On the summability factors of infinite series

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1. Let $\{p_n\}$ be a sequence of non-negative real constants such that $P_n = \sum_0^n p_\nu$ tends to infinity with n. The sequence

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu ,$$

where s_n is the n-th partial sum of a given infinite series Σa_n, defines the $(\bar{N}, p_n)$ means of $\{s_n\}$.

The series Σa_n is said to be summable $|\bar{N}, p_n|$, if the sequence $\{t_n\}$ is of bounded variation.

If $p_n = \dfrac{1}{n+1}$, we have $P_n \sim \log n$ and $P_n/\log n \in BV$, and therefore summability $\left|\bar{N}, \dfrac{1}{n+1}\right|$ is equivalent to the summability $|R, \log n, 1|$.

2. Recently BHATT ([1]) has proved the following theorem.

Theorem A. *If $\{\lambda_n\}$ is a convex sequence such that $\Sigma\lambda_n/n < \infty$ and the sequence $\{s_n\}$ is bounded, then the series $\Sigma\lambda_n a_n \log n$ is summable $|R, \log n, 1|$.*

Later on he [2] gave a generalisation of this theorem in the following form.

Theorem B. *If $\{\lambda_n\}$ is a convex sequence such that $\Sigma\lambda_n/n < \infty$, and if the sequence $\{k_n\}$, the $(R, \log n, 1)$ mean of the sequence $\{na_n \log (n+1)\}$, satisfies the condition*

$$|k_n| = O\{\log (n+1)\}^k \; (C, 1), \qquad k \geqq 0, \tag{2. 1}$$

then the series $\Sigma\lambda_n a_n\{\log (n+1)\}^{1-k}$ is summable $|R, \log n, 1|$.

The object of this paper is to obtain a further generalisation of this theorem.

3. We shall prove the following theorem.

Theorem 1. *If $\{\lambda_n\}$ is a convex sequence such that $\Sigma\lambda_n p_n < \infty$, where $\{p_n\}$ is a non-increasing positive sequence, and the sequence $\{\mu_n\}$, the $(\bar{N}, p_n)$ mean of $\{a_n P_n/p_n\}$, satisfies the condition*

$$|\mu_n| = O\{\gamma_n\} \; (C, 1), \tag{3. 1}$$

γ_n being a positive non-decreasing sequence such that $\left|\Delta^2 \dfrac{1}{\gamma_n}\right| \geqq 0$ and $\Delta\gamma_n = O\{p_n\gamma_n/P_n\}$, then the series $\Sigma a_n\lambda_n P_n/\gamma_n$ is summable $|\bar{N}, p_n|$.

It is clear that if we take $p_n = \frac{1}{n+1}$ and $\gamma_n = \{\log(n+1)\}^k$, Theorem B from our theorem.

4. The following lemmas are pertinent for the proof of this theorem.

Lemma 1. *If $\{\lambda_n\}$ is a convex sequence such that $\Sigma \lambda_n p_n < \infty$, where $\{p_n\}$ is a sequence of real positive constants such that $P_n \to \infty$, then $\{\lambda_n\}$ is a non-negative monotonic decreasing sequence tending to zero and $\lambda_n P_n = o(1)$, as $n \to \infty$.*

This generalises the following lemma of Chow ([3]).

Lemma A. *If $\{\lambda_n\}$ is a convex sequence such that $\Sigma \lambda_n / n < \infty$, then $\{\lambda_n\}$ is a non-negative decreasing sequence and $\lambda_n \log n = o(1)$, as $n \to \infty$.*

Proof of Lemma 1. Since $\Delta^2 \lambda_n \geqq 0$, it follows that $\Delta \lambda_n$ is non-increasing and λ_n either tends to a finite limit l or to $+\infty$ or $-\infty$. Also since $\Sigma \lambda_n p_n < \infty$ we have

$$\frac{1}{P_n} \sum_1^n \lambda_m p_m = o(1), \qquad (n \to \infty). \tag{4. 1}$$

Now let $\lim_{n\to\infty} \lambda_n = s$, where s is any number finite or infinite but not zero. Then by virtue of a well known result we have

$$\lim_{n\to\infty} \frac{1}{P_n} \sum_1^n \lambda_m p_m = s.$$

Since $s \neq 0$ we get a contradiction by virtue of (4. 1). Hence s must be zero so that $\lambda_n \to 0$ and therefore $\lim \Delta \lambda_n = 0$. Thus $\Delta \lambda_n \geqq 0$. This means that $\{\lambda_n\}$ is a non-increasing sequence and by virtue of the fact that $\lim_{n\to\infty} \lambda_n = 0$, it follows that $\{\lambda_n\}$ is non-negative and decreasing sequence tending to zero.

