The Brown-McCoy radicals of a hemiring

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1. *Introduction*. The concept of the F-radical of a ring, and a certain specialization thereof now known as the Brown—McCoy radical, was introduced by BROWN and McCOY ([3]). This paper investigates these concepts for hemirings (*i.e.*, additively commutative semirings with zero), and our methods are based largely on those in [3].

The notation and terminology used herein follow [7] and [8], and we shall often use results from these papers. However, we shall incorporate the more basic definitions and theorem statements from [7] and [8] whenever possible.

We begin in section 2 with the F-, FK-, and FH-radicals. These are defined in a general way and our results here are obtained under fairly weak hypothese.

We restrict our hypotheses somewhat in Section 3 and consider the FH-radical only. The main result in this section is Theorem 3. 2, which characterizes the FH-radical of a hemiring of type (H) as the intersection of a class of h-ideals. We also prove that for such a hemiring the Bourne factor hemiring modulo the radical has zero FH-radical. Section 4 contains our most interesting results, and deals with a special case of the FH-radical, namely, the H-radical of a hemiring of type (H). In case the hemiring is a ring, the H-radical is just the well-known Brown—McCoy radical. We first show that the H-radical of any hemiring of type (H) is the intersection of all h-ideals M such that the Bourne factor hemiring modulo M is a simple ring with identity, and then show that the Jacobson radical of any hemiring of type (H) is contained in the H-radical. If a hemiring is additively regular or additively periodic, and satisfies the minimal condition for right h-ideals, these two radicals coincide. Theorem 4. 5 is an analogue for hemirings of a generalization of the Wedderburn—Artin Theorem for rings. Theorem 4. 7 is more closely analogous to the Wedderburn—Artin Theorem in that it states for any hemiring of type (H), satisfying the minimal condition for h-ideals, that the H-radical is zero if and only if the hemiring is semi-isomorphic to a direct sum of finitely many simple rings with identity elements.

Turning to Section 5 we consider the H-radical of the hemiring S_n of all $n\times n$ matrices over a hemiring S of type (H). Our main result asserts that if S is a hemiring of type (H) with H-radical N, and if S_n is of type (H), then the H-radical of S_n is just N_n. We then give a condition sufficient to insure that S_n is of type (H); in particular, the condition holds in all hemirings with identity element.

In Section 6 we discuss other radicals. Specifically, we indicate how theories of a *K-radical* and another H-radical can be developed by methods analogous to those in Sections 2 through 5.

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2. *The Generalized F-Radicals.* A *left semi-ideal* I of a semiring S is a non-empty subset of S closed under premultiplication by elements of S and under addition. If S is additively commutative then a left semi-ideal I is called a *left k-ideal* if for each $a \in I$ and $x \in S$, $a+x \in I$ implies that $x \in I$; and I is called a *left h-ideal* of S if $i_1, i_2 \in I$ and $x, z, \in S$ with $x+i_1+z=i_2+z$ imply $x \in I$. Right semi-, k-, and h-ideals are defined dually, and a *two-sided ideal* (or simply *ideal*) of any type is both a left and a right ideal of that type. The concepts of k-ideal and h-ideal are due to Henriksen ([4]) and Iizuka ([5]), respectively.

The two congruence relations in semirings that we shall use were first given by Bourne ([1]) and Iizuka ([5]). By the *Bourne congruence* on an additively commutative semiring S, relative to a semi-ideal I, we mean the relation $a \equiv b\ (I)$ if and only if $a+i_1=b+i_2$ for some $i_1, i_2 \in I$. The *Iizuka congruence* on S, relative to I, is the relation $a[\equiv]b\ (I)$ if and only if $a+i_1+z=b+i_2+z$ for some $i_1,\ i_2 \in I$ and $z \in S$. Under the usual definitions of addition and multiplication of congruence classes, the Bourne classes form a semiring denoted by S/I, and the Iizuka classes form a semiring denoted by $S[/]I$. The semiring $S[/]I$ is additively cancellative. If S is a hemiring then both S/I and $S[/]I$ are hemirings. Also, I is the zero of S/I if and only if it is a k-ideal, and I is the zero if $S[/]I$ if and only if it is an h-ideal.

If M is any non-empty subset of an additively commutative semiring S, the intersection of all semi-ideals [k-ideals, h-ideals] of S containing M is called the *semi-ideal [k-ideal, h-ideal] generated by M*, and is denoted by $(M)[(M)_k,\ (M)_h]$. If M consists of a single element then these ideals are called *principal*.

Let C denote the class of all additively commutative semirings, and with each S in C associate a fixed mapping F_S of S into the collection of all non-empty subsets of S subject to the following condition:

(i) If S and Σ are in C, if $\varphi\colon a \to \bar{a}$ is a homomorphism of S onto Σ, and if $\overline{F_S(a)}$ is the image, under φ, of the subset $F_S(a)$ of S, then $\overline{F_S(a)} \subseteqq F_\Sigma(\bar{a})$.

Now for each S in C such a mapping F_S always exists, for we can define $F_S(a)=S$ for all $a \in S$, or $F_S(a)=a$ for all $a \in S$. Later on, we shall restrict the mappings F_S somewhat and impose an additional condition in order to get nicer theorems. Unless otherwise stated, all semirings considered will be members of C.

