On a generalization of homogeneous functions

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§ 1. Introduction The notion of homogeneous function introduced by L. EULER (1755, 1768, 1770) and nextly developed by C. HALPHEN (1911) [5] and A. FAVRE (1917) [3] was lately studied by V. ALACI (1923, 1943, 1950, 1952, 1955) [2].

The developement of the notion of homogeneous function, received up to now, can be presented as follows:

Definition 1. A solution $\varphi(x)$ of the functional equation

$$\varphi(\vartheta x)=\vartheta\varphi(x) \qquad x\in R_n,\ \vartheta\in R_1$$

is named a classical homogeneous function (of the first order).

Definition 2. A solution $\varphi(x)$ of the functional equation

$$\varphi(\vartheta x)=\vartheta\varphi(x) \qquad x\in R_n,\ \vartheta\in R_1^+$$

is named a classical positive-homogeneous function (of the first order).

Definition 3. A solution $\varphi(x)$ of the functional equations

$$\varphi(\vartheta x)=\vartheta^{\mu}\varphi(x) \qquad x\in R_n,\ \vartheta\in R_1^+$$

is named a classical positive-homogenous function of the order μ.

Definition 4. (given by V. ALACI in 1923)
A solution $\varphi(x)$ of the functional equation

$$(1) \qquad \varphi(x_1\vartheta^{\alpha_1}, \dots, x_n\vartheta^{\alpha_n}) = \vartheta^{\mu}\varphi(x_1, \dots, x_n) \qquad x\in R_n, \quad \vartheta\in R_1^+$$

in named a pseudo-homogeneous function.

Definition 5. (given by V. ALACI in 1923).
A solution $\varphi(x)$ of the functional equation

$$(2) \qquad \varphi\big(x_1 g_1(\vartheta), \dots, x_n g_n(\vartheta)\big)=k(\vartheta)\varphi(x_1, \dots, x_n) \qquad x\in R_n,\ \vartheta\in\theta$$

where $\theta\subset R_m$, $m\leqq n$, and functions $g_i(\vartheta)$, $k(\vartheta)$ are given, is named a generalized pseudo-homogeneous function.

Definition 6. (given by V. ALACI in 1952).
A solution $\varphi(x)$ of the functional equation

$$(3) \qquad \varphi\big(x_1+g_1(\vartheta), \dots, x_n+g_n(\vartheta)\big)=k(\vartheta)\varphi(x_1, \dots, x_n) \qquad x\in R_n,\ \vartheta\in\theta$$

where $\theta \subset R_m$, $m \leqq n$, and functions $g_i(\vartheta)$, $k(\vartheta)$ are given, is named a quasi-homogeneous function.

In the present paper we give a further generalization of the notion of homogeneous function.

Let be given three arbitrary sets X, P and θ and two functions

$$f: X \times \theta \to X, \qquad h: P \times \theta \to P.$$

Definition 7. A solution

$$\varphi: X \to P$$

of the functional equation

$$\varphi(f(x, \vartheta)) = h(\varphi(x), \vartheta) \qquad x \in X,\ \vartheta \in \theta \tag{4}$$

is named a general homogeneous function with respect to the given functions f and h.

The equation (4) may have no solutions if the functions f and h are arbitrary. In this paper we give a necessary and sufficient condition of the existence of solutions of the equation (4) under some initial assumptions.

The main purpose of this paper is to present some method of solving functional equations of the type (4). This method is based on the group theory so it does not require any regularity assumptions for given functions f, h and unknown function φ.

V. Alaci in solving functional equations of the type (2) or (3) has used differential equations. His method can not be used for functional equations of the general type (4).

We shall show in another work that in order to solve differential equations of some type it is more suitable to use functional equations of the type (4), than inversely.

It can be easily observed that the equation (4) contains the functional equation of concomitants of geometric objects (see [1], [6], [7]).

§ 2. Now we shall give some notions and facts of the theory of transformation groups (see [4], [8]).

Let in a set X be given a group $\mathscr{F}$ of transformations

$$y = f(x, \vartheta) \qquad x, y \in X,\ \vartheta \in \theta$$

with a parameter group θ.

A group operation in θ we denote by $\circ$.

We define a relation R in the set X by

$$xRy \Leftrightarrow \{\exists_{\vartheta \in \theta} : y = f(x, \vartheta)\}. \tag{5}$$

This relation is an equivalence.

Definition 8. The class of equivalence $[x]$ of a point $x \in X$ with respect to the relation R is named a domain of transitivity of the group $\mathscr{F}$, corresponding to the point x. Of course we have $x \in [x]$.

Definition 9. A set which is the sum of the domains of transitivity of the group $\mathscr{F}$ is named a bundle of domains of transitivity.

