theorem holds, provided $n > 2$, but it should be remarked that F^n is assumed to have a positive-valued fundamental function $L(x, y)$ and a positive-definite fundamental tensor $g_{ij}(x, y)$. In fact, these assumptions are essential to Deicke's theorem ([5] §24) which is applied in the proof.

In the two-dimensional case this theorem was really proved by Berwald [2]. In the proof he applied the so-called Berwald frame method [5],

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The manuscript of the paper was written during the period when the second author stayed in Debrecen between October 1 and October 19, 1995.
This research was supported by OTKA T-17261.
[1]Numbers in brackets refer to the references at the end of the paper.

which was developed under the assumption of the positive-definiteness of g_{ij}.

After some preliminary remarks on the modified theory of Berwald frames, the first purpose of the present paper is to show that Bácsó's and Berwald's reduction theorems hold also without this assumption.

The second section is devoted to the proof of a simple theorem, where the additional conditinons are such that the dimension number is equal to two and the T-tensor vanishes.

In section 4 another reduction theorem is presented on two-dimensional Finsler spaces with quartic metric. This represents a supplement to the theory of Finsler spaces with m-th root metric recently developed by the second author and his colleagues.

1. The modified theory of Berwald frames

The special and useful Berwald frame [2] was founded and developed method in oder to study two-dimensional Finsler spaces. It works under the assumption that the fundamental tensor $g_{ij}(x,y)$ is positive-definite. Then one can define a local field of orthonormal frame (l,m) called the *Berwald frame* ([5], §28), and then g_{ij} is written as $g_{ij} = l_i l_j + m_i m_j$. Positive-definiteness was an implicit assumption of Berwald which appeared rather natural in his time. However, in our days we have to pay attention to the recent rapid progress of Finsler geometry; we have various applications of this geometry to other fields of science [1]. Consequently it seems that positive-defineteness is too restrictive for the applications.

The modification of the Berwald frame method to the non-positive definite case has been given in [1], §3.5. We sketch it for later use.

We are concerned with a two-dimensional Finsler space F^2 with fundamental function $L(x,y)$, $x=(x^i)$, $y=(y^i)$, $i=1,2$. Then we have

$$
\begin{aligned}
&l^i = \frac{1}{L} y^i, \qquad l_i = \dot{\partial}_i L \qquad (\dot{\partial}_i = \partial/\partial y^i),\\
&h_{ij} = L \dot{\partial}_i \dot{\partial}_j L, \quad g_{ij} = l_i l_j + h_{ij}.
\end{aligned}
$$

Since the angular metric tensor h_{ij} has the matrix (h_{ij}) of rank one, we can define the vector $m=(m_1,m_2)$ by

$$h_{ij} = \varepsilon m_i m_j, \quad \varepsilon = \pm 1.$$

Then we get

$$g_{ij} = l_i l_j + \varepsilon m_i m_j, \ \det(g_{ij}) = \varepsilon (l_1 m_2 - l_2 m_1)^2.$$

The sign ε is called the *signature* of F^2.

Next, since the C-tensor $C_{ijk} = \dot{\partial}_i\dot{\partial}_j\dot{\partial}_k(L^2/4)$ has no components in the direction l^i ($C_{ijk}y^i = 0$), it can be written in the frame (l, m) as

$$LC_{ijk} = Im_i m_j m_k.$$

The scalar field I thus defined is called the *main scalar* of F^2. Then we have

$$\begin{cases} L\dot{\partial}_j l^i = \varepsilon m^i m_j, \\ L\dot{\partial}_j l_i = \varepsilon m_i m_j, \end{cases} \begin{cases} L\dot{\partial}_j m^i = -(l^i + \varepsilon I m^i)m_j, \\ L\dot{\partial}_j m_i = -(l_i - \varepsilon I m_i)m_j. \end{cases}$$