We know that if $\Sigma a_n < \infty$ and $\{\beta_n\}$ is any monotonic increasing sequence of positive numbers tending to infinity with n, then $\lim_{n\to\infty} \frac{1}{\beta_n} \sum_1^n a_m \beta_m = 0$. Taking $\beta_n = 1/\lambda_n$ and applying this result to the series $\Sigma\, p_n \lambda_n$, which is convergent, we have

$$\lambda_n \sum_1^n \lambda_m p_m \, 1/\lambda_m \to 0,$$

that is to say $P_n \lambda_n = o(1)$, as $n \to \infty$.

This completes the proof of the lemma.

Lemma 2. *If $\{\alpha_n \gamma_n\}$ satisfies the same condition as λ_n in Lemma 1, then*

$$\sum_1^m P_n \gamma_n \Delta \alpha_n = O(1), \qquad m \to \infty$$

where $\{\gamma_n\}$ is a positive non-decreasing sequence such that

$$\Delta \gamma_n = O(p_n \gamma_n / P_n).$$

If we take $p_n=\dfrac{1}{n+1}$ and $\gamma_n=1$, we get the following lemma due to PATI ([4]).

Lemma B. *If* $\{\lambda_n\}$ *is a convex sequence such that* $\Sigma\lambda_n/n<\infty$, *then*

$$\sum_1^m \log(n+1)\,\Delta\lambda_n=O(1),$$

as $m\to\infty$.

On the other hand, if we take $\gamma_n=\{\log(n+1)\}^k$, $k\geqq 0$ and $p_n=\dfrac{1}{n+1}$, we obtain the following result of PRASAD and BHATT ([6]).

Lemma C. *If* $\{(\log(n+1))^k\alpha_n\}$ *satisfies the same condition as* λ_n *in Lemma A, then*

$$\sum_1^m \{\log(n+1)\}^{k+1}\,\Delta\alpha_n = O(1), \qquad (m\to\infty).$$

PROOF OF LEMMA 2.

$$\sum_1^m \gamma_n\alpha_n p_n = \sum_1^{m-1} \Delta(\gamma_n\alpha_n)P_n+\gamma_m\alpha_m P_m =$$

$$=\sum_1^{m-1}(\gamma_n\Delta\alpha_n+\alpha_{n+1}\Delta\gamma_n)P_n+o(1)=\sum_1^{m-1}\gamma_n P_n\Delta\alpha_n+\sum_1^{m-1}\alpha_{n+1}\Delta\gamma_n P_n+o(1)=$$

$$=\sum_1^{m-1}\gamma_n P_n\Delta\alpha_n+O\left(\sum_1^{m-1}p_n\gamma_n\alpha_{n+1}\right)+o(1)=\sum_1^{m-1}\gamma_n P_n\Delta\alpha_n+O(1).$$

Therefore

$$\sum_1^m \gamma_n P_n\Delta\alpha_n = O(1) \qquad (m\to\infty).$$

Lemma 3. *If* $\{\gamma_n\alpha_n\}$ *satisfies the same condition as* λ_n *in Lemma* 1, *where* $\{\gamma_n\}$ *is a positive non-decreasing sequence such that* $\Delta^2\dfrac{1}{\gamma_n}\geqq 0$ *and*

$$\Delta\gamma_n=O(\gamma_n p_n/P_n),$$

and $\{p_n\}$ *is non-increasing sequence, then we have*

$$\sum_1^m nP_n\gamma_n\Delta^2\alpha_n = O(1),$$

and

$$mP_m\gamma_m\Delta\alpha_m=O(1) \qquad (m\to\infty).$$

The following lemmas are the special cases of this result.

Lemma D (PATI [5]). *If $\{\lambda_n\}$ satisfies the condition of Lemma A, then*

$$m \log (m+1) \Delta\lambda_m = O(1),$$

and

$$\sum_1^m n \log (n+1) \Delta^2 \lambda_n = O(1),$$

as $m \to \infty$.

