On ring classes defined by modules

By. P. VÁMOS (Sheffield)

1. *Introduction*

Let $\mathfrak{a}$ be a property of modules and define $\mathfrak{R}_\mathfrak{a}$ to be the class of rings for which every module has the property $\mathfrak{a}$. It is known that many types of rings can be defined in such a way i.e. with the aid of modules. The class of semi-simple, hereditary, Dekind, Prüfer, e.t.c. rings is a list of the most important examples. Let $\mathfrak{R}_n$ be the class of rings R with l.g.d. $R \leqq n$. This class is of great importance. The generalization to modules of certain theorems holding for Abelian groups is actually nothing else but the characterization of those rings with the property that the theorem holds for every R-module. Sometimes it occurs that if every R- and S-module has the same property, so does every $R \oplus S$-module and every module over the residue class rings of R or over the full matrix ring over R.

Our main object is to show that under certain conditions concerning property $\mathfrak{a}$, class $\mathfrak{R}_\mathfrak{a}$ is closed under the above ring operations. The conditions ragarding property $\mathfrak{a}$ depend only on the logical formalization, and not on the algebraic character of the module property $\mathfrak{a}$, although there is close connection between them. In order to show this it is necesary to create a "language". This is done in § 2.

We shall mean by a ring in this paper a ring with unit element, and by an R-module we shall mean a unitary left R-module.

2. *The language*

For the sake of simplicity we will not give syntactic rules and meaning for the language. This will be quite clear without saying anything after the following definitions.

Variables	$A, A_1, \ldots, A_n, B, C, \ldots, X$	modules
	$\alpha, \alpha_1, \ldots, \alpha_n, \beta, \gamma, \ldots$	homomorphisms
	o	the zero homomorphism

Atomic formulas (i) $\alpha(A, B)$ means $\alpha \in \operatorname{Hom}(A, B)$,

(ii) $\alpha \| \beta$ means $\operatorname{Im} \alpha = \operatorname{Ker} \beta$,

(iii) arithmetical formulas containing multiplication and addition in the ordinary sense,

(iv) Af.g. means A is finitely generated,

(v) Ac. means A is cyclic module.

If $\mathfrak{a}$ and $\mathfrak{b}$ are formulas which are already defined, then

$$\forall A(\mathfrak{a});\quad \forall\alpha(\mathfrak{a});\quad \exists A(\mathfrak{a});\quad \exists\alpha(\mathfrak{a});\quad \mathfrak{a}\wedge\mathfrak{b};\quad \mathfrak{a}\to\mathfrak{b}$$

are also formulas.

Thus we have got a well-defined language in which the well-formed formulas are obtained by combining the atomic formulas (i), ..., (v) using conjunction, implication, and quantification of the variables.

There is distinguised a module variable X. Any formula $\mathfrak{a}$ is understood as a one-variable formula in which X is a free variable. The other free variables which occur are considered as *constants*. Let $\mathscr{C}(R)$ be the category of R-modules. We assume that every variable of $\mathfrak{a}$ is in $\mathscr{C}(R)$ and in this case we shall say that $\mathfrak{a}$ is *over* $\mathscr{C}(R)$

If we neglect the atomic formulas (iv), (v) the language can be interpreted in any Abelian category.

To show the capacity of the language in formalization of properties of modules we give some examples.

1. X is projective

$$\mathfrak{a}(X)\Leftrightarrow\forall A\forall B\forall\alpha\forall\beta\big(\alpha(A,B)\wedge\beta(X,B)\wedge\alpha\|o\to\exists\gamma\big(\gamma(X,A)\wedge\alpha\gamma=\beta\big)\big).$$

2. X has injective dimension $\leqq n$

$$\mathfrak{a}(X)\Leftrightarrow\exists A_1\dots\exists A_n\,\exists\alpha_1\dots\exists\alpha_n\big(\alpha_1(X,A_1)\wedge o\|\alpha_1\wedge\alpha_2(A_1,A_2)\wedge\alpha_1\|\alpha_2\wedge\dots$$

$$\dots\wedge\alpha_n(A_{n-1},A_n)\wedge\alpha_{n-1}\|\alpha_n\wedge\alpha_n\|o\bigwedge_{i=1}^{n}\forall A\forall B\forall\alpha\forall\beta\big(\alpha(A,\beta)\wedge o\|\alpha\wedge$$

$$\wedge\beta(A,A_i)\to\exists\gamma\big(\gamma(B,A_i)\wedge\gamma\alpha=\beta\big)\big).$$