We deal with the covariant differentiations. Denote by $(;, .)$ and $(_|, |)$ the covariant differentiations in the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ and in the Cartan connection $(\Gamma^{*i}_{jk}, G^i_j, C^i_{jk})$ respectively. Then for a scalar field $S(x, y)$ we get

$$S_{;i} = S_{|i} = \partial_i S - (\dot{\partial}_r S)G^r_i,\ S_{.i} = S|_i = \dot{\partial}_i S.$$

We write $S_{|i}$ and $LS|_i$ in (l, m) as follows

$$S_{|i} = S_{,1}l_i + S_{,2}m_i,\ LS|_i = S_{;1}l_i + S_{;2}m_i.$$

$(S_{,1}, S_{,2})$ and $(S_{;1}, S_{;2})$ are called the *h- and the v-scalar derivatives* of S respectively.[2] If $S(x, y)$ is positively homogeneous of degree r in y, then we have $S|_i y^i = rS$. For zero degree scalar fields, we have $LS|_i = S_{;2}m_i$. For l_i and m_i of (l, m) we have

$$\begin{cases} l_{i;j} = 0, \\ Ll_{i.j} = \varepsilon m_i m_j, \end{cases} \qquad \begin{cases} m_{i;j} = -\varepsilon I_{,1} m_i m_j, \\ Lm_{i.j} = -(l_i - \varepsilon I m_i)m_j, \end{cases}$$
$$\begin{cases} l_{i|j} = 0, \\ Ll_i|_j = \varepsilon m_i m_j, \end{cases} \qquad \begin{cases} m_{i|j} = 0, \\ Lm_i|_j = -l_i m_j. \end{cases}$$

Next, the torsion and curvature tensors appear in the commutation formulae of covariant differentiations. For $C\Gamma$ we have

h-curvature $R_h{}^i{}_{jk} = \varepsilon R(l_h m^i - l^i m_h)(l_j m_k - l_h m_j)$,

hv-curvature $P_h{}^i{}_{jk} = \frac{1}{L} I_{,1}(l_h m^i - l^i m_h)m_j m_k$,

(v)h-torsion $R^i{}_{jk} = \varepsilon LRm^i(l_j m_k - l_k m_j)$,

(v)hv-torsion $P^i{}_{jk} = I_{,1} m^i m_j m_k$.

R is called the *curvature* of F^2. The v-curvature $S_h{}^i{}_{jk} = C_h{}^r{}_k C_r{}^i{}_j - C_h{}^r{}_j C_r{}^i{}_k$ vanishes identically for F^2.

[2] In Berwald's notation $(S_{,1}, S_{,2}, S_{;2}) = (S_s, S_b, S_\theta)$.

On the other hand, for $B\Gamma$ we have

(1.1) h-curvature: $H_h{}^i{}_{jk}=\varepsilon\{R(l_hm^i-l^im_h)+R_{;2}m_hm^i\}(l_jm_k-l_km_j)$,

(1.2) hv-curvature: $G_h{}^i{}_{jk}=\dfrac{1}{L}(-2I_{,1}l^i+I_2m^i)m_hm_jm_k$,

where

$$I_2=I_{,1;2}+I_{,2}.$$

The (v)h-torsion $R^i{}_{jk}$ coincides with that of $C\Gamma$.

The commutation formulae for scalar derivatives are written in the form

$$(1.3)\qquad \begin{cases} (1) & S_{,1,2}-S_{,2,1}=-RS_{;2}, \\ (2) & S_{,1;2}-S_{;2,1}=S_{,2} \\ (3) & S_{,2;2}-S_{;2,2}=-\varepsilon(S_{,1}+IS_{,2}+I_{,1}S_{;2}). \end{cases}$$

Finally the Bianchi identites for an F^2 reduces to the single identity:

$$(1.4)\qquad I_{,1,1}+RI+\varepsilon R_{;2}=0.$$

2. Landsberg spaces with vanishing Douglas tensor

A Finsler space F^n is called a *Landsberg space*, if $G_h{}^i{}_{jk}y_i=0$ or equivalently $C_{hij|k}y^k=0$. It is well-known that F^n is a Berwald (affinely connected) space, if $G_j{}^i{}_k$ are functions of position alone, that is $G_h{}^i{}_{jk}=0$, or equivalently $C_{hij|k}=0$.

