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Continuous multifunction from $[-1,0]$ to $\mathbb{R}$ having no continuous selection

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Abstract. The paper presents an example of a continuous and Hausdorff continuous multifunction $F:[-1,0]\to\mathbb{R}$ with closed values which has no continuous selection.

1. Introduction

A classical condition in selection theory is the convexness of values ([5, 6]). Some papers dealing with the problem of existence of continuous and quasicontinuous selections considered multifunctions with compact values (see e.g. [2, 4]). But it is well known, that even finite-valued continuous multifunctions need not have a continuous selection. In 1976 Carbone gave an example of a continuous multifunction F from a circle C onto the boundary of Möbius band such that for each x in C the set $F(x)$ has exactly two points and F has no continuous selection ([1]).

But what can be said about continuous multifunctions $F:X\to Y$ when X and Y are extremely "nice"? Let us consider a Hausdorff continuous multifunction $F:\mathbb{R}\to\mathbb{R}$ with closed values. It is easy to see, that if there is a point t in $\mathbb{R}$ such, that the value $F(t)$ has an upper bound in $\mathbb{R}$ then all values of F have this property. So, the function defined by $f(x)=\max F(x)$ for each x in $\mathbb{R}$ would be a continuous selection of F.

Maybe a little trick could help us to prove that for every Hausdorff continuous multifunction F from $\mathbb{R}$ to $\mathbb{R}$ there is a Hausdorff-continuous

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multifunction G with a bounded value and such, that $G(x) \subseteq F(x)$ for each x in $\mathbb{R}$?

No such little trick exists. To see this, it suffices to present an example of a Hausdorff continuous multifunction $F : \mathbb{R} \to \mathbb{R}$ with closed values which has no continuous selection.

2. Result

For definitions of basic notions: multifunction, selection, l.s.c., u.s.c. and Hausdorff continuous multifunction, Hausdorff metric etc. see e.g. [3] and [7]. A multifunction F is called continuous, if it is l.s.c. and u.s.c. (lower and upper semicontinuous).

The following example presents construction of a continuous and Hausdorff continuous multifunction $F : [-1, 0] \to \mathbb{R}$ with closed values which has no continuous selection.

Example. Let $S : [-1, 0] \to \mathbb{R}$ be defined as follows:

$$S(0) = \mathbb{R}$$

$$S(x) = \left\{ \frac{n(n+1)}{2} x + \frac{k}{2^n};\ k \in \mathbb{Z} \right\} \cup \left\{ n(n+1)\frac{2^n+1}{2^{n+1}} x + \frac{n+1}{2^{n+1}} + \frac{k}{2^n};\ k \in \mathbb{Z} \right\}$$

for every positive integer n and every $x \in \left\langle -\frac{1}{n}, -\frac{1}{n+1} \right\rangle$

In other words: the intersection of the graph of S with the set $\left\langle -\frac{1}{n}, -\frac{1}{n+1} \right\rangle \times \mathbb{R}$ is the system of segments joining the following couples of points: the point $[-\frac{1}{n}, \frac{m}{2^n}]$ with the point $[-\frac{1}{n+1}, \frac{m}{2^n} + \frac{1}{2}]$ and $[-\frac{1}{n}, \frac{m}{2^n}]$ with the point $[-\frac{1}{n+1}, \frac{m}{2^n} + \frac{1}{2} + \frac{1}{2^{n+1}}]$ where m is an arbitrary integer.

Of course, S is Hausdorff continuous on $[-1, 0)$; so, it is l.s.c. on this set. Now, it suffices to show, that S is Hausdorff continuous in 0. But it is easy to see that for every $t \in \left\langle -\frac{1}{n}, 0 \right)$ the following holds: if $s \in S(t)$ then $s + \frac{k}{2^n} \in S(t)$ for every integer k, so $H(S(t), \mathbb{R}) \leq \frac{1}{2^n}$, where H denotes the Hausdorff metric defined on $2^{\mathbb{R}}$. S has no continuous selection on $\mathbb{R}$ while every continuous selection g of S defined on the set $[-1, 0)$ has the property $\lim\limits_{t \to 0^-} g(t) = +\infty$.

The multifunction S is not u.s.c. To see this, define a set $U = \bigcup\limits_{k \in \mathbb{Z}} (\frac{k}{2} - \frac{1}{2^{|k|}}; \frac{k}{2} + \frac{1}{2^{|k|}})$. Then U is an open neighborhood of the set $S(-1)$

and for every neigborhood V of the point -1 there exists $t \in V$ such that $S(t)$ is not a subset of U. A problem of this kind will not appear when we make the set $\mathbb{R} - S(x)$ "sufficiently small", i.e., a subset of a compact interval.

Let $G : [-1, 0) \to \mathbb{R}$ be defined as follows:

$$G(x) = \left(-\infty, \frac{1}{x}\right\rangle \cup \left\langle -\frac{1}{x}, +\infty\right) \quad \text{for } x \in (-\infty, 0).$$

Now, let us define $F : [-1, 0] \to \mathbb{R}$ as follows:

$$F(x) = S(x) \cup G(x) \quad \text{for } x \in [-1, 0)$$
$$F(0) = S(0) = \mathbb{R}.$$

It is easy to verify that F is u.s.c. and Hausdorff continuous at the point 0.

Since both S and G are Hausdorff continuous on the set $[-1, 0)$, $F = S \cup G$ is Hausdorff continuous, too. F is u.s.c. on $[-1, 0)$. For example let $x \in \left\langle -\frac{1}{n}, -\frac{1}{n+1}\right\rangle$ and let W be an open neighborhood of the set $F(x)$.

Let us denote $A = F(x) - \big((-\infty, \frac{1}{x}) \cup (-\frac{1}{x}, +\infty)\big)$.

Let $A(\alpha) = \bigcup\limits_{a \in A} (a - \alpha, a + \alpha) \quad \text{for } \alpha > 0$.

Then there exists an $\varepsilon > 0$ such that the set $Z = (-\infty, \frac{1}{x} + \varepsilon) \cup (-\frac{1}{x} - \varepsilon, +\infty) \cup A(\varepsilon)$ is a subset of W. Let I be the set of such indices $k \in \mathbb{N}$, that there exists $t \in \left\langle -\frac{1}{n}, -\frac{1}{n+1}\right\rangle$ for which the set

$$\left\{\frac{n(n+1)}{2}t + \frac{k}{2^n}, n(n+1)\frac{2^n+1}{2^{n+1}}t + \frac{k}{2^{n+1}}\right\} \cap \langle -n-1, n+1\rangle$$

is nonempty.

Each of the functions $\frac{1}{x}$, $-\frac{1}{x}$, $\frac{n(n+1)}{2}x + \frac{k}{2^n}$ and $n(n+1)\frac{2^n+1}{2^{n+1}}x + \frac{n+1}{2^{n+1}} + \frac{k}{2^n}$ $(k \in I)$ is uniformly continuous on the interval $\langle -\frac{1}{n}, -\frac{1}{n+1}\rangle$. The set I is finite. So, considering the form of the set $F(x)$, it is easy to see that there exists an $\delta > 0$ (i.e. $\delta = \frac{\varepsilon}{2(n+1)^2}$) such that for every $t \in \mathbb{R}$ satisfying $| t - x | < \delta, F(t) \subset Z \subset W$ holds.

So, F is Hausdorff continuous, l.s.c. and u.s.c. on the interval $[-1, 0]$. Of course, F has no continuous selection on $[-1, 0]$.

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