Lemma E (BHATT [2]). *If $\{(\log (n+1))^k \alpha_n\}$, $k \geqq 0$, satisfies the same condition as λ_n in Lemma A, then*

$$m\{\log (m+1)\}^{k+1} \Delta\alpha_m = O(1),$$

and

$$\sum_1^m n\{\log (n+1)\}^{k+1} \Delta^2 \alpha_n = O(1),$$

as $m \to \infty$.

PROOF OF LEMMA 3. We have

$$\sum{}' \equiv \sum_1^m \gamma_n P_n \Delta\alpha_n = \sum_1^{m-1} (n+1) \Delta(\gamma_n P_n \Delta\alpha_n) + (m+1)\gamma_m P_m \Delta\alpha_m =$$

$$= \sum_1^{m-1} (n+1)\{\gamma_n P_n \Delta^2 \alpha_n + \Delta\alpha_{n+1} \Delta(\gamma_n P_n)\} + (m+1)\gamma_m P_m \Delta\alpha_m =$$

$$= \sum_1^{m-1} (n+1)\gamma_n P_n \Delta^2 \alpha_n + \sum_1^{m-1} (n+1)\Delta\alpha_{n+1} \Delta\gamma_n P_n +$$

$$+ \sum_1^{m-1} (n+1) \Delta\alpha_{n+1} \gamma_{n+1}(-p_{n+1}) + (m+1)\gamma_m P_m \Delta\alpha_m = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4.$$

Applying Lemma 2 we have $\Sigma' = O(1)$ and

$$\Sigma_2 = O\left(\sum_1^{m-1} np_n \gamma_n P_n(\Delta\alpha_{n+1})/P_n\right) = O\left(\sum_1^{m-1} P_n \gamma_n \Delta\alpha_{n+1}\right) = O(1),$$

since $\Delta^2\alpha_n \geqq 0$. Similarly

$$\Sigma_3 = O\left(\sum_1^{m-1} P_{n+1} \gamma_{n+1} \Delta\alpha_{n+1}\right) = O(1),$$

so that $\Sigma_1 + \Sigma_4 = O(1)$. Since Σ_1 and Σ_4 are positive the results follow.

5. PROOF OF THEOREM 1. Let $C_n = a_n P_n \lambda_n / \gamma_n$, $\alpha_n = \lambda_n / \gamma_n$,

$$T_n = \sum_0^n C_m \quad \text{and} \quad t_n^* = \frac{1}{P_n} \sum_0^n p_m T_m.$$

Then we have

$$t_n^* - t_{n+1}^* = \frac{1}{P_n} \sum_0^n p_m T_m - \frac{1}{P_{n+1}} \sum_0^{n+1} p_m T_m =$$

$$= \Delta(1/P_n) \sum_0^n p_m T_m - p_{n+1} T_{n+1}/P_{n+1} =$$

$$= -\Delta(1/P_n) \sum_0^{n-1} P_m C_{m+1} + \Delta(1/p_n) P_n T_n - \frac{p_{n+1} T_{n+1}}{P_{n+1}} =$$

$$= -\Delta(1/P_n) \sum_0^n P_m C_{m+1}.$$

Now

$$\sum_0^n P_m C_{m+1} = \sum_0^n P_m \lambda_{m+1} P_{m+1} a_{m+1}/\gamma_{m+1} =$$

$$= \sum_0^{n-1} \Delta\left(\frac{P_m \lambda_{m+1}}{\gamma_{m+1}}\right) \sum_{\mu=0}^m P_{\mu+1} a_{\mu+1} + \frac{P_n \lambda_{n+1}}{\gamma_{n+1}} \sum_{\mu=0}^n P_{\mu+1} a_{\mu+1} =$$

$$= \sum_0^{n-1} \Delta\left(\frac{P_m \lambda_{m+1}}{\gamma_{m+1}}\right) (\mu_{m+1} P_{m+1} - a_0 P_0) + \frac{P_n \lambda_{n+1}}{\gamma_{n+1}} (\mu_{n+1} P_{n+1} - a_0 P_0) =$$

$$= -a_0 P_0 (P_0 \lambda_1/\gamma_1 - P_n \lambda_{n+1}/\gamma_{n+1}) + \sum_0^{n-1} \Delta(P_m \lambda_{m+1}/\gamma_{m+1}) \mu_{m+1} P_{m+1} -$$