From $G_h{}^i{}_{jk}$ we get a projective invariant $D_h{}^i{}_{jk}$, called the *Douglas tensor* ([2], [4]):

$$D_h{}^i{}_{jk}=G_h{}^i{}_{jk}-\frac{1}{n+1}\left(y^iG_{hj\cdot k}+\delta^i_hG_{jk}+\delta^i_jG_{kh}+\delta^i_kG_{hj}\right),$$

where $G_{hj}=G_h{}^r{}_{jr}$ and $G_{hj\cdot k}=\dot\partial_kG_{hj}$. In particular the $D_h{}^i{}_{jk}$ of a two-dimensional Finsler space F^2 can be written in the form

$$3LD_h{}^i{}_{jk}=-(6I_{,1}+\varepsilon I_{2;2}+2II_2)m_hl^im_jm_k.$$

The purpose of the present section is to prove the following theorem without the assumption of positive-definiteness:

Theorem 1. *If a Finsler space F^n is a Landsberg space and has vanishing Douglas tensor, then it is a Berwald space.*

Proof. In the case of $n > 2$, an almost complet proof has been given by Bácsó and his colleagues [3]. From $G_h{}^i{}_{jk} y_i = 0$ and $D_h{}^i{}_{jk} = 0$ they derived

$$(2.1) \qquad \begin{cases} (1) \quad G_{hijk} = \frac{1}{n+1}(h_{hi}G_{jk} + h_{hj}G_{ki} + h_{hk}G_{ij}), \\ (2) \quad G_{hj} = \frac{G}{n-1} h_{hj}, \\ (3) \quad (n-2)GC_i = 0, \quad C_i = C_i{}^r{}_r. \end{cases}$$

(3) implies $G = 0$ or $C_i = 0$. From $G = 0$ and (2) we immediately get $G_{hijk} = 0$. On the other hand, from $C_i = 0$ and $G_{hj} = C_{h|j}$ (p. 144 of [3]) it follows that (1) and (2) imply $G_h{}^i{}_{jk} = 0$. In both cases the space reduces to a Berwald space. We note that originally (in [3]) Deicke's theorem was applied to get $C_i = 0$. This however is not necessary here.

In the case of $n = 2$ the theorem was proved by Berwald [2]. Now we modify his proof for the case of g_{ij} with arbitrary signature.

From (1.2) it follows that F^2 is a Landsberg space if and only if

$$(2.2) \qquad I_{,1} = 0.$$

Let us remarke that F^2 is a Berwald space if and only if $I_{,1} = I_{,2} = 0$, as shown by (1.2). The Douglas tensor of F^2 vanishes if and only if $6I_{,1} + \varepsilon I_{2;2} + 2II_2 = 0$ where $I_2 = I_{,1;2} + I_{,2}$. Consequently (2.2) leads to

$$(2.3) \qquad I_{,2;2} = -2\varepsilon I I_{,2}.$$

Further we must pay attention to (1.3) and (1.4). Then the latter reduces to

$$(2.4) \qquad R_{;2} = -\varepsilon RI.$$

Now we are concerned with $I_{,2,1}$ and $I_{,2,2}$. Applying (1) of (1.3) to $S = I$, we get

$$(2.5) \qquad I_{,2,1} = RI_{;2}.$$

Next, applying (2) of (1.3) to $S = I_{,2}$ and making use of (2.2), (2.3), (2.4) and (2.5), we have

$$\begin{aligned} I_{,2,2} &= I_{,2,1;2} - I_{,2;2,1} = (RI_{;2})_{;2} + 2\varepsilon(II_{,2})_{,1} \\ &= -\varepsilon RII_{;2} + RI_{;2;2} + 2\varepsilon I(RI_{;2}). \end{aligned}$$