Definition 2. 1. An element a in a semiring S is called *F_S-regular* provided $a \in F_S(a)$, and a complex M of S is said to be an *F_S-regular subset* of S if every element of M is an F_S-regular element of S. The *F-radical* [*FK-radical*, *FH-radical*] of a semiring S is the set of all elements b in S for which the principal semi-ideal (b) [k-ideal $(b)_k$, h-ideal $(b)_h$] is an F_S-regular subset of S; it is denoted by $N_F[N_{FK}, N_{FH}]$.

Since $(b) \subseteqq (b)_k \subseteqq (b)_h$, we have $N_{FH} \subseteqq N_{FK} \subseteqq N_F$. However, without additional information about the mapping F_S, we do not know whether any of these radicals is non-empty; for the remainder of this section we assume N_{FH} non-empty. The proof of the following lemma is straightforward and is omitted.

Lemma 2. 2. *Let φ be a homomorphism of a semiring S onto a semiring T, and let M be any complex in S. Then $(M)\varphi=(M\varphi)$, $(M)_k\varphi \subseteqq (M\varphi)_k$, and $(M)_h\varphi \subseteqq (M\varphi)_h$. If $(M)_k\varphi[(M)_h\varphi]$ is a k-ideal [h-ideal] of T then $(M)_k\varphi=(M\varphi)_k[(M)_h\varphi=(M\varphi)_h]$.*

We have remarked in [7] that it is not necessarily true that a homomorphic image of a k-ideal [h-ideal] be a k-ideal [h-ideal]. The proof of the next theorem parallels that of the corresponding theorem for rings.

Theorem 2. 3. *If φ is a homomorphism of a semiring S onto a semiring T then φ maps the F-radical of S into the F-radical of T. If φ preserves k-ideals [h-ideals] then φ maps the FK-radical [FH-radical] of S into that of T.*

Theorem 2. 4. *If S is a hemiring then N_F is contained in the intersection $\bigcap M$ of all k-ideals M of S such that S/M has zero F-radical; and N_F is contained in the intersection $\bigcap M^*$ of all h-ideals M^* of S such that $S[/]M^*$ has zero F-radical.*

Proof. Let $b \in N_F$ and M be any k-ideal of S such that S/M has zero F-radical. Let $\nu: a \to \bar{a}$ be the natural homomorphism of S onto S/M. By Theorem 2. 3, $\bar{b}$ is the F-radical of S/M, whence $\bar{b} = \bar{0}$. Since M is a k-ideal, $\bar{0} = M$, whence $\bar{b} = M$ implies $b \in M$. Our assertion now follows immediately, and a similar proof establishes the other assertion.

Frequently, as in the next theorem, we shall say F-regular and $F(a)$ instead of F_S-regular and $F_S(a)$ if it is clear what semiring S is under consideration.

Theorem 2. 5. *If S is a semiring then $N_{FK}[N_{FH}]$ is an F-regular subset of S and contains every F-regular k-ideal [h-ideal] of S.*

Proof. If $x \in N_{FK}$ then $(x)_k$ is F-regular. Thus $x \in F(x)$, whence N_{FK} is F-regular. Now let I be an F-regular k-ideal of S and $x \in I$; then $(x)_k \subseteqq I$. If $a \in (x)_k$ then $a \in I$, whence $a \in F(a)$. Thus $(x)_k$ is F-regular, so $x \in N_{FK}$. Therefore $I \subseteqq N_{FK}$.

A subdirect sum of semirings is defined analogously to a subdirect sum of rings, and a brief discussion can be found in [8]. In view of Theorem 2. 3 the following result is trivial.

Theorem 2. 6. *If a hemiring T is a subdirect sum of hemirings S_α, each of which has zero F-radical, then T has zero F-radical.*

A semiring S with zero element 0 is said to be *semi-isomorphic* to a semiring T with zero if there is a homomorphism from S onto T with 0 as its kernel.

Corollary 2. 7. *If a hemiring S is semi-isomorphic to a subdirect sum T of hemirings S_α, each of which has zero F-radical, then S has zero F-radical.*

Proof. Immediate from Theorems 2. 6 and 2. 3.

Definition 2. 8. A hemiring S is said to be of *type* (H) provided that if I is an h-ideal of S, and ν is the natural homomorphism of S onto S/I, then the image, under ν, of any h-ideal of S is an h-ideal of S/I.

As indicated in [7], not every hemiring is of type (H) but every additively regular hemiring and every additively periodic hemiring (in particular, any finite hemiring) is of this type; also, if S is of type (H) and I is any h-ideal of S, then S/I is of type (H). From Theorem 2. 3 we see that if I is any h-ideal of a hemiring S of type (H), then the natural homomorphism of S onto S/I maps the FH-radical of S into that of S/I. This last remark is essential to the proof of the next theorem, which parallels that of Theorem 2. 4.

Theorem 2. 9. *If S is a hemiring of type (H), then N_{FH} is contained in the intersection $\cap M$ of all h-ideals M such that S/M has zero FH-radical.*

3. *The F-radicals.* We now restrict the mappings F_S and impose an additional condition. Let C be defined as in Section 2, and with each S in C associate a fixed mapping F_S of S into the collection of all *h-ideals* of S subject to the following *two* conditions:

(i) Same as in Section 2;
(ii) If I is an h-ideal of S, if $\Sigma = S/I$, if ν: $a \to \bar{a}$ is the natural homomorphism of S onto Σ, and if $\overline{F_S(a)}$ is the image, under ν, of the h-ideal $F_S(a)$ of S, then $\overline{F_S(a)} = F_\Sigma(\bar{a})$.