Here we used the following familiar definition: X has injective dimension$\leqq n$ if X has injective resolution of length n.

$$0\to X\xrightarrow{\alpha_1}A_1\xrightarrow{\alpha_2}A_2\to\dots\xrightarrow{\alpha_n}A_n\to 0$$

3. X is a direct sum of cyclic modules.

To formalize the above property our language is insufficient. We have to use infinite formulas; e.g. if $\exists^I A_i$ means "there exist a family of $\{A_i\}_{i\in I}$"; $\forall^I A_i$ means „for any family of $\{A_i\}_{i\in I}$"; and $\bigwedge_{i\in I}$ denotes infinite conjunction, then the formula

$$\mathfrak{a}(X)\Leftrightarrow\exists^I A_i\exists^I\alpha_i\Big(\bigwedge_{i\in I}\big(\alpha_i(A_i,X)\wedge A_i\text{c.}\big)\wedge\forall B\forall^I\beta_i\Big(\bigwedge_{i\in I}\beta_i(A_i,B)\to$$

$$\to\exists\gamma\Big(\gamma(X,B)\wedge\bigwedge_{i\in I}\gamma\alpha_i=\beta_i\Big)\Big)\Big).$$

is equivalent to the statement that X is a direct sum of the cyclic modules $\{A_i\}_{i\in I}$.[1])

In spite of the above example, for the sake of brevity we will use only the "finite language". However, our results are valid for the infinite case as well.

We shall say that a formula is of *length n* if n sign occur in it.

3. *Functors*

Let $\mathscr{A}$ and $\mathscr{B}$ be categories of modules and $\mathfrak{a}$ a formula over $\mathscr{A}$. Given a functor $F: \mathscr{A} \to \mathscr{B}$ let $\mathfrak{a}_F$ be the formula over $\mathscr{B}$ obtained by changing the constantes in $\mathfrak{a}$ for the coresponding ones in $F(\mathscr{A})$. The quantified variables are automatically understood to be in the same category. If the functor F is onto, i.e. for every diagram $\beta: B_1 \to B_2$ in $\mathscr{B}$ there is a diagram $\alpha: A_1 \to A_2$ in $\mathscr{A}$ such that $F(A_1) = B_1$, $F(A_2) = B_2$, $F(\alpha) = \beta$, and if $\mathfrak{a}_F(X) \Leftrightarrow \mathfrak{a}(X)$ whenever $\mathfrak{a}$ is an atomic formula, then F is said to be *completely faithful.*

1. Theorem. *If a functor* $F: \mathscr{A} \to \mathscr{B}$ *is completely faithful then, for any formula* $\mathfrak{a}(X)$ *over* $\mathscr{A}$ $\mathfrak{a}(A) \Leftrightarrow \mathfrak{a}_F(F(A))$, $A \in \mathscr{A}$.

Our PROOF is based on induction. If $\mathfrak{a}(X)$ is one of the atomic formulas then the theorem is valid by hypothesis. Let $\mathfrak{a}(X)$ be of length $n > 1$. Now suppose that the theorem is valid for any formula of length $\leqq n$. According to the structure of $\mathfrak{a}(X)$ we have four cases.

1. $\mathfrak{a}(X) = \forall A_1 \mathfrak{a}^1(X)$ or $\mathfrak{a}(X) = \forall \alpha_1 \mathfrak{a}^1(X)$.

Obviously it is sufficient to discuss only one of the above formulas, say the former. Assume that for some $A \in \mathscr{A}$ $\mathfrak{a}(A)$ is true, and let $B \in \mathscr{B}$ be arbitrary. Since F is onto there exist an $A_2 \in \mathscr{A}$ such that $F(A_2) = B$. By the inductive hypothesis, $\mathfrak{a}^1(A)$ (puting A_2 as constant) implies $\mathfrak{a}_F^1(F(A))$ with $F(A_2) = B$. Since B was arbitrary $\forall A_1 \mathfrak{a}_F(F(A))$ true. The verification of the converse statement is similar.