The bundle of the domains of transitivity of the group $\mathscr{F}$ is identic with the factor-set X/R of the set X with respect to the relation of equivalence R defined above.

Let $x, y \in X$ be settled. In the set θ we consider a subset

$$\Xi(x, y) \overset{\mathrm{Df}}{=} \{\vartheta : \vartheta \in \theta \wedge y = f(x, \vartheta)\}.$$

The set $\Xi(x, y)$ is not empty if and only if xRy.

If we put

$$\Xi(x, y) \circ \Xi(y, z) \overset{\mathrm{Df}}{=} \{\vartheta \circ \mu : \vartheta \in \Xi(x, y) \wedge \mu \in \Xi(y, z)\}$$

then for all $x, y, z \in X$ we have

$$\Xi(x, y) \circ \Xi(y, z) = \Xi(x, z).$$

For each point $x \in X$ the set

$$\Xi(x) \overset{\mathrm{Df}}{=} \Xi(x, x) \tag{6}$$

is a subgroup of the group θ.

Definition 10. The subgroup $\Xi(x)$ of the group θ, defined by (6), is named the stationary subgroup of θ corresponding to the point x.

Definition 11. For $x \in X$ and $\mu \in \theta$ the set

$$\Xi(x) \circ \mu \overset{\mathrm{Df}}{=} \{\vartheta \circ \mu : \vartheta \in \Xi(x)\}$$

being a subset of θ, is named a right coset of the group θ with respect to the subgroup $\Xi(x)$.

It is known that

$$\mu \in \Xi(x, y) \Rightarrow \Xi(x, y) = \Xi(x) \circ \mu.$$

Definition 12. A set $\underset{0}{X} \subset X$ which has exactly one point in common with each domain of transitivity of the group $\mathscr{F}$ is named a generator of the bundle X/R.

Definition 13. We name X/R a homogeneous bundle of domains of transitivity if it exists a generator $\underset{0}{X}$ of the bundle X/R for which the stationary subgroup $\Xi(x_0)$ of the group θ is still the same subgroup for all points $x_0 \in \underset{0}{X}$. Such a generator we name a homogeneous generator.

The definitions 12 and 13 are the propositions of the author.

We have the following

Theorem 1. *If for given transformation group $\mathscr{F}$ the bundle X/R is a homogeneous bundle and $\underset{0}{X}$ is one of it's homogeneous generators, then for each $\mu \in \theta$ the set $\underset{\mu}{X}$ defined by*

$$\underset{\mu}{X} = \{x : x \in X \wedge x = f(x_0, \mu), x_0 \in \underset{0}{X}\}$$

is also a homogeneous generator of the homogeneous bundle X/R.

PROOF. We have to prove that the stationary subgroup $\Xi(x)$ of the group θ does not depend on $x \in \underset{\mu}{X}$.

We state that

$$f(x_0, \omega) = f(x_0, \mu) \Leftrightarrow \omega \in \Xi(x_0) \circ \mu.$$

Let $x \in \underset{\mu}{X}$. Then we have $x = f(x_0, \mu)$, $x_0 \in \underset{0}{X}$, and

$$f(x, \nu) = x \Leftrightarrow f(f(x_0, \mu), \nu) = f(x_0, \mu) \Leftrightarrow$$
$$\Leftrightarrow f(x_0, \mu \circ \nu) = f(x_0, \mu) \Leftrightarrow \mu \circ \nu \in \Xi(x_0) \circ \mu \Leftrightarrow$$
$$\Leftrightarrow (\exists_{\vartheta \in \Xi(x_0)} : \nu = \mu^{-1} \circ \vartheta \circ \mu) \Leftrightarrow \Xi(x) = \mu^{-1} \circ \Xi(x_0) \circ \mu$$

what was to be proved.

Now we introduce some transformations which play an important role in the announced method of solving functional equations of the type (4).

Let be given a transformation group $\mathscr{F}$ such that the bundle X/R (R defined by (5), $f \in \mathscr{F}$) of the domains of transitivity is the homogeneous bundle and $\underset{0}{X}$ be one of it's homogeneous generators.

The right cosets $\Xi(x_0, x)$, $x_0 \in \underset{0}{X}$, $x \in X$ of the θ we denote shortly by $\mathfrak{x}, \mathfrak{y}, \dots$ and the set of them we denote by $\mathfrak{H}$. According to this notation the stationary subgroup $\Xi(x_0)$ we denote by $\mathfrak{x}_0$. Of course we have: $\mathfrak{x}_0, \mathfrak{x}, \dots \in \mathfrak{H}$.