As before, for each $S \in C$ such a mapping F_S always exists, and unless we specify otherwise all semirings considered are to be members of C.

Since C and condition (i) are unchanged, our earlier results are still valid. From this point on, however, our work will be affected by the fact that F_S maps S into its set of h-ideals and by condition (ii). If we require only that F_S map S into its set of k-ideals, and change the word h-ideal in condition (ii) to k-ideal, then results can be obtained that are different from, but similar to, those we are about to give. We shall say more about alternatives for F_S and condition (ii) in the final section.

Bourne and Zassenhaus ([2]) define the *zeroid* Z of S as $\{z \in S: z + x = x$ for some $x \in S\}$. Clearly Z is non-empty if S has an additive idempotent. Iizuka ([5]) gave an equivalent definition of Z for hemirings, and proved that in any hemiring Z is the intersection of all h-ideals. However, using the original definition it is easy to prove Iizuka's result in additively commutative semiring for which Z is non-empty. Thus for any $S \in C$ for which Z is non-empty, $Z \subseteqq F_S(a)$ for each $a \in S$, whence Z is an F_S-regular h-ideal. If $z \in Z$ then $(z)_h \subseteqq Z$, so that $(z)_h$ is an F_S-regular subset of S and $z \in N_{FH}$. Hence N_{FH} is non-empty if Z is non-empty.

The main result in this section is Theorem 3. 2, from which it follows that for a hemiring of type (H), N_{FH} is an h-ideal. First a trivial lemma.

Lemma 3. 1. *Let φ be a homomorphism of a semiring S onto a semiring T, and let A and M be complexes in S such that every h-ideal I that includes M properly most include A also. Then, if J is an h-ideal of T such that $M\varphi \subset J$, we have $A\varphi \subseteqq J$.*

We point out that this lemma still holds if we replace the world *h-ideal* by *k-ideal*, *semi-ideal*, or *subset*.

The reader is referred to [8] for a discussion of subdirect h-irreducibility in hemirings. For our present purposes it is sufficient to recall that a hemiring is subdirectly h-irreducible if and only if the intersection of all its non-zero two-sided h-ideals is non-zero. Also, we shall use the fact, established in [7], that if I is h-ideal of a hemiring S then S/I has zeroid equal to zero. Although the proof of Theorem 3. 2 closely parallels the proof for rings, we include it for completenes.

Theorem 3. 2. *If S is a hemiring then N_{FH} contains the intersection $\cap M$ of all h-ideals M of S such that S/M is subdirectly h-irreducible and has zero FH-radical.*

PROOF. Suppose $b \notin N_{FH}$; then for some $a \in (b)_h$, $a \notin F_S(a)$. If H denotes the class of all h-ideals of S that contain $F_S(a)$ but not a, then Zorn's lemma shows that H contains a maximal member, say M^*. Thus every h-ideal of S that properly contains M^* must contain a. Let $v; c \to \bar{c}$ be the natural mapping of S onto S/M^*, and consider the congruence class $\bar{a}$. Since $a \notin M^*$, $\bar{a} \neq \bar{0}$ $(= M^*)$. Furthermore, every h-ideal of S/M^*, in particular every non-zero h-ideal, contains $\overline{M^*} = \bar{0}$. Applying Lemma 3. 1, where the φ, T, A, and M of that lemma are our v, S/M^*, a, and M^*, respectively, we see that every non-zero h-ideal of S/M^* contains $\bar{a} (\neq \bar{0})$; S/M^* is subdirectly h-irreducible. Now let $T = S/M^*$. From $F_S(a) \subseteqq M^*$ and condition (ii), we have $F_T(\bar{a}) = \overline{F_S(a)} \subseteqq M^* = \bar{0}$, whence $F_T(\bar{a}) = \bar{0}$. Thus $\bar{a} \notin F_T(\bar{a})$ and every non-zero h-ideal of T contains the element $\bar{a}$ which is not F_T-regular. Since T has zeroid equal to zero, the zero of T is an h-ideal and is F_T-regular. It follows that T has zero FH-radical. Finally, if $b \in M^*$ then $(b)_h \subseteqq M^*$, whence $a \in (b)_h \subseteqq M^*$, contrary to $a \notin M^*$. Therefore if $b \notin N_{FH}$ then $b \notin M^*$ for some h-ideal M^* such that S/M^* is subdirectly h-irreducible and has zero FH-radical. Our assertion follows.

From Theorems 2. 9 and 3. 2 we immediately obtain

Corollary 3. 3. *If S is a hemiring of type (H) then N_{FH} is the intersection $\cap M$ of all h-ideals M of S such that S/M is subdirectly h-irreducible and has zero FH-radical, and hence is an h-ideal.*

If $N_{FH} = S$ then S is called an *FH-radical hemiring*. From Theorem 3. 2 we see that if the hemiring S itself is the only h-ideal M such that S/M is subdirectly h-irreducible with zero FH-radical, then S is an FH-radical hemiring. Moreover, Corollary 3. 3 yields

Corollary 3. 4. *Let S be a hemiring of type (H). Then S is an FH-radical hemiring if and only if S itself is the only h-ideal M such that S/M is subdirectly h-irreducible with zero FH-radical.*

Corollary 3. 5. *If S is a hemiring of type (H) then N_{FH} is the union of all F-regular h-ideals, the maximal F-regular h-ideal, and the h-ideal generated by the set of all F-regular h-ideals.*

Proof. Since, by Corollary 3. 3, N_{FH} is an h-ideal the assertions follow immediately from Theorem 2. 5.