2. $\mathfrak{a}(X) = \exists A_1 \mathfrak{a}^1(X)$ or $\mathfrak{a}(X) = \exists \alpha_1 \mathfrak{a}^1(X)$.

Suppose that $\mathfrak{a}(A)$ is true. Then there exists an appropriate $A_1 \in \mathscr{A}$. If we take $F(A_1)$, the formula $\mathfrak{a}_F^1(F(A))$ will be true. Since F is onto $\mathfrak{a}_F(F(A)) \Rightarrow \mathfrak{a}(A)$ can be proven similarly.

3. $\mathfrak{a}(X) = \mathfrak{a}^1(X) \wedge {}^2(X)$

Trivial.

4. $\mathfrak{a}(X) = \mathfrak{a}^1(X) \to \mathfrak{a}^2(x)$

Suppose that $\mathfrak{a}(A)$ is true. We wish to show that $\mathfrak{a}_F(F(A))$ is also true. Evidently the only interesting case is when $\mathfrak{a}_F^1(F(A))$ is true. By induction this implies that $\mathfrak{a}^1(A)$ is true. Whence $\mathfrak{a}^2(A)$ is true. Again by induction we can complete the proof. The verification of the converse statement is similar.

---

[1]) This is the formalized definition of direct sum given by families of maps.

REMARK. Observe that, except in the last case 4. the verification that $\mathfrak{a}(A)$ implies $\mathfrak{a}_F(F(A))$ needs only the assumption that the same holds for atomic formulas. In case 4. we used the assumption that $\mathfrak{a}_F^1(F(A)) \Rightarrow \mathfrak{a}^1(A)$. We obtain the

COROLLARY 1. *Let $F: \mathscr{A} \to \mathscr{B}$ be a functor which satisfies all conditions for a completely faithful functor except that there may exist one atomic formula $\mathfrak{a}^0$ such that $\mathfrak{a}_F^0 \not\Rightarrow \mathfrak{a}^0$. Then*

(i) *if the formula $\mathfrak{a}$ over $\mathscr{A}$ does not contain $\mathfrak{a}^0$ then $\mathfrak{a}(A) \Leftrightarrow \mathfrak{a}(F(A))$;*

(ii) *if $\mathfrak{a}^0$ is not contained in the conclusion of any implication in $\mathfrak{a}$, then $\mathfrak{a}(A) \Leftarrow \mathfrak{a}_F(F(A))$;*

(iii) *if $\mathfrak{a}^0$ is not contained in any premise of any implication in $\mathfrak{a}$, then $\mathfrak{a}(A) \Rightarrow \mathfrak{a}_F(F(A))$.*

Let $\varphi: A_1 \to A_2$ be an isomorphism. We can define a functor $T_\varphi: \mathscr{A} \to \mathscr{A}$ such that $T_\varphi(A_1) = A_2$, $T_\varphi(A_2) = A_1$, $T_\varphi(A) = A$ otherwise. T_φ is a completely faithful functor and naturally equivalent to the identity functor. Thus the assumption in Theorem 1 that F is onto is not essential, and can be replaced by the condition that, for every diagram $B_1 \xrightarrow{\beta} B_2$ in $\mathscr{B}$ there is a diagram $A_1 \xrightarrow{\alpha} A_2$ in $\mathscr{A}$ and a completely faithful functor T from $\mathscr{B}$ to $\mathscr{B}$ which is naturally equivalent to the identity functor and which is such that $TF(A_1) = B_1$, $TF(A_2) = B_2$ and $TF(\alpha) = \beta$. (T depends on the chosen diagram). From this we conclude

COROLLARY 2. *Let $F_1: \mathscr{A} \to \mathscr{B}$, $F_2: \mathscr{B} \to \mathscr{A}$ be functors such that F_1F_2 and F_2F_1 are naturally equivalent to the identity functors, and such that $\mathfrak{a}(A) \Leftrightarrow \mathfrak{a}_{F_i}(F_i(A))$ for any atomic formula $(i = 1, 2)$. Then $\mathfrak{a}(A) \Leftrightarrow \mathfrak{a}_{F_i}(F_i(A))$ $(i = 1, 2)$ for any formula.*

Let R^n be the n-th order full matrix ring over R, and e_{ik} the matrix with 1 in the (i, k)th position and zero elsewhere. Thus $e_{ik}e_{jl} = \delta_{kj}e_{il}$. If $A \in \mathscr{C}(R^n)$ then we can give $e_{11}A$ the structure of an R-module by writing $r(e_{11}a) = (re_{11})a$ $r \in R$ $a \in A$.