From the definition of symbol $\Xi(x_0, x)$ it follows the following implication

$$x_0 \in \underset{0}{X} \wedge x \in [x_0] \Rightarrow (f(x_0, \mu) = x, \ \mu \in \Xi(x_0, x)).$$

In consequence of that we state that the function $f(x_0, \mu)$, $x_0 \in \underset{0}{X}$, $\mu \in \theta$ is constant with respect to the variable μ in each right coset $\mathfrak{x}$.

Therefore we can introduce a function $F(x_0, \mathfrak{x})$ defined in $\underset{0}{X} \times \mathfrak{H}$ by

$$F(x_0, \mathfrak{x}) = f(x_0, \vartheta) \qquad \vartheta \in \mathfrak{x}. \tag{7}$$

It may be observed that the transformation

$$x = F(x_0, \mathfrak{x}) \quad (x_0, \mathfrak{x}) \in \underset{0}{X} \times \mathfrak{H}, \ x \in X \tag{8}$$

gives a one-to-one correspondence between the points $(x_0, \mathfrak{x})$ and x. That means that this transformation is integrally inversible in the set $\underset{0}{X} \times \mathfrak{H}$.

The inverse transformation to the transformation (8) we write in the form

$$\begin{cases} x_0 = F_X(x) \\ \mathfrak{x} = F_{\mathfrak{H}}(x) \end{cases} \qquad x \in X, \quad (x_0, \mathfrak{x}) \in \underset{0}{X} \times \mathfrak{H}. \tag{9}$$

This yields the identities

$$\begin{aligned} &F(F_X(x), F_{\mathfrak{H}}(x)) = x && x \in X \\ &F_X(F(x_0, \mathfrak{x})) = x_0 && \\ &F_{\mathfrak{H}}(F(x_0, \mathfrak{x})) = \mathfrak{x} && (x_0, \mathfrak{x}) \in \underset{0}{X} \times \mathfrak{H}. \end{aligned} \tag{10}$$

From the implication

$$x_0 \in \underset{0}{X} \cap [x_*] (x_* \in X) \Rightarrow (F_X(x) = x_0, \ x \in [x_*])$$

it follows that the function $F_X(x)$ may be interpretated as an operation of the projection of the set X onto the generator $\underset{0}{X}$ by means of the bundle X/R.

The function $F_{\mathfrak{H}}(x)$ gives a one-to-one correspondence between the domains of transitivity of the transformation group $\mathscr{F}$ and the right cosets of the group θ with respect to the subgroup $\mathfrak{x}_0$.

Let us put now

$$\mathfrak{x}\circ\mu \overset{\mathrm{Df}}{=} \{\vartheta\circ\mu:\ \vartheta\in\mathfrak{x}\} \qquad \mu\in\theta.$$

Then we may write the equivalence

$$\mathfrak{y}=\mathfrak{x}\circ\mu\in\mathfrak{H} \Leftrightarrow \mathfrak{x}\in\mathfrak{H}.$$

Now we shall prove the

Theorem 2. *If $\underset{0}{X}$ is a homogeneous generator of the homogeneous bundle of the domains of transitivity of a transformation group $\mathscr{F}$ then each transformation $f\in\mathscr{F}$ may be written in the form*

$$f(x,\mu)=F\big(F_X(x),\ F_{\mathfrak{H}}(x)\circ\mu\big) \qquad x\in X,\ \ \mu\in\theta \tag{11}$$

where F, F_X and $F_{\mathfrak{H}}$ are defined by (7) *and* (9).

Proof. It follows from the definition of a transformation group that

$$f\big(f(x,\vartheta),\mu\big)=f(x,\vartheta\circ\mu) \qquad x\in X,\ \ \vartheta,\mu\in\theta. \tag{12}$$

If we make in (12) the substitution $x=x_0\in X$ then we obtain

$$f\big(f(x_0,\vartheta),\mu\big)=f(x_0,\vartheta_0\,\mu) \qquad x_0\in\underset{0}{X},\quad \vartheta,\mu\in\theta.$$

Hence in view of the definition in (7) it follows that

$$f\big(F(x_0,\mathfrak{x}),\mu\big)=F(x_0,\mathfrak{x}\circ\mu) \qquad x_0\in\underset{0}{X},\quad \mathfrak{x}\in\mathfrak{H},\quad \mu\in\theta. \tag{13}$$

If we use the formulas (8) and (9) then from (13) we obtain the formula (11).

§ 3. Now we are going to the problem of the solvability of equations of the type (4) and to a method of solving this equations.

We take the following assumptions about the functions $f(x,\vartheta)$ and $h(p,\vartheta)$ in the equation (4):

1. Let transformations

$$y=f(x,\vartheta) \qquad x,y\in X,\ \ \vartheta\in\theta$$

and

$$q=h(p,\vartheta) \qquad p,q\in P,\ \ \vartheta\in\theta$$

form transformation groups $\mathscr{F}$ and $\mathscr{H}$, respectively, with the same parameter group θ.