Theorem 3. 6. *If a hemiring S has zero FH-radical then S is semi-isomorphic to a subdirect sum of subdirectly h-irreducible hemirings S/M, each of which has zero FH-radical.*

Proof. If $N_{FH} = 0$, and $\cap M$ is the intersection of all h-ideals M such that S/M is subdirectly h-irreducible with zero FH-radical, then by Theorem 3. 2, $\cap M \subseteqq N_{FH} = 0$. Thus $\cap M = 0$, whence it follows from Theorem 2. 2 of [8] that S is semi-isomorphic to a subdirect sum of the hemirings S/M.

The proof of the next theorem parallels that of the corresponding theorem for rings. Note that the hypothesis that S be of type (H) is not needed to establish the necessity of the condition.

Theorem 3. 7. *Let $S(\neq 0)$ be a subdirectly h-irreducible hemiring of type (H). Then N_{FH} is zero if and only if the minimal h-radical A (the intersection of all non-zero h-ideals of S) contains an element $b \neq 0$ such that $F(b) = 0$.*

The proof of our final theorem in this section is facilitated by the theorem, proved in [7], stating that if I and M are h-ideals in a hemiring S of type (H), and $I \subseteq M$, then S/M is isomorphic to $(S/I)/(M/I)$. Again, we omit the proof since it is analogous to a familar one in ring theory.

Theorem 3. 8. *If S is a hemiring of type (H) then S/N_{FH} has zero FH-radical.*

4. *The H-radical.* In this section we consider hemirings of type (H) and restrict the mappings F_S still more, thereby obtaining a radical analogous to the Brown—McCoy radical of a ring.

Let C be the class of all hemirings of type (H), and with each hemiring S in C associate a fixed mapping of S into the collection of all h-ideals of S in the following way: if $a \in S$, let $I_a = \{ax + x\}$ where $x \in S$, and then let $F_S(a) = (I_a)_h$, *i.e.*, the h-ideal of S generated by I_a. We now show that the mapping F_S so defined meets the conditions laid down at the beginning of Section 3.

For condition (i), suppose that S and Σ are in C and $\varphi\colon a \to \bar{a}$ is a homomorphism of S onto Σ. Let $\overline{F_S(a)}$ be the image, under φ, of the h-ideal $F_S(a)$ of S, and let $F_\Sigma(\bar{a})$ be the ideal of Σ associated with $\bar{a}$. We must show that $\overline{F_S(a)} \subseteq F_\Sigma(\bar{a})$. Now $F_S(a) = (I_a)_h$ and $F_\Sigma(\bar{a}) = (I_{\bar{a}})_h$. By Lemma 2. 2 $(I_a)_h\varphi \subseteq (I_a\varphi)_h$, that is, $\overline{(I_a)_h} \subseteq (\bar{I}_a)_h$. But $\bar{I}_a = \{\overline{ax+x}\} = \{\bar{a}\bar{x} + \bar{x}\} = I_{\bar{a}}$. Thus $(\bar{I}_a)_h = (I_{\bar{a}})_h$, so that

$$\overline{F_S(a)} = \overline{(I_a)_h} \subseteq (\bar{I}_a)_h = (I_{\bar{a}})_h = F_\Sigma(\bar{a}). \tag{1}$$

For condition (ii), suppose $S \in C$, and M is an h-ideal of S. Let $\nu\colon a \to \bar{a}$ be the natural homomorphism of S onto $\Sigma = S/M$. Now $\Sigma = S/M \in C$. Let $\overline{F_S(a)}$ be the image, under ν, of the h-ideal $F_S(a)$ of S, and let $F_\Sigma(\bar{a})$ be the h-ideal of Σ associated with $\bar{a}$. We must show that $\overline{F_S(a)} = F_\Sigma(\bar{a})$. Since S is of type (H), it follows from Lemma 2. 2 that $(I_a)_h\nu = (I_a\nu)_h$, that is, $\overline{(I_a)_h} = (\bar{I}_a)_h$, whence I becomes $\overline{F_S(a)} = \overline{(I_a)_h} = (\bar{I}_a)_h = (I_{\bar{a}})_h = F_\Sigma(\bar{a})$.

Thus for each $S \in C$ the F_S defined above meets conditions (i) and (ii) of section 3, and for the rest of this section we assume that every S in C has a mapping F_S associated with it in this way.

Definition 4. 1. Let S be a hemiring of type (H). The *LH-radical* of S is the set of all elements b in S for which the principal h-ideal $(b)_h$ is an F_S-regular subset of S, it is denoted by N_{LH}.

The L occurring in this definition in place of the (perhaps expected) F is a reminder of the asymmetry in the definition of F_S at the beginning of this section. Since the LH-radical of a hemiring of type (H) is but a special case of the FH-radical, all results obtained for the FH-radical apply here. It is the special character of the mapping F_S, however, that enables us to obtain additional results.

The Brown—McCoy radical of a *ring* R can be defined as follows (see [9]). For each $a \in R$, let $G(a)$ be the ideal of R generated by $\{ax + x\}$, where $x \in R$. The Brown—McCoy radical of R is the set of all elements $b \in R$ for which $a \in G(a)$ for every element a in the ideal of R generated by b. Thus if the hemiring S of type (H) is actually a ring, the LH-radical of S is just the Brown—McCoy radical.

