A note on the Jacobson radical of a hemiring

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1. *Introduction*. The Jacobson radical of a hemiring (*i.e.*, an additively commutative semiring with zero) was first introduced by BOURNE ([1]) in an internal way, and has subsequently been studied by IIZUKA ([5]) from the point of view of representation theory.

The present paper establishes several new results for the Jacobson radical of a hemiring by considering h-ideals and k-ideals. These ideals are of a more restricted type than the familiar concept of ideal in semirings.

Section 2 contains a brief discussion of subdirect sums of hemirings, and gives analogues of several results well-known in rings.

Our main results are in section 3. We first characterize the Jacobson radical J of a hemiring S in terms of h- and k-ideals, and then prove that if S is a ring, the Jacobson ring and hemiring radicals coincide. An analogue for hemirings of a theorem familiar in ring theory is also proved, from which it follows that any additively regular hemiring S (*i.e.*, for $a \in S$ there exists $x \in S$ such that $a+x+$ $+a=a$) for which J is zero is a ring.

2. *Subdirect Sums*. The *complete direct sum* $\sum_\alpha S_\alpha$ of a collection of semirings $\{S_\alpha\}$, $\alpha \in \Omega$, is defined analogously to the complete direct sum of a collection of rings. For convenience, we shall denote an arbitrary element of $\sum_\alpha S_\alpha$ by $(s_1, s_2, \ldots)$ where $s_\alpha \in S_\alpha$ and addition and multiplication are defined term by term. However, in adopting this notaion we do not intend to imply that the collection $\{S_\alpha\}$ is necessarily countable. It is immediate that if each S_α is a hemiring, or if each S_α is additively regular, then so is $\sum_\alpha S_\alpha$.

Now let T be a subsemiring of $S=\sum_\alpha S_\alpha$, $\alpha \in \Omega$, and $t=(t_1, t_2, \ldots) \in T$. For each $\alpha \in \Omega$ the mapping $Q_\alpha\colon t \to t_\alpha$ is a homomorphism of T into S_α. If, for each α, $TQ_\alpha = S_\alpha$, we call T a *subdirect sum* of the semirings S_α.

A *left semi-ideal* of a semiring S is a non-empty subset I closed under premultiplication by elements of S and under addition. A left semi-ideal I of an additively commutative semiring S is called a *left k-ideal* if for each $a \in I$ and $x \in S$, $a+x \in I$ implies that $x \in I$; and I is called a *left h-ideal* of S if $i_1, i_2 \in I$ and $x, z \in S$ with $x+i_1+z=i_2+z$ implies $x \in I$. Right semi-, k-, and h-ideals are defined dually and a *two-sided ideal* (or simply *ideal*) of any type is both a left and a right ideal of that type. The concepts of k-ideal and h-ideal are due to HENRIKSEN ([4]) and IIZUKA ([5]), respectively.

By the *Bourne congruence* on an additively commutative semiring S, relative to a given semi-ideal I, we mean the relation $a \equiv b(I)$ if and only if $a+i_1=b+i_2$ for some $i_1, i_2 \in I$. The *Iizuka congruence* on S, relative to I, is defined by $a[\equiv]b\ (I)$ if and only if $a+i_1+z=b+i_2+z$ for some $i_1, i_2 \in I$ and $z \in S$. Under the usual definitions of addition and multiplication of congruence classes, the Bourne classes and the Iizuka classes form semirings denoted by S/I and $S[/]I$, respectively. The semiring $S[/]I$ is additively cancellative, and both S/I and $S[/]I$ are hemirings in case S is a hemiring. A final definition, again due to Bourne [2], is the following. A semiring S with zero element 0 is said to be *semi-isomorphic to* a semiring T with zero if there is a homomorphism of S onto T with kernel 0. A detailed treatment of k-and h-ideals, semi-isomorphisms, and the semirings S/I and $S[/]I$ can be found in [8].

The proofs of the next two theorems are omitted since they parallel those given in [9] for the corresponding results in rings.