2. Let the factor-set X/R (R defined by (5)) be a homogeneous bundle of the domains of transitivity of the transformation group $\mathscr{F}$ and $\underset{0}{X}$ be one of it's homogeneous generators to which it corresponds a stationary subgroup $\mathfrak{x}_0$ of the group θ.

Let us denote by $\pi(p)$ the stationary subgroup of the group θ corresponding to a point $p \in P$. We introduce the following subset of the set P:

$$P_{\mathfrak{x}_0} \stackrel{\mathrm{Df}}{=} \{p : p \in P \wedge \mathfrak{x}_0 \subset \pi(p)\} \tag{14}$$

where $\mathfrak{x}_0$ is the subgroup of the group θ which appears in the assumption 2.

We shall prove by the assumptions 1. and 2. the following two theorems (3 and 4).

Theorem 3. *If a function φ is a solution of the functional equation* (4) *then the following condition holds*

$$\varphi(\underset{0}{X}) \subset P_{\mathfrak{x}_0} \tag{15}$$

where $\varphi(\underset{0}{X})$ denotes the image of the generator $\underset{0}{X}$ by the function φ.

Proof. Let us suppose that φ is a solution of the functionnal equation (4) by the assumptions 1. and 2. Then we have the identity

$$\varphi\big(f(x, \vartheta)\big) = h\big(\varphi(x), \vartheta\big) \qquad x \in X, \quad \vartheta \in \theta. \tag{16}$$

If in (16) we make the substitution $x = x_0 \in \underset{0}{X}$, then for each $\vartheta \in \theta$ we have (we rememeber that $f(x_0, \vartheta) = x_0$ for each $\vartheta \in \mathfrak{x}_0$)

$$\varphi(x_0) = h\big(\varphi(x_0), \vartheta\big) \qquad x_0 \in \underset{0}{X}, \quad \vartheta \in \mathfrak{x}_0$$

or

$$p_0 = h(p_0, \vartheta) \qquad \vartheta \in \mathfrak{x}_0 \tag{17}$$

where $p_0 = \varphi(x_0)$.

It follows from the definition of the symbol $\pi(p_0)$ that the identity (17) must be also true for all $\vartheta \in \pi(p_0)$. But we surely know that it is true for all $\vartheta \in \mathfrak{x}_0$ so in consequence of that we may conclude the following relation

$$\big(\vartheta \in \mathfrak{x}_0 \Rightarrow \vartheta \in \pi(p_0)\big) \Leftrightarrow \mathfrak{x}_0 \subset \pi(p_0). \tag{18}$$

The relation (18) is true for all points $x_0 \in \underset{0}{X}$ (we remember that $p_0 = \varphi(x_0)$). From that and the definition in (14) we get immediately the condition (15), which was to be proved.

Theorem 4. *If we assume that*

$$P_{\mathfrak{x}_0} \neq \emptyset \qquad (\emptyset \textit{ empty-set}) \tag{19}$$

then for an arbitrary function $\varphi(x_0)$, $x_0 \in \underset{0}{X}$, of the property that

$$\Phi(\underset{0}{X}) \subset P_{\mathfrak{x}_0} \tag{20}$$

there exists one and only one solution φ of the equation (4) *such that the condition*

$$\varphi(x_0) = \Phi(x_0) \qquad x_0 \in \underset{0}{X} \tag{21}$$

is satisfied.

PROOF. Let us observe that the following implication is true

$$p_0 \in P_{\mathfrak{x}_0} \Rightarrow \big(h(p_0, \vartheta) = p_0,\ \vartheta \in \mathfrak{x}_0\big)$$

(see (4) and (14)). In consequence of that we get the implication

$$p_0 \in P_{\mathfrak{x}_0} \Rightarrow \big(h(p_0, \vartheta) = \mathrm{const} \in P,\ \vartheta \in \mathfrak{x}\big).$$

In fact, if we take one of the points of the set $\mathfrak{x}$ ($\mathfrak{x} \in \mathfrak{H}$), for example $\overset{*}{\mu}$, then each another point $\mu \in \mathfrak{x}$ can be represented in the form $\mu = \vartheta \circ \overset{*}{\mu}$ where $\vartheta \in \mathfrak{x}_0$. Thus we get (we remember that $h(p_0, \vartheta) = p_0$ for all $\vartheta \in \mathfrak{x}_0$)

$$h(p_0, \mu) = h(p_0, \vartheta \circ \overset{*}{\mu}) = h\big(h(p_0, \vartheta), \overset{*}{\mu}\big) = h(p_0, \overset{*}{\mu})$$

what was to be shown.