which implies

$$I_{,2,2} = R(I_{;2;2} + \varepsilon II_{;2}). \tag{2.6}$$

Applying (3) of (1.3) to $I_{,2}$, we get similarly

$$2\varepsilon(I_{,2})^2 + R\{I_{;2;2;2} + 3\varepsilon II_{;2;2} + \varepsilon(I_{;2})^2 + 2I^2I_{;2} + \varepsilon I_{;2}\} = 0. \tag{2.7}$$

If we apply the scalar differentiation $(_{;2})$ to (2.7) and substitute from (2.3) and (2.4), then we easily obtain

$$\begin{aligned} R\{I_{;2;2;2;2} + 6\varepsilon II_{;2;2;2} + (5\varepsilon I_{;2} + 11I^2 + \varepsilon)I_{;2;2} \\ +(7I_{;2} + 6\varepsilon I^2 + 3)II_{;2}\} = 0. \end{aligned} \tag{2.8}$$

From $R = 0$ and (2.7) we get $I_{,2} = 0$, so that F^2 becomes a Berwald space with $R = 0$, that is, a locally Minkowski space. In the case of $R \neq 0$ we apply the differentiation $(_{,2})$ to $\{\dots\}$ of (2.8). Then we get the terms $I_{;2,2}$, $I_{;2;2,2}$, $I_{;2;2;2,2}$ and $I_{;2;2;2;2,2}$. We will use the following formulae:

$$\begin{cases} (1) \quad I_{;2,1} = -I_{,2}, \\ (2) \quad I_{;2;2,1} = 3\varepsilon II_{,2}, \\ (3) \quad I_{;2;2;2,1} = (4\varepsilon I_{;2} - 7I^2 + \varepsilon)I_{,2}, \end{cases} \tag{2.9}$$

$$\begin{cases} (1) \quad I_{;2,2} = -\varepsilon II_{,2}, \\ (2) \quad I_{;2;2,2} = (-\varepsilon I_{;2} + I^2 - \varepsilon)I_{,2} \\ (3) \quad I_{;2;2;2,2} = (-\varepsilon I_{;2;2} + 3II_{;2} - \varepsilon I^3 + 4I)I_{,2}, \\ (4) \quad I_{;2;2;2;2,2} = \big\{ -\varepsilon I_{;2;2;2} + 4II_{;2;2} + 3(I_{;2})^2 - 6\varepsilon I^2 I_{;2} \\ \qquad\qquad + 8I_{;2} + I^4 - 11\varepsilon I^2 + 1\big\} I_{,2}. \end{cases} \tag{2.10}$$

The proof of these relations is simple. We establish one of them only say (3) of (2.10). Applying (3) of (1.3) to $S = I_{;2;2}$, we get

$$I_{;2;2;2,2} = I_{;2;2,2;2} + \varepsilon I_{;2;2,1} + \varepsilon II_{;2;2,2}.$$

Substituting from (2) of (2.10), (2) of (2.9) and then (2.3), we obtain (3) of (2.10) immediately.

Now, applying $(_{,2})$ to the $\{\dots\}$ of (2.8) and substituting from (2.10), we finally obtain

$$\{\varepsilon I_{;2;2;2} + 3II_{;2;2} + (I_{;2})^2 + 2\varepsilon I^2 I_{;2} + I_{;2}\} I_{,2} = 0.$$

Comparing this with (2.7), we can get $I_{,2} = 0$. Therefore the proof of Theorem 1 has been completed.

3. Some remarks

As it was shown by Berwald ([2],[4]), a Finsler space F^n is projectively flat, if and only if

$$
\begin{aligned}
&1) \quad n>2: \quad (a)\ W_h{}^i{}_{jk}=0, && (b)\ D_h{}^i{}_{jk}=0,\\
&2) \quad n=2: \quad (a)\ 3R_{,2}-R_{;2,1}=0, && (b)\ D_h{}^i{}_{jk}=0,
\end{aligned}
$$

where $W_h{}^i{}_{jk}$ is the Weyl projective curvature tensor, a projectively invariant tensor and R is the curvature.