$$- a_0 P_0 P_n \lambda_{n+1}/\gamma_{n+1} + P_n \lambda_{n+1} \mu_{n+1} P_{n+1}/\gamma_{n+1} =$$

$$= \sum_0^{n-1} \Delta(P_m \lambda_{m+1}/\gamma_{m+1}) \mu_{m+1} P_{m+1} + \frac{P_n \lambda_{n+1} \mu_{n+1} P_{n+1}}{\gamma_{n+1}} + O(1) =$$

$$= \sum_0^n \Delta(P_m \lambda_{m+1}/\gamma_{m+1}) \mu_{m+1} P_{m+1} + P_{n+1}^2 \mu_{n+1} \lambda_{n+2}/\gamma_{n+2} + O(1) =$$

$$= \sum_0^{n-1} (m+1) \Delta\{P_{m+1} \Delta(P_m \lambda_{m+1}/\gamma_{m+1})\} \frac{1}{(m+1)} \cdot \sum_0^m \mu_{r+1} +$$

$$+ P_{n+1} \Delta(P_n \lambda_{n+1}/\gamma_{n+1})(n+1) \cdot \frac{1}{(n+1)} \cdot \sum_0^n \mu_{r+1} +$$

$$+ P_{n+1}^2 \mu_{n+1} \lambda_{n+2}/\gamma_{n+2} + O(1) =$$

$$= L_1 + L_2 + L_3 + O(1).$$

It is therefore sufficient to prove that

$$\sum \Delta(1/P_n)|L_r| = O(1) \qquad (r=1, 2, 3).$$

Now

$$\sum_1^m \Delta(1/P_n)|L_1| = O\left\{\sum_1^m \Delta(1/P_n) \sum_0^{n-1} (r+1)\gamma_{r+1}\, |\Delta\{P_{r+1}\, \Delta(P_r\, \lambda_{r+1}/\gamma_{r+1})\}|\right\} =$$

$$= O\left\{\sum_0^{m-1} \frac{(r+1)\gamma_{r+1}}{P_{r+1}}\, |\Delta\{P_{r+1}\, \Delta(P_r\, \alpha_{r+1})\}|\right\} =$$

$$= O\left\{\sum_0^{m-1} \frac{(r+1)\gamma_{r+1}}{P_{r+1}} \cdot P_r\, P_{r+1}\, \Delta^2\, \alpha_{r+1}\right\} +$$

$$+ O\left\{\sum_0^{m-1} \frac{(r+1)\gamma_{r+1}}{P_{r+1}}\, P_{r+1}\, p_{r+1}\, \Delta\alpha_{r+2}\right\} +$$

$$+ O\left\{\sum_0^{m-1} \frac{(r+1)\gamma_{r+1}}{P_{r+1}}\, \alpha_{r+3}\, |\Delta p_{r+1}\, P_{r+1}|\right\} =$$

$$= O\left\{\sum_0^{m-1} (r+1)\gamma_{r+1}\, P_{r+1}\, \Delta^2\, \alpha_{r+1}\right\} + O\left\{\sum_0^{m-1} \gamma_{r+1}\, P_{r+1}\, \Delta\alpha_{r+1}\right\} +$$

$$+ O\left\{\sum_0^{m-1} (r+1)\lambda_{r+1}\, |\Delta(p_{r+1}\, P_{r+1})|/P_{r+1}\right\} =$$

$$= O(1) + O\left\{\sum_0^{m-1} (r+1)\lambda_{r+1}\, P_{r+1}(\Delta p_{r+1})/P_{r+1}\right\} +$$

$$+ O\left\{\sum_0^{m-1} (r+1)\lambda_{r+1}\, p_{r+2}^2/P_{r+1}\right\} =$$

$$= O(1) + O\left(\sum_0^{m-1} \Delta p_{r+1} \sum_0^r \lambda_s\right) + O\left(\sum_0^{m-1} p_{r+1}\, \lambda_{r+1}\right) = O(1),$$

by virtue of lemmas 2 and 3, the hypotheses and the fact that

$$\sum_0^{m-1} \Delta p_r \sum_0^r \lambda_s = \sum_0^m \lambda_r\, p_r - p_m \sum_0^m \lambda_r =$$