In view of this situation we may introduce a function $H(p_0, \mathfrak{x})$ defined in $P_{\mathfrak{x}_0} \times \mathfrak{H}$ by the relation

$$H(p_0, \mathfrak{x}) = h(p_0, \mu) \qquad \mu \in \mathfrak{x}. \tag{22}$$

Let us make now in the equation (4) the substitution $x = x_0 \in \underset{0}{X}$. Then we get the relation

$$\varphi\big(f(x_0, \vartheta)\big) = h\big(\varphi(x_0), \vartheta\big) \qquad x_0 \in \underset{0}{X},\ \ \vartheta \in \theta$$

from which in view of the condition (21) we obtain

$$\varphi\big(f(x_0, \vartheta)\big) = h\big(\Phi(x_0), \vartheta\big) \qquad x_0 \in X,\ \ \vartheta \in \theta. \tag{23}$$

But we have assumed that $\Phi(x_0) \in P_{\mathfrak{x}_0}$ (see (20)). Therefore using the definition of the function $H(p_0, \mathfrak{x})$ (given by the relation (22)) and the definition of the function $F(x_0, \mathfrak{x})$ (given by the relation (7)) we can write the relation (23) in the form

$$\varphi\big(F(x_0, \mathfrak{x})\big) = H\big(\Phi(x_0), \mathfrak{x}\big) \qquad x_0 \in \underset{0}{X},\quad \mathfrak{x} \in \mathfrak{H}. \tag{24}$$

If now we use the formulas (8) and (9) then in consequence of (24) we obtain

$$\varphi(x) = H\big(\Phi(F_X(x)),\ F_{\mathfrak{H}}(x)\big). \tag{25}$$

The function $\varphi(x)$, given by the formula (25), is a solution of the equation (4). In fact, if we use the representation (11) for the function $f(x, \vartheta)$ and the identities (10), then after putting the right-hand side expression of the formula (25) instead of the function $\varphi(x)$ in the equation (4) we get

1. On the left side:

$$\begin{aligned}\varphi\big(f(x, \vartheta)\big) &= \varphi\big(F\big(F_X(x),\ F_{\mathfrak{H}}(x) \circ \vartheta\big)\big) = \\ &= H\big(\Phi\big(F_X(F(F_X(x),\ F_{\mathfrak{H}}(x) \circ \vartheta))\big),\ F_{\mathfrak{H}}\big(F(F_X(x),\ F_{\mathfrak{H}}(x) \circ \vartheta)\big)\big) = \\ &= H\big(\Phi\big(F_X(x)\big),\ F_{\mathfrak{H}}(x) \circ \vartheta\big).\end{aligned}$$

2. On the right side (let μ_F denotes an element of $\mathfrak{x} = F_{\mathfrak{H}}(x)$)

$$\begin{aligned}h\big(\varphi(x), \vartheta\big) &= h\big(H\big(\Phi(F_X(x)),\ F_{\mathfrak{H}}(x)\big), \vartheta\big) = \\ &= h\big(h\big(\Phi(F_X(x)),\ \mu_F\big), \vartheta\big) = \\ &= h\big(\Phi\big(F_X(x)\big),\ \mu_F \circ \vartheta\big) = \\ &= H\big(\Phi\big(F_x(x)\big),\ F_{\mathfrak{H}}(x) \circ \vartheta\big).\end{aligned}$$

The expressions obtained in 1. and 2. are identical what proves that the function $\varphi(x)$ determined by the formula (25) satisfies the equation (4).

We easy observe that the formula (25), where instead of the function Φ may be taken an arbitrary function satisfying the condition (20), gives us the general solution of the functional equation (4) by the assumptions 1. and 2.

Corollary. *A necessary and sufficient condition for the existence of solutions of the functional equation* (4) *under the assumptions* 1. *and* 2. *is that to be*

$$P_{\mathfrak{x}_0} \neq \emptyset.$$

Necessity is a direct consequence of Theorem 3 and the sufficiency was proved in the Theorem 4.

Remark. In the method of solving functional equations of the type (4), presented in this paper, it is necessary to possess the inverse transformation (9) to a transformation (8). This inverse transformation may be obtained by the following way:

We take an arbitrary generator $\underset{0}{\theta}$ of the set $\mathfrak{H}$ (of the right cosets $\mathfrak{x}$ of the group θ with respect to the subgroup $\mathfrak{x}_0$) — which may be treated as a bundle of right cosets — and in the transformation (8) we put instead of $\mathfrak{x}$ it's representative element ϑ_0 such that it belongs to $\underset{0}{\theta}$. We get the transformation

$$x = f(x_0, \vartheta_0) \qquad x_0 \in \underset{0}{X}, \quad \vartheta_0 \in \theta, \quad x \in X \tag{26}$$

where

$$f(x_0, \vartheta_0) = F(x_0, \mathfrak{x}) \qquad \vartheta_0 \in \mathfrak{x}, \quad \underset{x_0}{x} \in \underset{0}{X}$$

(it may be observed that the function $f(x_0, \vartheta_0)$ is identic with the restriction of the function $f(x, \vartheta)$ to the set $\underset{0}{X} \times \underset{0}{\theta}$).