(1) $W_h{}^i{}_{jk}$ vanishes identically in the case of $n=2$.
(2) It has been shown by Z. Szabó ([8],[4]) that F^n $(n>2)$ is of scalar curvature K, if and only if its $W_h{}^i{}_{jk}$ vanishes.
(3) It has been shown by S. Numata ([5], §30) that, if a Landsberg space F^n $(n>2)$ is of non-zero scalar curvature K, then F^n is a Riemannian space of constant curvature K, provided that F^n has a positive-definite metric.

Theorem 1 is concerned with Landsberg spaces satisfying the conditions (b) above. What is a two-dimensional Landsberg space satisfying (a) of 2)? This is an open problem.

Next we shall be concerned with the so-called *T-tensor* (§28 of [5]):

$$T_{hijk}=LC_{hij}|_k+l_hC_{ijk}+l_iC_{hjk}+l_jC_{hik}+l_kC_{hij}.$$

In the case of $n=2$ this is written in the form

$$LT_{hijk}=I_{;2}m_hm_im_jm_k.$$

The following theorem has been shown by Szabó [9]: If F^n has the vanishing T-tensor, then it is a Riemannian space, provided that $n>2$ and the metric is positive-definite. In the case of $n=2$ we show a reduction theorem:

Theorem 2. *If a two-dimensional Finsler space F^2 is a Landsberg space and has vanishing T-tensor, then F^2 is a Berwald space.*

Proof. Our assumptions are written as

$$(1)\ I_{,1}=0, \qquad (2)\ I_{;2}=0.$$

Then (2) of (1.3) immediately implies $I_{,2}=0$. Thus F^2 is a Berwald space.

So we have a conclusive theorem on two-dimensional Berwald spaces. See [5], §28 (positive-definite case alone) and [1], §3.5.

4. Two-dimensional Landsberg spaces with quartic metric

Let F^n_4 be a Finsler space with a fundamental function given by

$$L^4 = a_{hijk}(x)y^h y^i y^j y^k,$$

where $a_{hijk}(x)$ are components of a covariant symmetric tensor of degree four ([6], [7]). A metric defined by such an L is called a quartic metric. The second author has proved the following theorem in the second paper [6] of a series concerned with m-th root metrics:

If a Finsler space F^n_3 with a *cubic metric* is a Landsberg space, then it is a Berwald space.

This theorem holds without any assumption on the dimension or on the metric. The purpose of the present section is to show the

Theorem 3. *If a two-dimensional Finsler space F^2_4 with a quartic metric is a Landsberg space, then it is a Berwald space.*

Proof. As has been shown in [6], F^2_4 has a quartic metric, if and only if the main scalar I satisfies

$$I_{;2;2} + 10\varepsilon I I_{;2} + 4I(3I^2 + 4\varepsilon) = 0. \tag{4.1}$$

Since our F^2_4 is a Landsberg space, (2.2) holds also.

First, applying (2) and (3) of (1.3) to $S = I$, we have

$$I_{;2,1} = -I_{,2}, \tag{4.2}$$

$$I_{,2;2} = I_{;2,2} - \varepsilon I I_{,2}. \tag{4.3}$$

Let us apply the differentiation $(_{,1})$ to (4.1), then we get $I_{;2;2,1}{=}-10\varepsilon I I_{;2,1}$. Now (4.2) leads to

$$I_{;2;2,1} = 10\varepsilon I I_{,2}. \tag{4.4}$$

Next, applying (2) of (1.3) to $S = I_{;2}$ and substituting from (4.2) and (4.4), we get