Theorem 4. 2. *A hemiring S of type (H) that is subdirectly h-irreducible has zero LH-radical if and only if S is a simple ring with identity element.*

Proof. The theorem is trivial if $S=\{0\}$. Thus suppose $S\neq\{0\}$. If S is a simple ring with identity element e, then $F_S(-e)=(\{-ex+x\})_h=(0)_h=0$, whence, by Theorem 3. 7, S has zero LH-radical. Conversely, suppose that S has zero LH-radical. By Theorem 3. 7 the minimal h-ideal A of S contains an element $b\neq 0$ such that $F_S(b)=0$. Then $0=F_S(b)=(I_b)_h\supseteqq I_b=\{bx+x\}$, so $bx+x=0$ for all $x\in S$. Thus bx is an additive inverse for x, so S is a ring. Since $x=(-b)x$, we see that $-b$ is a left identity element of S. Now if $x\in S$ then $x=(-b)x\in Ax\subseteqq A$, so $S=A$ and S is a simple ring. That $-b$ is actually a two-sided identity element for S has been shown by Brown and McCoy [3]. The next theorem follows immediately from Corollary 3. 3 and Theorem 4. 2.

Theorem 4. 3. *If S is a hemiring of type (H) then the LH-radical of S is the intersection $\cap M$ of all h-ideals M of S such that S/M is a simple ring with identity element.*

By Theorem 2. 3 in [7] it follows that if S is a hemiring then the set of all h-ideals M of S such that S/M is a simple ring is precisely the set of all k-ideals M of S such that S/M is a simple ring. Thus we may change the word *h-ideals* to *k-ideals* in Theorem 4. 3.

We point out that if, when defining F_S, we take $I_a=\{xa+x\}$, where $x\in S$, and then define the RH-radical in the expected manner, dual arguments lead to Theorem 4. 3 for the RH-radical. Thus the asymmetry introduced when defining I_a affects nothing, and we henceforth drop the prefixes R and L and speak only of the H-radical N_H.

Theorem 4. 3 may be applied to show that the H-radicals of the hemiring I^+ of non-negative integers and the hemiring E^+ of non-negative even integers are zero.

We refer the reader to [5] for several definitions and results used in the proof of the next theorem.

Theorem 4. 4. *The Jacobson radical J of any hemiring S of type (H) is contained in the H-radical N_H of S.*

Proof. By Theorem 4. 3, N_H is the intersection $\cap M$ of all h-ideals M of S such that S/M is a simple ring with identity; let M be any such ideal and $S/M\neq\bar{0}$. The Brown—McCoy radical of the ring S/M is zero by Theorem 4. 2. Since, as shown in [3], this radical contains the Jacobson *ring* radical of S/M, which, from [8], coincides with the Jacobson *hemiring* radical of S/M, the Jacobson hemiring radical $\bar{J}$ of S/M is $\bar{0}$. By Theorem 6 of [5], $\bar{J}$ is the intersection of all primitive (in the semiring sense) h-ideals of S/M. Thus there is a primitive h-ideal N of S/M that is a proper subset of S/M. By Lemma 5 of [5], $N=(0{:}\,A)$ for some irreducible S/M-semimodule A. But, since N is an h-ideal (a ring ideal) of S/M, and since S/M is simple, $N=\bar{0}$. Therefore A is a faithful irreducible S/M-semimodule (see [5]). Thus S/M is primitive in the semiring sense and hence so is M (see Definition 7 of [5]). Since M is a primitive h-ideal of S, it follows from Theorem 6 of [5] that $J\subseteqq M$, whence $J\subseteqq\cap M=N_H$.

The following theorem is an analogue for hemirings of Theorem 8 of [3], which is a generalization of the Wedderburn—Artin theorem for rings.

Theorem 4. 5. *A hemiring S of type (H) has zero H-radical if and only if S is semi-isomorphic to a subdirect sum (in the semiring sense) of simple rings with identity elements.*

PROOF. If $N_H=0$ then the intersection of all h-ideals M of S such that S/M is a simple ring with identity is zero. By Theorem 2. 2 in [8], S is semi-isomorphic to a subdirect sum T of the rings S/M. Conversely, suppose S is semi-isomorphic to a subdirect sum T (in the semiring sense) of simple rings R_α, each of which has an identity element. Let $\varphi\colon s\to t=(r_1, r_2, \ldots)$ be a semi-isomorphism of S onto T. Since $(r_1, r_2, \ldots)\to r_\alpha$ maps T homomorphically onto R_α for each α, the mappings $\varphi_\alpha\colon s\to t\to r_\alpha$ are homomorphisms of S onto the R_α. If K_α is the kernel of φ_α then, by Theorem 2. 5 in [7], K_α is an h-ideal and S/K_α is semi-isomorphic to R_α under a mapping Ψ_α. Since any hemiring semi-isomorphic to a ring is itself a ring, each S/K_α is a simple ring with identity, and thus has zero H-radical. Now $\bigcap\limits_\alpha K_\alpha=0$. Let $b\in N_H$ and let v_α be the natural homomorphism of S upon S/K_α. By the remarks preceding Theorem 2. 9, bv_α is in the H-radical of S/K_α, that is, $bv_\alpha=\bar{0}_\alpha(=K_\alpha)$. Thus $b\in K_\alpha$ for each α, whence $b=0$ and our result follows.