If the inverse transformation to the transformation (26), which is integrally inversible, we note in the form

$$\begin{cases} x_0 = f_X(x) \\ \vartheta_0 = f_\theta(x) \end{cases} \qquad x \in X, \quad x_0 \in \underset{0}{X}, \quad \vartheta_0 \in \underset{0}{\theta}$$

then the transformation (9) inverse to the transformation (8) may be written in the form

$$\begin{cases} x_0 = F_X(x) \equiv f_X(x) \\ \mathfrak{x} = F_{\mathfrak{H}}(x) \ni \vartheta_0 = f_\theta(x). \end{cases}$$

Of course, in the case when $\mathfrak{x}_0$ is equal to the identity of the group θ the generator $\underset{0}{\theta}$ is identic with the set θ and the transformations (8) and (26) are the same.

Now we shall give an example for illustration of the described method of solving functional equations of the type (4). This example concerns the case when $\mathfrak{x}_0$ is different from the identity element of the group θ.

The functional equation, or saying more exactly, the system of functional equations which will be solved there, is connected with the problem of determining the algebraic concomitants of the mixed tensor of the valency (1, 1) in the 2-dimensional space. Such a problem, treated in n-dimensional space, was solved with using an another method by A. Zajtz [10].

Example. Let X be the set of the real matrices $\|x^i_j\|$ of the 2^{nd}-degree with two different real characteristic roots, which will be noted by t_1, t_2. The transformations of the transformation group $\mathscr{F}$ are given by

$$(28)\qquad x^{i'}_{j'} = A^{i'}_i\, x^i_j\, A^j_{j'} \qquad x, x' \in X, \quad \det \ \|A^j_{j'}\| \neq 0 \quad (i, j = 1, 2)$$

where $\|A^{i'}_i\|$ denotes the inverse matrix to a matrix $\|A^j_{j'}\|$.

The parameter group θ of the transformation group $\mathscr{F}$ is the group of non-singular matrices $\|A^j_{j'}\|$ which is usually denoted by $\mathfrak{L}^1_2$.

Furthermore let be given a group $\mathscr{H}$ of transformations

$$(29)\qquad p'_l = h_l(p_k, A^j_{j'}) \quad p, p' \in P, \qquad \|A^j_{j'}\| \in \theta \quad (l = 1, \dots, m)$$

with the parameter group θ equal to $\mathfrak{L}^1_2$.

We consider the following system of functional equations

$$(30)\qquad \varphi_l(A^{i'}_i\, x^i_j\, A^j_{j'}) = h_l(\varphi_k(x^i_j), A^s_{s'}) \qquad \|x^i_j\| \in X, \quad \|A^j_{j'}\| \in \theta \quad (l = 1, \dots, m)$$

where φ_l are the unknown functions.

In order to find the general solution of the system (30) by meance of using the described method, we must go to built a transformation of the type (26) and to determine it's inverse transformation.

The domains of transitivity of the transformation group $\mathscr{F}$ of transformations (28) are the classes of similar matrices. We know that two similar matrices have the same characteristic roots. Hence it follows from that that the set

$$(31)\qquad \underset{0}{X} = \left\{ \begin{Vmatrix} t_1, & 0 \\ 0, & t_2 \end{Vmatrix} : t_1 \neq t_2 \right\}$$

is one of the generators of the bundle X/R.

For all points $x_0 \in \underset{0}{X}$ the stationary subgroup $\mathfrak{x}_0$ of the group θ is steel the same subgroup and is given by

$$(32)\qquad \mathfrak{x}_0 = \left\{ \begin{Vmatrix} \lambda_1, & 0 \\ 0, & \lambda_2 \end{Vmatrix} : \lambda_1 \cdot \lambda_2 \neq 0 \right\}$$

from where it follows that the bundle X/R (R defined by (5), f given by (28)) and it's generator $\underset{0}{X}$ are homogeneous.