(5. 1)

$$= O(1) + O\left(p_m \frac{1}{p_m} \sum_0^m \lambda_r\, p_r\right) = O(1).$$

Next

$$\sum_1^m \Delta(1/P_n)|L_2| = O\left(\sum_1^m \Delta(1/P_n)P_{n+1}(n+1)\gamma_{n+1}|\Delta P_n \alpha_{n+1}|\right) =$$

$$= O\left(\sum_1^m p_{n+1}(n+1)\gamma_{n+1}p_{n+1}\alpha_{n+2}/P_n\right) +$$

$$+ O\left(\sum_1^m p_{n+1}(n+1)\gamma_{n+1}P_n(\Delta\alpha_{n+1})/P_n\right) =$$

$$= O\left(\sum_1^m p_{n+1}\lambda_{n+1}\right) + O\left(\sum_1^m P_{n+1}\gamma_{n+1}\Delta\alpha_{n+1}\right) = O(1),$$

by the hypotheses and lemma 2.

Again

$$\sum \Delta(1/P_n)|L_3| \leqq \sum_1^m \Delta(1/P_n)\cdot P_{n+1}^2|\mu_{n+1}|\lambda_{n+2}/\gamma_{n+2} =$$

$$= \sum_1^m \frac{p_{n+1}}{\gamma_{n+2}}\lambda_{n+2}P_{n+1}|\mu_{n+1}|/P_n = \sum_1^{m-1}\Delta(p_{n+1}P_{n+1}\alpha_{n+2}/P_n)\sum_0^n|\mu_{r+1}| +$$

$$+(p_{m+1}P_{m+1}\alpha_{m+2}/P_m)\sum_0^m|\mu_{r+1}| =$$

$$= O\left(\sum_1^{m-1}(n+1)\gamma_{n+1}|\Delta(p_{n+1}P_{n+1}\alpha_{n+2}/P_n)|\right) +$$

$$+ O(p_{m+1}P_{m+1}(m+1)\alpha_{m+2}\gamma_{m+2}/P_m) = L_{31} + L_{32}.$$

Now

$$L_{32} = O(P_{m+1}\lambda_{m+1}) = o(1) \qquad (m\to\infty).$$

Since

$$\Delta(p_{n+1}P_{n+1}/P_n) = O(p_{n+1}^2/P_n) + O(\Delta p_{n+1}),$$

we have

$$L_{31} = O\left(\sum_1^{m-1}(n+1)\gamma_{n+1}p_{n+1}P_{n+1}(\Delta\alpha_{n+2})/P_n\right) +$$

$$+ O\left(\sum_1^{m-1}(n+1)\gamma_{n+1}\alpha_{n+3}|\Delta(p_{n+1}P_{n+1}/P_n)|\right) =$$

$$= O\left(\sum_1^{m-1}\gamma_{n+1}P_{n+1}\Delta\alpha_{n+1}\right) + O\left(\sum_1^{m-1}(n+1)\lambda_{n+1}p_{n+1}^2/P_n\right) +$$

$$+ O\left(\sum_1^{m-1}(n+1)\lambda_{n+1}\Delta p_{n+1}\right) =$$

$$= O(1) + O\left(\sum_1^{m-1}\lambda_{n+1}p_{n+1}\right) + O\left(\sum_1^{m-1}\Delta p_{n+1}\sum_0^n\lambda_r\right) = O(1),$$

by Lemma 2, the hypotheses and (5. 1).

Finally, we have

$$\sum_{1}^{m} \Delta(1/P_n) = O(1) \qquad (m \to \infty).$$

This completes the proof of Theorem 1.

6. We deduce the following theorem for $|C, 1|$ summability factors of infinite series.

Theorem 2. *If* $\{\lambda_n\}$ *is a convex sequence such that* $\Sigma\lambda_n < \infty$ *and the sequence* $\{t_n^1\}$, *the* $(C, 1)$ *mean of* $\{na_n\}$, *satisfies the condition*

$$|t_n^1| = O(\gamma_n) \ (C, 1),$$

γ_n *being a positive non-decreasing sequence such that* $\Delta^2 \dfrac{1}{\gamma_n} \geqq 0$ *and* $\Delta\gamma_n = O(\gamma_n/n)$, *then the series* $\Sigma na_n\lambda_n/\gamma_n$ *is summable* $|C, 1|$.

This generalises a result of PRASAD and BHATT [6] for the summability $|C|$ of order 1. A theorem of TRIPATHI [7] can also be deduced as a corollary from this theorem.

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