We have previously remarked that every additively regular hemiring is of type (H). If, in Theorem 4. 5, we require that S be additively regular, the following result is obtained.

Theorem 4. 6. *Let S be an additively regular hemiring. If S has zero H-radical then S is a ring isomorphic to a subdirect sum (in the ring sense) of simple rings with identity elements. Conversely, if S is semi-isomorphic to a subdirect sum (in the semiring sense) of simple rings with identity elements, then S is a ring with zero H-radical.*

PROOF. If $N_H=0$ then, by Theorem 4. 4, S has zero Jacobson radical. The last result in [8] states that any additively regular hemiring with zero Jacobson radical is a ring. Since $N_H=0$, the intersection of all h-ideals M such that S/M is a simple ring with identity is zero, whence it follows that S is semi-isomorphic to a subdirect sum T of the rings S/M. Since S is a ring, any semi-isomorphism from S to T is an isomorphism. Conversely, if S is semi-isomorphic to a subdirect sum (in the semiring sense) of simple rings with identity elements, then $N_H=0$ by Theorem 4. 5. Thus $J\subseteq N_H$ implies $J=0$, whence as above S is a ring.

If S is any semiring with zero satisfying the minimal condition for semi-ideals [k-ideals, h-ideals], and C is any class of semi-ideals [k-ideals, h-ideals] of S with intersection zero, then some finite subclass of C has intersection zero. By using this result and by paralleling the proof of Theorem 9 in [3], we obtain the following analogue of the Wedderburn—Artin theorem.

Theorem 4. 7. *Let S be a hemiring of type (H) satisfying the minimal condition for h-ideals. The H-radical N_H of S is zero if and only if S is semi-isomorphic to the direct sum of a finite number of simple rings with identity elements.*

It is shown in [3] that the Brown—McCoy and Jacobson radicals coincide in any ring satisfying the minimal condition for right ideals. Although we have been unable to obtain such a result for arbitrary hemirings of type (H), we do have the following theorem.

Theorem 4. 8. *If S is an additively regular or additively periodic hemiring that satisfies the minimal condition for right h-ideals, then the Jacobson radical J of S coincides with the H-radical N_H of S.*

PROOF. Now S/J has zero Jacobson radical by [1], and since J is an h-ideal [8], S/J is of type (H). Also, S/J is a ring. For if S is additively regular, the last result in [8] shows S/J is a ring, while if S is additively periodic then, S/J having zeroid equal to zero by Theorem 2. 9 of [7], the proof of Lemma 9 in [2] shows that S/J is a ring. Since S/J is a ring, it follows from Theorem 3. 2 of [8] that S/J has zero Jacobson ring radical, and, since S/J also satisfies the minimal condition for right ideals, the Wedderburn—Artin theorem for rings shows that S/J is isomorphic to the direct sum of finitely many simple rings with identity elements. By Theorem 4. 5, S/J has zero H-radical. Now if $b \in N_H$ then, since S is of type (H) and J is an h-ideal, Theorem 2. 3 shows that if v is the natural mapping of S onto S/J then bv is in the H-radical of S/J. Thus $bv = \bar{0}$ $(=J)$, so $b \in J$ and $N_H \subseteqq J$. Since $J \subseteqq N_H$ by Theorem 4. 4, we have $J = N_H$.

5. *The H-radical of a Matrix Hemiring.* Most of the proofs in this section follow those given in [3] for rings.

If S is any semiring, the semiring of all matrices of order n with entries from S will be denoted by S_n. If S is a hemiring, or if S is an additively regular semiring, then S_n is clearly such a semiring. We shall later see that a hemiring S with an identity element is of type (H) if and only if S_n is of type (H). Given a semiring S, the following three statements are easily verified.

(a) If M is a semi-ideal [k-ideal, h-ideal] of S then M_n is a semi-ideal [k-ideal, h-ideal] of S_n.

(b) The mapping $M \to M_n$ is one-to-one from the semi-ideals [k-ideals, h-ideals] of S into those of S_n.

(c) If M and N are semi-ideals of S, then $M \subseteqq N$ if and only if $M_n \subseteqq N_n$.

The following two lemmas are proved in [3].

Lemma 5. 1. *Let S be a semiring with a zero element, i, j, p, q, and n positive integers with $i, j, p, q \leqq n$, $\mathscr{L}$ a semi-ideal of S_n and a an element in the (i, j) position of a matrix A in $\mathscr{L}$. If x and y are elements in S then $\mathscr{L}$ contains the matrix with xay in the (p, q) position and zero elsewhere.*

Lemma 5. 2. *A ring R is a simple ring with an identity element if and only if R_n is a simple ring with an identity element.*

Lemma 5. 3. *If S is a hemiring, and S_n is a ring, then S is a ring.*

PROOF. If $a \in S$, the matrix $(a)^{11}$ with a in the $(1, 1)$ position and zero elsewhere is in S_n, so there is a matrix B in S_n such that $(a)^{11} + B = 0$. Thus $a + b_{11} = 0$, and it follows that S is a ring.

The proof of the next lemma is detailed but quite straightforward, and we omit it.

Lemma 5. 4. *If S is a hemiring, and M is a semi-ideal of S, then $(S/M)_n \cong S_n/M_n$.*

Theorem 5. 5. *Let S be a hemiring and n a positive integer. If $\mathscr{L}$ is a k-ideal of S_n such that $S_n/\mathscr{L}$ has an identity element, and M is the set of all elements of S*

that appear as entries in at least one matrix in $\mathscr{L}$, then M is a k-ideal of S and $\mathscr{L}=M_n$. If $\mathscr{L}$ is an h-ideal of S_n then M is an h-ideal of S.

