On the form of real solutions of the matrix functional equation $\Phi(x)\Phi(y) = \Phi(xy)$ for non-singular matrices Φ

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§ 1. In the present paper we shall give the general real measurable solution of the matrix functional equation

$$\Phi(x)\Phi(y)=\Phi(xy) \tag{1}$$

for arbitrary real values of $x \neq 0$. We shall assume that the values of Φ are real, non-singular $n \times n$ matrices (only non-singular solutions are of importance for applications), whereas arguments are real numbers different from zero. Measurable solutions of equation (1) have been essentially given by D. Ž. ĐOKOVIĆ ([1]) and S. KUREPA ([3]). However, S. KUREPA considered equation (1) in the set

$$\tilde{D} = \exp D = \left\{x : x = \exp \frac{l}{2^k},\ \ l,\ k \quad \text{integers}\right\} \tag{2}$$

of exponential diadic fractions, and D. Ž. ĐOKOVIĆ in the set R^+ of positive real numbers. But for various purposes (e. g. for applications in the theory of geometric objects) it is necessary to know the solution of (1) *for arbitrary real* $x \neq 0$. Admitting also negative values of x presents some new difficulties.

The solution obtained is given an explicit form, suitable for immediate applications [1]). Such a form is not found in the papers by ĐOKOVIĆ and KUREPA, though it may be obtained by elementary considerations from the formulae given by those authors (but only for $x > 0$!).

The main point in the argument is the remark that, since the multiplicative group R of reals is commutative, $\Phi(R)$ must also be commutative. But since in general the multiplication of matrices is not commutative, this allows one to get fairly precise informations about the form of the function Φ.

In the last section we give a decomposition of the general solution of (1) into a product of a regular solution and a totally non-measurable solution.

§ 2. Matrices Φ may be regarded as linear transforms of a real n-dimensional vector space $\mathfrak{R}^n$. Their form thus depends on the choice of the basis $\mathscr{B}$ in $\mathfrak{R}^n$. A change of the basis is analytically expressed by the formula

$$\tilde{\Phi} = C^{-1}\Phi C,$$

[1]) The classification of linear homogeneous geometric objects in $\mathfrak{X}_1$, based on the results of the present note, is the subject of the paper [5].

where C is a real non-singular $n\times n$ matrix. Since the transformation

(3) $$\tilde{\Phi}(x)=C^{-1}\Phi(x)C$$

leads from a solution $\Phi(x)$ of (1) to a solution $\tilde{\Phi}(x)$ of (1), we may seek the solution $\Phi(x)$ in a basis $\mathscr{B}$ in which it has a particularly simple form. The general solution $\tilde{\Phi}(x)$ is then obtained with the aid of formula (3).

Solutions of equation (1) are homomorphisms of the multiplicative group R of real numbers different from zero into the linear group $GL(n, \mathfrak{R})$ of order n over reals:

$$\Phi\colon R\to GL(n, \mathfrak{R}).$$

The group R is a direct product $U_2\times R^+$ of the group $U_2=\{-1, 1\}$ of the square roots of unity, and of the multiplicative group R^+ of positive real numbers. Consequently the homomorphism Φ admits a decomposition into a product of homomorphisms

(4) $$\Phi(x)=\Phi_1(\operatorname{sgn} x)\,\Phi_2(|x|)$$

with

$$\Phi_1\colon U_2\to GL(n, \mathfrak{R}),$$

$$\Phi_2\colon R^+\to GL(n, \mathfrak{R}),$$

and it follows from (1) that Φ_1 and Φ_2 commute:

(5) $$\Phi_1\Phi_2=\Phi_2\Phi_1.$$

We start with determining the homomorphism Φ_1.

§ 3. Setting $x=y=1$ in the relation $\Phi_1(x)\Phi_1(y)=\Phi_1(xy)$ we obtain $\Phi_1(1)^2=\Phi_1(1)$, whence, since $\Phi_1(1)$ is non-singular, $\Phi_1(1)=E_n$ (the unit matrix of order n). Similarly, for $x=y=-1$ we obtain

(6) $$\Phi_1(-1)^2=\Phi_1(1)=E_n.$$

Writing $F=\Phi_1(-1)$ we have

(7) $$\Phi_1(\operatorname{sgn} x)=\begin{cases}E_n & \text{for}\quad x>0,\\ F & \text{for}\quad x<0.\end{cases}$$

By (6) $F^2=E_n$, i. e. the minimal polynomial of F is $\lambda^2-1=(\lambda-1)(\lambda+1)$. Hence it follows that the characteristic roots of F are $+1$ (say, of multiplicity p) and -1 (of multiplicity $q=n-p$), and all Jordan boxes of F are simple (of order 1). Therefore there exists a basis $\mathscr{B}_0$ in $\mathfrak{R}^n$ in which F has the diagonal form