Assume that $\alpha: A \to B$ is a homomorphism in $\mathscr{C}(R\ )$. Then there is a mapping $\alpha': e_{11}A \to e_{11}B$ in which $\alpha'(e_{11}a) = e_{11}(\alpha a)$.

1. Lemma *The mappings $A \to e_{11}A$, $\alpha \to \alpha'$ define a functor M from $\mathscr{C}(R^n)$ to $\mathscr{C}(R)$ such that $\mathfrak{a}(A) \Leftrightarrow \mathfrak{a}_M(M(A))$ for the atomic formulas* (i)—(iv) *and there exist a functor $\overline{M}$ from $\mathscr{C}(R)$ to $\mathscr{C}(R^n)$ with the same property for atomic formulas* (i)—(iv) *such that $M\overline{M}$ and $\overline{M}M$ are naturally equivalent to the identity functors.*

It is easy to see that M is an additive covariant functor. Let

$$(1) \qquad A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3$$

be an exact sequence in $\mathscr{C}(R^n)$. For the exactness of the sequence

$$(2) \qquad M(A_1) \xrightarrow{M(\alpha_1)} M(A_2) \xrightarrow{M(\alpha_2)} M(A_3)$$

the only thing we have to show is that $\mathrm{Ker}\,(\alpha_2') \subseteq \mathrm{Im}\,(\alpha_1')$. If $o = \alpha_2'(e_{11}a_2) =$

$=e_{11}\alpha_2(a_2)=\alpha_2(e_{11}a_2)$ $a_2\in A_2$ then by the exactness of (1) there exist $a_1\in A_1$ such that $\alpha_1(a_1)=e_{11}a_2$. Hence

$$\alpha_1'(e_{11}a_1)=e_{11}\alpha_1(a_1)=e_{11}^2a_2=e_{11}a_2.$$

This proves that (2) is exact.

In order ro define $\overline{M}$, let $C\xrightarrow{\beta}D$ be a diagram in $\mathscr{C}(R)$ and $C^n=\bigoplus_{i=1}^{n}C$. C^n becomes an R^n-module when the product is defined by

$$e_{ik}(c_1,c_2,\dots,c_n)=(o,\dots,\underbrace{c_k}_{i},\dots,o)$$

and β becomes an R^n-homomorphism from C^n to D^n defined by

$$\beta^n(c_1,\dots,c_n)=\big(\beta(c_1),\dots,\beta(c_n)\big).$$

It follows immediately from the definitions that the mappings $C\to C^n$, $\beta\to\beta^n$ define an additive covariant exact functor $\overline{M}$ from $\mathscr{C}(R)$, to $\mathscr{C}(R^n)$. It is easy to check that $M\overline{M}$ is naturally equivalent to the identity functor. On the other hand, if $A\in\mathscr{C}(R^n)$, then the mappings $\varphi_A\colon\overline{M}M(A)\to A$, $\mu_A\colon A\to\overline{M}M(A)$ defined respectively by

$$\varphi_A(e_{11}a_1,\dots,e_{11}a_n)=e_{11}a_1+e_{21}a_2+\dots+e_{n1}a_n$$

$$\mu_A(a)\qquad=(e_{11}a,\ e_{12}a,\dots,e_{1n}a)$$

are well-defined because $e_{i1}=e_{i1}e_{11}$ and $e_{1i}=e_{11}e_{1i}$. Furthermore φ_A and μ_A are R^n-homomorphisms and $\varphi_A\mu_A=1_A$ and $\mu_A\varphi_A=1_{\overline{M}M(A)}$. The diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \mu_A\downarrow & & \downarrow\mu_B \\ \overline{M}M(A) & \xrightarrow{\overline{M}M(\alpha)} & \overline{M}M(B)\end{array}$$