Proof. Since it is clear that $\mathscr{L}\subseteq M_n$, we need only show $M_n\subseteq\mathscr{L}$ to have $\mathscr{L}=M_n$. Since $S_n/\mathscr{L}$ has an identity element, S_n contains at least one matrix $U=(u_{pq})$ such that, for every element X of S_n, $UXU\equiv X(\mathscr{L})$. Let $A=(a_{ij})$ be an element of M_n; if, in general, $(x_{pq})^0$ denotes the matrix with x_{pq} in the (p, q) position and zero elsewhere, then $U(a_{ij})^0U=(b_{kl})$ where $b_{kl}=u_{ki}a_{ij}u_{jl}(k, l=1, ..., n)$. For arbitrary k and l it follows from Lemma 5. 1 that $(b_{kl})^0\in\mathscr{L}$, so that $U(a_{ij})^0U$ is the sum of n^2 matrices in $\mathscr{L}$ and hence is itself in $\mathscr{L}$. Since $\mathscr{L}$ is a k-ideal of S_n, $U(a_{ij})^0U\equiv(a_{ij})^0$ modulo $\mathscr{L}$ implies that $(a_{ij})^0\in\mathscr{L}$. Then, since $A=\sum_{i,j=1}^{n}(a_{ij})^0$, we have $A\in\mathscr{L}$, whence $M_n\subseteq\mathscr{L}$.

Now let M^* denote the set of all elements of S that appear in the $(1, 1)$ position in some matrix in $\mathscr{L}$. Clearly $M^*\subseteq M$. Moreover, if $\mathscr{L}$ is an h-ideal of S_n then M^* is an h-ideal of S. For suppose there are elements $a, b_{11}\in M^*$ and $x, z\in S$ such that $x+a+z=b_{11}+z$. Let (b_{ij}) be a matrix in $\mathscr{L}$ with b_{11} in the $(1, 1)$ position, X the matrix with x in the $(1, 1)$ position and b_{ij} in the (i, j) position for i and j not both 1, and, for arbitrary $y\in S$, $(y)^{11}$ the matrix with y in the $(1, 1)$ position and zero elsewhere. Then $X+(a)^{11}+(z)^{11}=(b_{ij})+(z)^{11}$. Since $M^*\subseteq M$, we have $(a)^{11}\in M_n^*\subseteq M_n=\mathscr{L}$. Thus, since $\mathscr{L}$ is an h-ideal, since (b_{ij}) and $(a)^{11}$ are in $\mathscr{L}$, and $X+(a)^{11}+(z)^{11}=(b_{ij})+(z)^{11}$, we have $X\in\mathscr{L}$, and so $x\in M^*$. Thus M^* is an h-ideal of S. Similarly, if $\mathscr{L}$ is assumed to be only a k-ideal of S_n then M^* is a k-ideal of S. Since $M^*\subseteq M$, we need only show that $M\subseteq M^*$ to complete the proof.

We recall that for every matrix X in S_n, $UXU\equiv X(\mathscr{L})$. If, in particular, X is chosen as $X=(x)^{11}$ for arbitrary $x\in S$, it follows from $UXU\equiv X(\mathscr{L})$ that $u_{11}xu_{11}\equiv\equiv x(M^*)$. Now if $b\in M$ there is a matrix in $\mathscr{L}$ with b in some position, whence Lemma 5. 1 shows that $(u_{11}bu_{11})^{11}\in\mathscr{L}$, and so $u_{11}bu_{11}\in M^*$. Thus $u_{11}bu_{11}\equiv b(M^*)$ implies that $b\in M^*$. Therefore $M\subseteq M^*$.

Corollary 5. 6. *If M is a semi-ideal in a hemiring S, and S/M is a simple ring with an identity element, then S_n/M_n is a simple ring with an identity element. If S is a hemiring and $\mathscr{L}$ is an h-ideal [k-ideal] of S_n such that $S_n/\mathscr{L}$ is a simple ring with an identity element, then the set M of all elements of S that appear as entries in at least one matrix in $\mathscr{L}$ is an h-ideal [k-ideal] of S, $\mathscr{L}=M_n$, and S/M is a simple ring with an identity element.*

Proof. The first assertion follows from Lemma 5. 2 and Lemma 5. 4, and the next two assertions follow from Theorem 5. 5. Finally,since $(S/M)_n\cong S_n/M_n==S_n/\mathscr{L}$ by Lemma 5. 4, and since $S_n/\mathscr{L}$ is a simple ring with an identity element, Lemma 5. 2 and Lemma 5. 3 show that S/M is a simple ring with an identity element.

Theorem 5. 7. *Let S be a hemiring of type (H) with H-radical N. If S_n is of type (H) then the H-radical of S_n is N_n.*

Proof. Under the one-to-one mapping $M\to M_n$ of the h-ideals M of S into the h-ideals of S_n, the class of all h-ideals M of S such that S/M is a simple ring with identity is mapped one-to-one onto the class of all h-ideals $\mathscr{L}$ of S_n such that $S_n/\mathscr{L}$ is a simple ring with identity. This follows from Lemmas 5. 2, 5. 4, and Corollary 5. 6. Our result now follows readily from Theorem 4. 3.