(8) $$F=\{\underbrace{1, \dots, 1}_{p},\quad \underbrace{-1, \dots, -1}_{q}\}.$$

In view of (7) we get

(9) $$\Phi_1(\operatorname{sgn} x)=\{\underbrace{1, \dots, 1}_{p},\quad \underbrace{\operatorname{sgn} x, \dots, \operatorname{sgn} x}_{q}\},\qquad p+q=n.$$

In the sequel we assume that in $\mathfrak{R}^n$ a basis $\mathscr{B}_0$ has been fixed so that formula (9) gives the form of the homomorphism Φ_1.

§ 4. Now we pass to the determination of $\Phi_2(x)$. Throughout this section x may take values in R^+. Hence it follows (cf. [2], and also [1]) that we may write (Φ_2 being non-singular)

$$\Phi_2(x) = \exp \Lambda(x) \tag{10}$$

and it follows from (1) that

$$\exp[\Lambda(x) + \Lambda(y)] = \exp \Lambda(xy).$$

This implies, Λ being real, that

$$\Lambda(x) + \Lambda(y) = \Lambda(xy). \tag{11}$$

Since x, y are positive real numbers, we may set $x = e^u$, $y = e^v$. Introducing a new function $\Xi(u) = \Lambda(e^u)$ we obtain from (11)

$$\Xi(u) + \Xi(v) = \Xi(u+v). \tag{12}$$

Hence

$$\Phi_2(x) = \exp \Xi(\ln x), \qquad x \in R^+, \tag{13}$$

where $\Xi(u)$ satisfies Cauchy's additive equation (12). Formula (13) gives the general solution of equation (1) in R^+.

From the commutativity condition (5) it follows that $\Phi_2(x)$ must commute with matrix (8) and consequently it must be of the form

$$\Phi_2(x) = \begin{bmatrix} \varphi_p(x) & 0 \\ 0 & \varphi_q(x) \end{bmatrix} = \{\varphi_p(x), \varphi_q(x)\},$$

where $\varphi_p(x)$ and $\varphi_q(x)$ are matrices of order p and q, respectively. In virtue of (10) also $\Lambda(x)$ must have a similar, generalized diagonal form $\Lambda(x) = \{\lambda_p(x), \lambda_q(x)\}$ and hence also $\Xi(u) = \{\xi_p(u), \xi_q(u)\}$. Consequently, according to (13),

$$\Phi_2(x) = \{\exp \xi_p(\ln x), \exp \xi_q(\ln x)\}. \tag{14}$$

The functions $\xi_p(u)$ and $\xi_q(u)$ (with values in $GL(p, \mathfrak{R})$ and $GL(q, \mathfrak{R})$, respectively) satisfy, similarly as $\Xi(u)$, the Cauchy equation (12).

The measurability condition implies now that

$$\xi_p(u) = uA_p, \qquad \xi_q(u) = uA_q, \tag{15}$$

where A_p and A_q are constant real matrices of order p and q, respectively.

In the p- and q-dimensional linear subspaces of $\mathfrak{R}^n$ determined by the first p and by the last q vectors of the basis $\mathscr{B}_0$, respectively, we choose new bases $\mathscr{B}_p$ and $\mathscr{B}_q$ in which matrices A_p and A_q have Jordan's canonical form. This does not change the form of the matrix F and consequently formula (9) is still valid in the new basis $\mathscr{B}_0^* = \mathscr{B}_p \times \mathscr{B}_q$.

Since A_p and A_q are real, their Jordan boxes will have the form

$$K=\begin{bmatrix} a & 1 & & & & \\ & a & 1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & a & 1 \\ & & & & & a \end{bmatrix}, \quad L=\begin{bmatrix} B & E_2 & & & & \\ & B & E_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & B & E_2 \\ & & & & & B \end{bmatrix}, \tag{16}$$

where

$$B=\begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \qquad E_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and a, b are real constants (cf. e. g. [4], p. 87).

Inserting (15) into (14) we obtain

$$\Phi_2(x)=\{x^{A_p}, x^{A_q}\}, \qquad p+q=n, \tag{17}$$

where A_p and A_q have Jordan boxes of form (16).