is commutative which means that μ_A is a natural isomorphism $1_{\mathscr{C}(R^n)}\to\overline{M}M$. By application of the functor $\overline{M}$ we can obtain that the exactness of (2) implies the exactness of (1), and if $M(\alpha)=o$ then $\alpha=o$. Thus in the above we have proved that if $\mathfrak{a}$ is one of the atomic formulas (i)—(iii) then $\mathfrak{a}\Leftrightarrow\mathfrak{a}_M$ and $\mathfrak{a}\Leftrightarrow\mathfrak{a}_{\overline{M}}$. It is evident that $M(A)$f.g. (finitely generated) implies Af.g. Let A be an R^n-module and let the elements $a_1,\dots,a_k$ form a system of generators for A. Since an element r_n of R^n can be written in the form

$$r_n=\sum_{i,k=1}^{n}r_{ik}e_{ik}\qquad(r_n\in R^n,\ r_{ik}\in R),$$

the elements $e_{11}a_1,\dots,e_{1n}a_1,e_{11}a_2,\dots,e_{1n}a_2,\dots,e_{11}a_k,\dots,e_{1n}a_k$ form a system of generators for the R-module $e_{11}A$.

If the R-module $e_{11}A$ is cyclic then the R^n-module A is also cyclic, but the converse statement is not necessarily true.

The next case we propose to discuss is the direct sum of rings.

Let R and S be rings and let A be an $R\oplus S$-module. Every element $a\in A$ can be written as $a=(1,0)a+(0,1)a$. Moreover, if $(1,0)a=(0,1)a'$ $(a,a'\in A)$ then $(1,0)a=(1,0)(1,0)a=(1,0)(0,1)a'=0$. Hence $A=(1,0)A\oplus(0,1)A$. Also we may regard $(1, 0)A$ and $(0, 1)A$ as R- and S-modules respectively. If $\alpha: A \to B$ is an $R \oplus S$-homomorphism, then

$$(1, 0)a \to (1, 0)\alpha(a), \quad (0, 1)a \to (0, 1)\alpha(a)$$

define the respective R- and S-homomorphisms,

$$\alpha': (1, 0)A \to (1, 0)B \quad \text{and} \quad \alpha'': (0, 1)A \to (0, 1)B.$$

2. Lemma *The mappings $A \to (1,0)A$, $\alpha \to \alpha'$ and $A \to (0,1)A$, $\alpha \to \alpha''$ define functors D', D'' from $\mathscr{C}(R \oplus S)$ onto $\mathscr{C}(R)$ and $\mathscr{C}(S)$ respectively. If $\mathfrak{a}$ is any of the atomic formulas, then* $\mathfrak{a} \Leftrightarrow \mathfrak{a}_{D'} \wedge \mathfrak{a}_{D''}$.

For atomic formulas of types (iv), (v) the result is immediate.

Next, both D', D'' are additive, covariant, exact functors from $\mathscr{C}(R \oplus S)$ onto $\mathscr{C}(R)$ and $\mathscr{C}(S)$ respectively. It is sufficient to prove this only for D'. The first two statement are trivial. Let

$$A_1 \xrightarrow{\alpha} A_2 \xrightarrow{\beta} A_3$$

be an exact sequence of $R \oplus S$-modules. Take an element $(1, 0)a_2 \in \operatorname{Ker} \beta'$. Then $\beta'(1, 0)a_2 = \beta(1, 0)a_2$, which implies the existence of an element $a_1 \in A_1$ such that $\alpha(a_1) = (1, 0)a_2$. Thus $\alpha'(1. 0)a_1 = (1, 0)a_2$, and this shows that $\operatorname{Ker} \beta' \subseteqq \operatorname{Im} \alpha'$. Since $\beta\alpha = 0$ implies $\operatorname{Im} \alpha' \subseteqq \operatorname{Ker} \beta'$, we have proved that $\alpha' \| \beta'$.

Before we prove the converse we make a number of observations. An R-module A' can be regarded as an $R \oplus S$-module by the definition

$$(r, s)a' = ra' \qquad (a' \in A', (r, s) \in R \oplus S).$$

Let A'' be an S-module which is regarded as an $R \oplus S$-module, then $A' \oplus A''$ also an $R \oplus S$-module. If $\alpha': A' \to B'$ and $\alpha'': A'' \to B''$ are R-and S-homomorphisms respectively then the mapping $\alpha = \alpha' \oplus \alpha''$ given by $\alpha(a', a'') = (\alpha' a', \alpha'' a'')$ $(a' \in A', a'' \in A'')$ is an $R \oplus S$-homomorphism. It is easy to see that

$$D'(A' \oplus A'') = A' \quad D''(A' \oplus A'') = A''$$

and

$$D'(\alpha' \oplus \alpha'') = \alpha' \qquad D''(\alpha' \oplus \alpha'') = \alpha''.$$