The right cosets $\mathfrak{x}$ of the group θ with respect to the subgroup $\mathfrak{x}_0$ are of the form

$$(33)\qquad \mathfrak{x} = \left\{ \begin{Vmatrix} \lambda_1, & 0 \\ 0, & \lambda_2 \end{Vmatrix} \cdot \begin{Vmatrix} \alpha_{11}, & \alpha_{12} \\ \alpha_{21}, & \alpha_{22} \end{Vmatrix} : \lambda_1 \cdot \lambda_2 \neq 0 \right\} \qquad \|\alpha_{ij}\| \in \mathfrak{x}.$$

One of the generators, $\underset{0}{\theta}$, of the family $\mathfrak{H}$ of right cosets (33) may be, for example, represented by

$$(34)\qquad \underset{0}{\theta} = \left\{ \begin{Vmatrix} \cos\mu_1, & \sin\mu_1 \\ \cos\mu_2, & \sin\mu_2 \end{Vmatrix} : (\mu_1, \mu_2) \in M \right\}$$

where

$$(35)\qquad M = \{(\mu_1, \mu_2) : 0 \leqq \mu_1 < \pi \wedge 0 \leqq \mu_2 < \pi \wedge \mu_1 \neq \mu_2\}.$$

To see this it suffices to verify the following facts:

1. For every non-singular matrix $\|\alpha_{ij}\| \in \theta$ there exists the matrix $\begin{Vmatrix} \lambda_1, 0 \\ 0, \ \lambda_2 \end{Vmatrix} \in \underset{0}{\theta}$ and the point $(\mu_1, \mu_2) \in M$ such that the following relation holds

$$\begin{Vmatrix} \cos\mu_1, \ \sin\mu_1 \\ \cos\mu_2, \ \sin\mu_2 \end{Vmatrix} = \begin{Vmatrix} \lambda_1, 0 \\ 0, \ \lambda_2 \end{Vmatrix} \cdot \begin{Vmatrix} \alpha_{11}, \alpha_{12} \\ \alpha_{21}, \alpha_{22} \end{Vmatrix}.$$

2. If is $(\mu_1, \mu_2) \neq (\mu_1', \mu_2')$ then the elements of $\underset{0}{\theta}$ corresponding to them belong to two different right cosets of θ.

Therefore from the transformation regula (28) it follows that the transformation (26) may be written in the form

$$\begin{Vmatrix} x_1^1, \ x_2^1 \\ x_1^2, \ x_2^2 \end{Vmatrix} = \begin{Vmatrix} \cos\mu_1, \ \sin\mu_1 \\ \cos\mu_2, \ \sin\mu_2 \end{Vmatrix}^{-1} \cdot \begin{Vmatrix} t_1, \ 0 \\ 0, \ t_2 \end{Vmatrix} \cdot \begin{Vmatrix} \cos\mu_1, \ \sin\mu_1 \\ \cos\mu_2, \ \sin\mu_2 \end{Vmatrix} \tag{36}$$

which the transformation gives a one-to-one correspondence between the elements $x \in X$ and $(x_0, (\mu_1, \mu_2)) \in \underset{0}{X} \times \underset{0}{\theta}$.

Thus way a transformation of the type (26) is obtained. Now we are going to get the inverse transformation to it.

Using the left-side multiplication in (36) by matrix $\begin{Vmatrix} \cos\mu_1, \ \sin\mu_1 \\ \cos\mu_2, \ \sin\mu_2 \end{Vmatrix}$ we receive

$$\begin{cases} x_1^1 \sin\mu_1 = (t_1 - x_1^1)\cos\mu_1 \\ x_2^1 \cos\mu_1 = (t_1 - x_2^2)\sin\mu_1 \\ x_1^2 \sin\mu_2 = (t_2 - x_1^1)\cos\mu_2 \\ x_2^2 \cos\mu_2 = (t_2 - x_2^2)\sin\mu_2 . \end{cases} \tag{37}$$

Because of the fact that t_1, t_2 are the characteristic roots of the matrix $\|x_j^i\|$ we may write the formulas

$$\begin{cases} t_1 = \dfrac{1}{2}\left[(x_1^1 + x_2^2) - \sqrt{(x_1^1 + x_2^2)^2 - 4(x_1^1 x_2^2 - x_2^1 x_1^2)}\,\right] \\ t_2 = \dfrac{1}{2}\left[(x_1^1 + x_2^2) + \sqrt{(x_1^1 + x_2^2)^2 - 4(x_1^1 x_2^2 - x_2^1 x_1^2)}\,\right]. \end{cases} \tag{38}$$

For calculation μ_1, μ_2 it is necessary to consider the following four cases:

$$\begin{cases} \text{I} \quad x_2^1 = 0 \wedge x_1^2 = 0 \Rightarrow \mu_1 = 0, & \mu_2 = \dfrac{\pi}{2} \\ \text{II} \quad x_2^1 = 0 \wedge x_1^2 \neq 0 \Rightarrow \mu_1 = 0, & \mu_2 = \operatorname{arc\,ctg} \dfrac{x_1^2}{x_2^2 - x_1^1} \\ \text{III} \quad x_2^1 \neq 0 \wedge x_1^2 = 0 \Rightarrow \mu_1 = \operatorname{arc\,ctg} \dfrac{x_1^1 - x_2^2}{x_2^1}, & \mu_2 = \dfrac{\pi}{2} \\ \text{IV} \quad x_2^1 \neq 0 \wedge x_1^2 \neq 0 \Rightarrow \mu_1 = \operatorname{arc\,ctg} \dfrac{x_1^2}{t_1 - x_1^1}, & \mu_2 = \operatorname{arc\,ctg} \dfrac{x_1^2}{t_2 - x_1^1} \end{cases} \tag{39}$$

where t_1, t_2 are given by (38).