§ 5. Taking into account (4), (9) and (17) we obtain

$$\Phi(x)=\{|x|^{A_p}, (\operatorname{sgn} x)\,|x|^{A_q}\}, \; p+q=n. \tag{18}$$

As results from the general theory of functions of a matrix argument (cf. [2]), to boxes (16) of matrices A_p, A_q there correspond the following boxes of matrices $|x|^{A_p}$, $|x|^{A_q}$:

$$|x|^K=\begin{bmatrix} |x|^a & |x|^a \ln|x| & \frac{1}{2!}|x|^a \ln^2|x| \ldots \frac{1}{(m-1)!}|x|^a \ln^{m-1}|x| \\ & |x|^a & |x|^a \ln|x| \ldots \frac{1}{(m-2)!}|x|^a \ln^{m-2}|x| \\ & & \cdots\cdots\cdots\cdots \\ & & \cdots\cdots\cdots \\ & & \cdots\cdots \\ & & |x|^a \end{bmatrix}, \tag{19}$$

$$|x|^L=\begin{bmatrix} |x|^B & |x|^B(b\ln|x|) & \frac{1}{2!}|x|^B(b\ln|x|)^2 \ldots \frac{1}{(s-1)!}|x|^B(\ln|x|)^{s-1} \\ & |x|^B & |x|^B(b\ln|x|) \ldots \frac{1}{(s-2)!}|x|^B(b\ln|x|)^{s-2} \\ & & \cdots\cdots\cdots\cdots \\ & & \cdots\cdots\cdots \\ & & \cdots\cdots \\ & & |x|^B \end{bmatrix}, \tag{20}$$

where

$$|x|^B = \begin{bmatrix} |x|^a \cos(b \ln |x|) & |x|^a \sin(b \ln |x|) \\ -|x|^a \sin(b \ln |x|) & |x|^a \cos(b \ln |x|) \end{bmatrix}.$$

Here a and b are the constants occurring in (16).

Thus we arrive to the following result.

Theorem 1. *For every real non-singular measurable solution $\Phi(x)$ of equation* (1) *in R there exists a basis in which $\Phi(x)$ is a generalized diagonal matrix function with boxes of the four possible forms:*

$$(21) \qquad |x|^K, \quad |x|^L, \quad (\operatorname{sgn} x)|x|^K, \quad (\operatorname{sgn} x)|x|^L,$$

where $|x|^K$ and $|x|^L$ are given by formulae (19) *and* (20). *On the other hand, every generalized diagonal matrix function $\Phi(x)$* [2]) *with boxes of form* (21) *evidently satisfies equation* (1).

§ 6. Instead assuming measurability we may restrict the domain of definition of Φ to the group $U_2 \times \tilde{D}$, where $\tilde{D}$ is the multiplicative group of exponential diadic fractions (cf. (2)). The same argument as previously leads now to formula (18) as the general real solution of equation (1) in the set considered.

Now, if $\Phi(x)$ is any solution of equation (1) in R, then (18) must hold for x of the form $\pm \exp \frac{l}{2^k}$. The function

$$\Psi(x) = \Phi(x)\{|x|^{-A_p}, (\operatorname{sgn} x)|x|^{-A_q}\}$$

also is a solution of (1) and in view of (18) we have $\Psi(x) = E_n$ for $x = \pm \exp \frac{l}{2^k}$, and moreover

$$(22) \qquad \Psi(-1) = E_n.$$

Hence the decomposition

$$\Phi(x) = \Psi(x)\{|x|^{A_p}, (\operatorname{sgn} x)|x|^{A_q}\}$$

results, where $\Psi(x)$ is a solution of (1) in R which is identically E_n on $\pm \tilde{D}$. According to the considerations in § 4 we must have (cf. (22))

$$\Psi(x) = \exp \Xi(\ln |x|),$$

whence finally

$$(23) \qquad \Phi(x) = (\exp \Xi(\ln |x|))\{|x|^{A_p}, (\operatorname{sgn} x)|x|^{A_q}\}.$$

In (23) $\Xi(u)$ is a function satisfying equation (12) and vanishing on the set D of diadic fractions. Unless $\Xi(u) \equiv 0$, when (23) reduces to (18), the matrix function $\Xi(u)$ is totally non-measurable (all its entries are non-measurable functions). Thus we are led to the following decomposition theorem:

[2]) Or, more generally, every function (3), where $\Phi(x)$ is a generalized diagonal matrix function with boxes of form (21).

Theorem 2. *Every real non-singular solution of equation* (1) *in R may be given (in a suitable basis) form* (23), *where* $\{|x|^{A_p}, (\operatorname{sgn} x)|x|^{A_q}\}$ *is a measurable solution of* (1) *as described in Theorem* 1, *and* $\Xi(u)$ *is a totally non-measurable matrix-valued function satisfying equation* (12), *or* $\Xi(u) \equiv 0$.

This decomposition may turn out useful for the purpose of classification of geometric objects with not necessarily measurable transformation law.

References

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