We wish to show that, for atomic formulas

$$\mathfrak{a}_{D'} \wedge \mathfrak{a}_{D''} \Rightarrow \mathfrak{a} \tag{3}$$

Let $\alpha', \beta': A' \to B'$ and $\alpha'', \beta'': A'' \to B''$ be homomorphisms. Then

$$\big((\alpha' + \beta') \oplus (\alpha'' + \beta'')\big)(a', a'') = \big((\alpha' + \beta')a', (\alpha'' + \beta'')a''\big) = (\alpha' a', \alpha'' a'') +$$
$$+ (\beta' a', \beta'' a'') = \big((\alpha' \oplus \alpha'') + (\beta' \oplus \beta'')\big)(a', a''), \quad \text{i. e.} \quad (\alpha' + \beta') \oplus (\alpha'' + \beta'') =$$
$$= (\alpha' \oplus \alpha'') + (\beta' \oplus \beta'').$$

Now consider $A' \xrightarrow{\alpha'} B' \xrightarrow{\beta'} C'$ in $\mathscr{C}(R)$ and $A'' \xrightarrow{\alpha''} B'' \xrightarrow{\beta''} C''$ in $\mathscr{C}(S)$. Then $(\beta'\alpha' \oplus \beta''\alpha'')(a', a'') = (\beta' \oplus \beta'')(\alpha' \oplus \alpha'')(a', a'')$ i.e. $(\beta'\alpha' \oplus \beta''\alpha'') = (\beta' \oplus \beta'')(\alpha' \oplus \alpha'')$. Therefore if $\mathfrak{a}$ is an arithmetical formula then (3) valid. Finally let $A' \to B' \to C'$ and $A'' \to B'' \to C''$ be exact sequences of R- and S-modules. They remain exact if we regard them as sequences of $R \oplus S$-modules. The sequence $A' \oplus A'' \to B' \oplus B'' \to C' \oplus C''$ will be exact by the exactness of the direct sum functor. This completes the proof of the lemma.

Let N be a two-sided ideal of the ring R. Every module $A \in \mathscr{C}(R/N)$ can be regarded as an R-module by the definition $ra = (r+N)a$ $(a \in A)$. The class

$$\mathscr{C}_N(R) = \{A \in \mathscr{C}(R) | NA = 0\}$$

is a category, and every module in $\mathscr{C}_N(R)$ can be regarded as an R/N-module. We can now define a functor H from $\mathscr{C}_N(R)$ to $\mathscr{C}(R/N)$. The R-module A in $\mathscr{C}_N(R)$ corresponds to A in $\mathscr{C}(R/N)$. If $\alpha\colon A \to B$ is an R-homomorphisms (A, B in $\mathscr{C}_N(R)$) then α is also an R/N-homomorphism. Under H, α corresponds to itself. The following lemma does not need proof.

3. Lemma. *The functor $H\colon \mathscr{C}_N(R) \to \mathscr{C}(R/N)$ is completely faithful.*

4. *The Ring Class $\mathfrak{R}_\mathfrak{a}$*

Now we are in the position to apply Theorem 1. in the case of special functors discussed in 3.

The functor $\overline{M}$ and M fulfill the conditions of Corollary 2. except in respect of the atomic formula of type (v). Using the corollaries to Theorem 1., we deduce the following result

2. Theorem. *If the formula $\mathfrak{a}$ does not contain the atomic formula* (v) *then*

$$\mathfrak{a}(X) \Leftrightarrow \mathfrak{a}_{\overline{M}}(X)$$

If the formula $\mathfrak{a}$ satisfies the condition that the atomic formula (v) *is not contained in the premise of any implication in $\mathfrak{a}$, then*

$$\mathfrak{a}(X) \Rightarrow \mathfrak{a}_{\overline{M}}(X), \quad \textit{and} \quad R^n \in \mathfrak{R}_\mathfrak{a} \quad \textit{whenever} \quad R \in \mathfrak{R}_\mathfrak{a}$$