Now we proceed to constract the general solution of the system (30) of functional equations.

From Corollary, given in the papar, it follows that for the existence of the system (30) it is necessary to assumpt that the set $P_{\mathfrak{x}_0}$ (given by the definition (14)) corresponding to the stationary subgroup $\mathfrak{x}_0$ (given by (29)) — is not the empty-set.

So suppose that:

$$P_{\mathfrak{x}_0} \neq \emptyset.$$

We make now the following substitutions for the equations of the system (30):

$$\|x_j^i\| = \begin{Vmatrix} t_1, & 0 \\ 0, & t_2 \end{Vmatrix} \in \underset{0}{X}$$

$$\|A_{j'}^j\| = \begin{Vmatrix} \cos\mu_1, & \sin\mu_1 \\ \cos\mu_2, & \sin\mu_2 \end{Vmatrix} \in \underset{0}{\theta}.$$

In consequence of that we obtain the system of functional equations

$$\varphi_l\left(\begin{Vmatrix} \cos\mu_1, & \sin\mu_1 \\ \cos\mu_2, & \sin\mu_2 \end{Vmatrix}^{-1} \cdot \begin{Vmatrix} t_1, 0 \\ 0, t_2 \end{Vmatrix} \cdot \begin{Vmatrix} \cos\mu_1, & \sin\mu_1 \\ \cos\mu_2, & \sin\mu_2 \end{Vmatrix}\right) = h_l\left(\varphi_k\left(\begin{Vmatrix} t_1, 0 \\ 0, t_2 \end{Vmatrix}\right), \begin{Vmatrix} \cos\mu_1, & \sin\mu_1 \\ \cos\mu_2, & \sin\mu_2 \end{Vmatrix}\right)$$

(40) $$(l = 1, \dots, m)$$

which is equivalent to the system (30). This fact follows from the theory of the equation (4).

If we put

$$\Phi_l(t_1, t_2) = \varphi_l\left(\begin{Vmatrix} t_1, & 0 \\ 0, & t_2 \end{Vmatrix}\right) \qquad (l = 1, \dots, m)$$

and use the formulas (36), (38) and (39) then from (40) we get, respectively to the cases I—IV, the formulas

$$(41) \qquad \varphi_l(\|x_j^i\|) = \begin{cases} h_l\left(\Phi_k(x_1^1, x_2^2), \begin{Vmatrix} 1, 0 \\ 0, 1 \end{Vmatrix}\right) \\ h_l\left(\Phi_k(x_1^1, x_2^2), \begin{Vmatrix} 1, & 0 \\ \cos \operatorname{arc\,ctg} \dfrac{x_1^2}{x_2^2 - x_1^1}, & \sin \operatorname{arc\,ctg} \dfrac{x_1^2}{x_2^2 - x_1^1} \end{Vmatrix}\right) \\ h_l\left(\Phi_k(t_1, t_2), \begin{Vmatrix} \cos \operatorname{arc\,ctg} \dfrac{x_1^1 - x_2^2}{x_2^1}, & \sin \operatorname{arc\,ctg} \dfrac{x_1^1 - x_2^2}{x_2^1} \\ 0, & 1 \end{Vmatrix}\right) \\ h_l\left(\Phi_k(t_1, t_2), \begin{Vmatrix} \cos \operatorname{arc\,ctg} \dfrac{x_1^2}{t_1 - x_1^1}, & \sin \operatorname{arc\,ctg} \dfrac{x_1^2}{t_1 - x_1^1} \\ \cos \operatorname{arc\,ctg} \dfrac{x_1^2}{t_2 - x_1^1}, & \sin \operatorname{arc\,ctg} \dfrac{x_1^2}{t_2 - x_1^1} \end{Vmatrix}\right) \end{cases}$$

where t_1, t_2 are given by (38).

From the theory of the equation (4) it follows that for arbitrary functions Φ_l such that is valid (20) the functions φ_l given by the formulas (41) are the solutions of the system (30). That means that the formulas (41) gives us the general solution of the system (30).

In the paper [9] I give an another example which concerns the equation (1). Further examples will be published.

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