Theorem 2. 1. *A semiring S with zero is semi-isomorphic to a subdirect sum T of semirings S_α with zero elements 0_α if and only if there exists homomorphisms φ_α of S onto S_α (for all α) such that, if $0 \neq s \in S$, then $s\varphi_\alpha \neq 0_\alpha$ for some α.*

THEOREM **2. 2.** *A hemiring S is semi-isomorphic to a subdirect sum of [additively cancellative] semirings $S_\alpha, \alpha \in \Omega$, with zero elements 0_α if and only if for each $\alpha \in \Omega$ there exists a k-ideal [h-ideal] I_α of S such that S/I_α [$S[/]I_\alpha$] is semi-isomorphic to S_α and $\bigcap_\alpha I_\alpha = 0$.*

Let T be a semiring with zero that is a subdirect sum of semirings S_α with zero, and let S be a semiring with zero that is semi-isomorphic to T under a mapping σ. As for rings, we call T a *reperesentation* of S as a subdirect sum of the semirings S_α. The mapping φ_α defined by $s\varphi_\alpha=(s\sigma)Q_\alpha$ is a homomorphism of S onto S_α, which may be a semi-isomorphism.

Definition 2. 3. A semiring S with zero is called *subdirectly irreducible* if in every representation of S as a subdirect sum of semirings S_α some φ_α is a semi-isomorphism.

By following the proof given for rings in [9] it is readily proved that *a hemiring is subdirectly irreducible if and only if the intersection of all its non-zero two-sided k-ideals is non-zero.* Unfortunately, there exist examples to show that the term *k-ideal* cannot here be replaced by *h-ideal* or *semi-ideal.* However, if we define a semiring S with zero to be *subdirectly h-irreducible* if in every representation of S as a subdirect sum of *additively cancellative* semirings S_α, some φ_α is a semi-isomorphism, then *a hemiring* is *subdirectly h-irreducible if and only if the intersection of all its non-zero two-sided h-ideals is non-zero.*

Finally, we remark that by again paralleling the proof for rings [9] it can be shown that *every hemiring $S \neq 0$ is semi-isomorphic to a subdirect sum of subdirectly irreducible hemirings.* Moreover, *if the zero of S is an h-ideal then S is semi-isomorphic to a subdirect sum of subdirectly h-irreducible additively cancellative hemirings.*

3. *The Jacobson Radical.* The first to appear [6] among the many equivalent definitions of the Jacobson radical of a ring is the following. The *Jacobson radical* of a ring R is the right ideal generated by the set of all right quasi-regular right

ideals of R. We recall that a right ideal I is called *right quasi-regular* if for $z \in I$ there exists $z' \in R$ such that $z+z'+zz'=0$.

For convenience, we state Bourne's definition ([1]) of the Jacobson radical of a hemiring. A left [right, two-sided] semi-ideal I of a hemiring S is called *right-semiregular* if for each pair of elements $i_1, i_2 \in I$ there exist elements $j_1, j_2 \in I$ such that

$$i_1+j_1+i_1j_1+i_2j_2 = i_2+j_2+i_1j_2+i_2j_1.$$

The *Jacobson radical* J of a hemiring S is the right semi-ideal generated by the set of all right-semiregular right semi-ideals of S.

Bourne showed the Jacobson radical, so defined, to be a right-semiregular right semi-ideal of S. Defining left-semiregularity dually, he proved that J is the left semi-ideal generated by the set of all left-semiregular left semi-ideals, and is left-semiregular. In [3] Bourne and Zassenhaus proved that J is a k-ideal, and Iizuka ([5]) proved that J is an h-ideal of S.

Theorem 3. 1. *The Jacobson radical J of a hemiring S is the right k-ideal [h-ideal] generated by the set of all right-semiregular right k-ideals [h-ideals] of S.*

PROOF. Let A be the set of all right-semiregular right semi-ideals of S and B the set of all right-semiregular right k-ideals of S. Let $(A)_s$ and $(A)_k$ be respectively the right semi-ideal and right k-ideal generated by A, and let $(B)_k$ be the right k-ideal generated by B. Now $(B)_k \subseteqq (A)_k$ since $B \subseteqq A$. Since $J=(A)_s$ is a k-ideal it follows that $(A)_k=(A)_s=J$, whence $(B)_k \subseteqq J$. But J is a right-semiregular right k-ideal and so $J \subseteqq (B)_k$. Thus $J=(B)_k$. Replacing the word k-ideal by h-ideal, and the letter k by h, the above argument gives $J=(B)_h$.