The following examples indicate the effectiveness of the above theorem, although most of them are well-known. (see e.g. [1])

a) $A \in \mathscr{C}(R)$ is projective (injective) if and only if $e_{11}A \in \mathscr{C}(R)$ is projective (injective),$\mathrm{l.dh.}_{R^n}(A) = \mathrm{l.dh.}_R(e_{11}A)$.
b) $R \in \mathfrak{R}_k$ if and only if $R^n \in \mathfrak{R}_k$, i.e. l.g.d.R$=$l.g.d.R^n.
c) If every R-module is a direct sum of cyclic modules, then so is every R^n-module.
d) If the ring R satisfies the condition that every finitely generate R-module is a direct sum of cyclic modules, then the ring R^n also satisfies this condition.

Let R and S be rings. Is the statemant $\mathfrak{a}_{D'} \wedge \mathfrak{a}_{D''} \Leftrightarrow \mathfrak{a}$ valid for any formula? We may try to prove this the same way as in Theorem 1., but the proof breaks down when $\mathfrak{a}$ has the form $\mathfrak{a} = \mathfrak{a}^1 \to \mathfrak{a}^2$. Thus to establish $\mathfrak{a} \Rightarrow \mathfrak{a}_{D'} \wedge \mathfrak{a}_{D''}$, consider just $\mathfrak{a} \Rightarrow \mathfrak{a}_{D'}$. We would assume in addition that $\mathfrak{a}^1_{D'}$ were true. But we can only deduce $\mathfrak{a}^1$ when we know that both $\mathfrak{a}^1_{D'}$ and $\mathfrak{a}^1_{D''}$ are true. This situation is similar to that of Theorem 1., Corollary 1., so we obtain

3. Theorem. *If the formula* $\mathfrak{a}$ *does not contain implication, then*

$$\mathfrak{a} \Leftrightarrow \mathfrak{a}_{D'} \wedge \mathfrak{a}_{D''}$$

If the formula $\mathfrak{a}$ *satisfies the condition that implication is not contained in the premise of any implication of* $\mathfrak{a}$, *then*[2])

$$\mathfrak{a}_{D'} \wedge \mathfrak{a}_{D''} \Rightarrow \mathfrak{a} \quad \textit{and} \quad R \oplus S \in \mathfrak{R}_{\mathfrak{a}} \quad \textit{whenever} \quad R, S \in \mathfrak{R}_{\mathfrak{a}}.$$

The examples in § 1 show that the most important properties satisfy these conditions. In particular it follows that, if $R, S \in \mathfrak{R}_n$ then $R \oplus S \in \mathfrak{R}_n$ and similar examples can be given as to Theorem 2.

We recall thet the functor $H: \mathscr{C}_N(R) \to \mathscr{C}(R)$ is not onto, since $H(\mathscr{C}_N(R)) \neq \neq \mathscr{C}(R/N)$. Thus $R \in \mathfrak{R}_{\mathfrak{a}}$ does not in general imply that $R/N \in \mathfrak{R}_{\mathfrak{a}}$.

4. Theorem. *Assume that the formula* $\mathfrak{a}$ *satisfies the conditions*

(i) $\mathfrak{a}$ *does not contain variables quantified by the quantifier* $\exists$, *or if it does then this quantifier is restricted to* $\mathscr{C}_N(R)$, *and the constans in* $\mathfrak{a}$ *belong to* $\mathscr{C}_N(R)$.

(ii) *The universal quantifier does not occur in the premise of any implication of* $\mathfrak{a}$, *then*

$$R \in \mathfrak{R}_{\mathfrak{a}} \quad \textit{implies} \quad R/N \in \mathfrak{R}_{\mathfrak{a}}.$$

The PROOF can be made in these same way as in Theorem 1. In particular the formula $\mathfrak{a}(X)$ in example 3 § 2 fulfills these conditions.

References

[1] L. LEVY, Torsion-free and divisible modules over non-integral domains. *Canad. J. Math.* **15** (1963), 132—151.

(Received August 9, 1965.)

[2]) Since $(a \to b) \to a = a \vee b$, the condition is necesary. In fact, if $\mathfrak{a} = \mathfrak{a}^1 \vee \mathfrak{a}^2$, it may happen that $\mathfrak{a}_{D'}$ and $\mathfrak{a}_{D''}$ are true but this obviously does not imply that $\mathfrak{a}$ is true.