$$I_{;2,2} = I_{;2,1;2} - I_{;2;2,1} = -I_{,2;2} - 10\varepsilon I I_{,2}.$$

Then (4.3) leads to

$$I_{;2,2} = -\frac{9}{2}\varepsilon I I_{,2}, \tag{4.5}$$

and (4.3) can be written in the form

$$I_{,2;2} = -\frac{11}{2}\varepsilon I I_{,2}. \tag{4.3'}$$

Next we consider $I_{;2;2,2}$. Applying (3) of (1.3) to $S = I_{;2}$, we get

$$I_{;2;2,2} = I_{;2,2;2} + \varepsilon I_{;2,1} + \varepsilon I I_{;2,2}.$$

Substituting from (4.2), (4.5) and then from (4.3'), we obtain

$$I_{;2;2,2} = \left(-\frac{9}{2}\varepsilon I_{;2} + \frac{81}{4}I^2 - \varepsilon\right) I_{,2}. \tag{4.6}$$

On the other hand, (4.1) yields

$$I_{;2;2,2} = -10\varepsilon I_{,2}I_{;2} - 10\varepsilon I I_{;2,2} - 4(3I^2 + 4\varepsilon)I_{,2} - 4I(6II_{,2}).$$

Substituting from (4.5), the above is written as

$$I_{;2;2,2} = (-10\varepsilon I_{;2} + 9I^2 - 16\varepsilon)I_{,2}. \tag{4.7}$$

Consequently (4.6) and (4.7) give $I_{,2} = 0$, or

$$22\varepsilon I_{;2} + 45I^2 + 60\varepsilon = 0.$$

This together with (2.2) yields $I_{;2,1} = 0$, that is $I_{,2} = 0$ results from (4.2).

In any case we obtain $I_{,2} = 0$, and hence we can conclude that F_4^2 reduces to a Berwald space.

Remark. As we have mentioned, Theorem 1 is now completely proved, 54 years after Berwald. The authors conjecture that Theorem 3 may be extended to arbitrary dimension.

References

[1] P. L. Antonelli, R. S. Ingarden and M. Matsumoto, The theory of sprays and Finsler spaces with applications in physics and biology, FTP 58, *Kluwer Academic Publishers*, *Dordrecht-Boston-London*, 1993.

[2] L. Berwald, On Cartan and Finsler geometries, III. Two-dimensional Finsler spaces with rectilinear extremal, *Ann. of Math.* **42** no. 2 (1941), 84–112.

[3] S. Bácsó, F. Ilosvay and B. Kis, Landsberg spaces with common geodesics, *Publ. Math. Debrecen* **42** (1993), 139–144.

[4] M. Matsumoto, Projective changes of Finsler metrics and projectively flat Finsler spaces, *Tensor (N.S.)* **34** (1980), 303–315.

[5] M. Matsumoto, Foundations of Finsler geometry and special Finsler spaces, *Kaiseisha Press*, *Saikawa, Otsu, Japan*, 1986.

[6] M. Matsumoto, On Finsler spaces with m-th root metric II, to appear in Publ. Math. Debrecen.

[7] M. Matsumoto and K. Okubo, On Finsler spaces with m-th root metric, to appear in Tensor (N.S.).

[8] Z. Szabó, Ein Finslerschen Raum ist gerade dann von skalarer Krümmung, wenn seine Weylsche Projektivkrümmung verschwindet, *Acta Sci. Math. Szeged* **39** (1977), 163–168.

[9] Z. Szabó, Positive definite Finsler spaces satisfying the T-condition are Riemannian, *Tensor (N.S.)* **35** (1981), 247–248.

S. BÁCSÓ
MATHEMATICAL INSTITUTE
UNIVERSITY OF LAJOS KOSSUTH
H–4010 DEBRECEN
P.O.BOX 12.
HUNGARY

M. MATSUMOTO
15, ZENBU–CHO, SHIMOGAMO
SAKYO–KU, KYOTO
606 JAPAN

(Received November 8, 1995)