Theorem 3. 2. *If R is a ring then the Jacobson hemiring radical J of R coincides with the Jacobson ring radical N of R.*

PROOF. If I is a right-semiregular right k-ideal of R and $i_1 \in I$ then, taking $i_2=0$, there exist elements $j_1, j_2 \in I$ such that $i_1+j_1+i_1j_1 = j_2+i_1j_2$, that is, $i_1+(j_1-j_2)+i_1(j_1-j_2) = 0$, so that i_1 is right quasi-regular. Since k-ideals in R coincide with ring ideals, it follows that the class A of all right-semiregular right k-ideals is contained in the class C of all right quasi-regular right ideals of R. Thus by Theorem 3. 1, $J=(A)_k \subseteqq (C)_k$. But $N=(C)_k$, so that $J \subseteqq N$. Conversely, if we can show that every right quasi-regular right ideal of R is right-semiregular, then $C \subseteqq A$ and so $N \subseteqq J$. Thus let I be a right quasi-regular right ideal of R and let $i_1, i_2 \in I$. Then $z=i_1-i_2 \in I$, so there is an element $z' \in R$ such that $z+z'+zz'=0$, *i.e.*, $(i_1-i_2)+z'+(i_1-i_2)z'=0$. Therefore $i_1+z'+i_1z'=i_2+i_2z'$, whence $i_1+z'+i_1z'+i_2 0=i_2+0+i_1 0+i_2z'$. Now $z' = -(z+zz') \in I$. Hence there are elements $j_1=z'$ and $j_2=0$ such that $i_1+j_1+i_1j_1+i_2j_2=i_2+j_2+i_1j_2+i_2j_1$, so that I is right-semiregular.

Since, for any homomorphism φ of a hemiring S onto a hemiring T, the image of a right-semiregular right semi-ideal under φ is again such an ideal, it follows that φ maps the Jacobson radical of S into that of T. Hence, it is fairly immediate that a subdirect sum of semirings S_α, each of which has zero Jacobson radical, has zero Jacobson radical. These remarks will be used in our next theorem.

The term *primitive*, well-known in rings, has been defined for hemirings by Iizuka ([5]). Theorem 6 of his paper states that *the Jacobson radical of any hemiring*

S that is not a radical hemiring (i.e., $J \neq S$) is the intersection of all primitive h-ideals of S. Using another of Iizuka's results, one can prove that *any ring primitive in the hemiring sense is also primitive in the ring sense.* We omit the proof since to include it here would necessitate a brief exposition of parts of Iizuka's paper. These remarks bring us to the next theorem.

Theorem 3. 3. *If a hemiring $S \neq 0$ has zero Jacobson radical then S is semi-isomorphic to a subdirect sum of primitive hemirings. Conversely, if a hemiring S is semi-isomorphic to a subdirect sum of additively cancellative primitive hemirings then S has zero Jacobson radical.*

PROOF. If $S \neq 0$ has zero Jacobson radical then the intersection of all primitive h-ideals I_α of S is zero. Since each such h-ideal is also a k-ideal, it follows from Theorem 2. 2 that S is semi-isomorphic to a subdirect sum of the hemirings S/I_α, each of which is primitive since each I_α is primitive. Conversely, suppose the hemiring S is semi-isomorphic to a subdirect sum T of additively cancellative primitive hemirings S_α. Since each S_α is additively cancellative, the zero 0_α of each is an h-ideal. Therefore, since each S_α is primitive and $S_\alpha \cong S_\alpha/0_\alpha$, 0_α is a primitive h-ideal. It follows that each S_α has zero Jacobson radical, whence, by our earlier remarks, T has zero Jacobson radical and so does S.

Theorem 3. 3 is an analogue for hemirings of the familiar theorem for rings due to Jacobson ([7]) stating that a ring $R \neq 0$ has zero Jacobson radical if and only if R is isomorphic to a subdirect sum of primitive (in the ring sense) rings. We have been unable to delete the hypothesis of additive cancellation since, in general, the zero of a hemiring need not be an h-ideal. However, for additively regular hemirings we have the following result, the immediate corollary of which will be useful to us elsewhere.

Theorem 3. 4. *An additively regular hemiring $S \neq 0$ has zero Jacobson radical if and only if S is semi-isomorphic to a subdirect sum (in the ring sense) of rings primitive in the ring sense.*

PROOF. Suppose the hemiring $S \neq 0$ is semi-isomorphic to a subdirect sum T (in the ring sense) of rings primitive in the ring sense. Since T is a ring, Theorem 2. 4 of [8] shows that S is a ring isomorphic to T. By the result mentioned above, S has Jacobson radical 0.

Conversely, suppose the additively regular hemiring $S \neq 0$ has zero Jacobson radical. Then by the result of Iizuka, the intersection of all h-ideals I_α of S is zero, whence it follows from Theorem 2. 2 that S is semi-isomorphic to a subdirect sum T of the hemirings S/I_α. By Corollary 2. 8 of [8], each S/I_α is a ring, and, since I_α is a primitive h-ideal of S, S/I_α is a ring primitive in the hemiring sense. Thus, using, a result stated earlier, each S/I_α is a ring primitive in the ring sense. The proof will be complete when we show that T is actually a subring of the complete direct sum of the rings S/I_α. To this end, we denote by v_α, for each α, the natural homomorphism of S onto S/I_α. Since $\bigcap_\alpha I_\alpha = 0$, the proofs of Theorems 2. 1 and 2. 2 show that $T = \{(sv_1, sv_2, \ldots) : s \in S\}$. Since each I_α is a k-ideal, I_α is the zero element 0_α of S/I_α, whence the zero of T is $(I_1, I_2, I_3, \ldots)$. Now if $s \in S$ and s' is an additive inverse of s in S, then $s'v_\alpha$ is an additive inverse of sv_α in S/I_α. Thus, for any element $(sv_1, sv_2, sv_3, \ldots)$ in T, an additive inverse in T is $(s'v_1, s'v_2, s'v_3, \ldots)$. Since $s+s'$ is an additive idempotent in S, its image is an additive idempotent in S/I_α, whence $(s+s')v_\alpha = 0_\alpha = I_\alpha$ since S/I_α is a ring. Therefore, if $(sv_1, sv_2, sv_3, \ldots) \in T$ we have $(sv_1, sv_2, sv_3, \ldots)+(s'v_1, s'v_2, s'v_3, \ldots) = (I_1, I_2, I_3, \ldots)$, so that $(sv_1, sv_2, sv_3, \ldots)$ has a group inverse in T with respect to addition. Thus T is a ring.

Corollary 3. 5. *An additively regular hemiring with zero Jacobson radical is a ring.*

PROOF. The proof of Theorem 3. 4 shows S is semi-isomorphic to a ring T, whence it follows that S itself is a ring.

Bibliography

[1] S. BOURNE, The Jacobson radical of a semiring, *Proc, Nat. Acad. Sci.*, U.S.A. **37** (1951), 163—170.

[2] S. BOURNE, On the Homomorphism Theorem for Semirings, *Proc, Nat. Acad. Sci.* U.S.A. **38** (1952), 118—119.

[3] S. BOURNE and H. ZASSENHAUS, On the Semiradical of a Semiring, *Proc. Nat. Acad. Sci.*, U.S.A. **44** (1958), 907—914.

[4] M. HENRIKSEN, Ideals in semirings with commutative addition, *Amer. Math. Soc. Notices* **6** (1958), 321.

[5] K. IIZUKA, On the Jacobson radical of a semiring, *Tohoku Math. J.* (2) **11** (1959), 409—421.

[6] N. JACOBSON, The radical and semi-simplicity for arbitrary rings, *Amer. J. Math.* **67** (1945), 300—320.

[7] N. JACOBSON, Structure of rings, *Providence, R. I.*, 1956.

[8] D. R. LATORRE, On h-ideals and k-ideals in hemirings, *Publ. Math. Debrecen* **12** (1965), 219—226.

[9] N. H. MCCOY, *The theory of rings, New York*, 1964.

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