Theorem 5. 8. *Let S be a hemiring, n a positive integer, and suppose that if $\mathscr{L}$ is an h-ideal of S_n then $\mathscr{L}=M_n$ for some h-ideal M of S. Then S is of type (H) if and only if S_n is of type (H).*

Proof. For any $x\in S$, and any pair of integers i, j $(1\leqq i, j\leqq n)$, we denote by $(x)^{ij}$ the matrix in S_n with x in the (i, j) position and 0 elsewhere.

Suppose that S is of type (H), $\mathscr{L}$ and ϑ are h-ideals of S_n, and Γ is the natural homomorphism of S_n onto $S_n/\mathscr{L}$. We must show that $\vartheta\Gamma$ is an h-ideal of $S_n/\mathscr{L}$. By hypothesis, there exist h-ideals I, M of S such that $I_n=\vartheta$ and $M_n=\mathscr{L}$. To show $I_n\Gamma$ an h-ideal of S_n/M_n, we suppose that $X\Gamma+U\Gamma+Z\Gamma=V\Gamma+Z\Gamma$ where $U, V\in I_n$ and then show that $X\Gamma\in I_n\Gamma$. From $(X+U+Z)\Gamma=(V+Z)\Gamma$, there exist matrices $A, B\in M_n$ such that $(X+U+Z)+A=(V+Z)+B$. If, for each pair i, j $(1\leqq i, j\leqq n)$, we denote by x_{ij}, u_{ij}, v_{ij}, z_{ij}, a_{ij}, and b_{ij} the (i, j) entries in X, U, V, Z, A, and B, respectively, then $x_{ij}+u_{ij}+z_{ij}+a_{ij}=v_{ij}+z_{ij}+b_{ij}$. Thus if v is the natural homomorphism of S onto S/M we have $x_{ij}v+u_{ij}v+z_{ij}v+a_{ij}v=v_{ij}v+z_{ij}v+b_{ij}v$. Since $a_{ij}, b_{ij}\in M$, $a_{ij}v=b_{ij}v=M$, and, since M is the zero of S/M, we have $x_{ij}v+u_{ij}v+z_{ij}v=v_{ij}v+z_{ij}v$. Since $u_{ij}, v_{ij}\in I$, and since, by our assumption that S is of type (H), Iv is an h-ideal of S/M, we have $x_{ij}v\in Iv$. Therefore $x_{ij}v=y_{ij}v$ for some $y_{ij}\in I$, whence there exist $m_1, m_2\in M$ such that $x_{ij}+m_1=y_{ij}+m_2$. Hence $(x_{ij})^0+(m_1)^{ij}=(y_{ij})^0+(m_2)^{ij}$, $(x_{ij})^0$ being the matrix with x_{ij} in the (i, j) position and 0 elsewhere. Since $(m_1)^{ij}, (m_2)^{ij}\in M_n$ we have $(x_{ij})^0\equiv(y_{ij})^0(M_n)$, whence $(x_{ij})^0\Gamma=(y_{ij})^0\Gamma\in I_n\Gamma$. Now $X=\sum_{i,j=1}^{n}(x_{ij})^0$ and $Y=\sum_{i,j=1}^{n}(y_{ij})^0\in I_n$. Thus

$$X\Gamma=\left(\sum_{i,j=1}^{n}(x_{ij})^0\right)\Gamma=\sum_{i,j=1}^{n}(x_{ij})^0\Gamma=\sum_{i,j=1}^{n}(y_{ij})_0\Gamma=\left(\sum_{i,j=1}^{n}(y_{ij})^0\right)\Gamma=Y\Gamma\in I_n\Gamma.$$

So $I_n\Gamma$ is an h-ideal of S_n/M_n and S_n is of type (H).

Conversely, suppose that S_n is of type (H). Let I and M be h-ideals of S and v the natural homomorphism of S onto S/M. Now I_n and M_n are h-ideals of S_n and if Γ is the natural homomorphism of S_n onto S_n/M_n then, by our supposition that S_n is of type (H), $I_n\Gamma$ is an h-ideal of S_n/M_n. To show that Iv is an h-ideal of S/M we suppose that $xv+i_1v+zv=i_2v+zv$, where $i_1, i_2\in I$, and then show $xv\in Iv$. From $(x+i_1+z)v(z)^{11}(i_2+z)v$ there exist elements $m_1, m_2\in M$ such that $(x+i_1+z)+m_1=(i_2+z)+m_2$. Therefore $(x)^{11}+(i_1)^{11}+(z)^{11}+(m_1)^{11}=(i_2)^{11}+(z)^{11}+(m_2)^{11}$, whence $(x)^{11}\Gamma+(i_1)^{11}\Gamma+(z)^{11}\Gamma+(m_1)^{11}\Gamma=(i_2)^{11}\Gamma+(z)^{11}\Gamma+(m_2)^{11}\Gamma$. Since $(m_1)^{11}$ and $(m_2)^{11}$ are in M_n, and M_n ist he zero of S_n/M_n, wehave $(x)^{11}\Gamma+(i_1)^{11}\Gamma+(z)^{11}\Gamma=(i_2)^{11}\Gamma+(z)^{11}\Gamma$. But $(i_1)^{11}$ and $(i_2)^{11}$ are in I_n, and, since $I_n\Gamma$ is an h-ideal of S_n/M_n, we have $(x)^{11}\Gamma\in I_n\Gamma$. Thus $(x)^{11}\Gamma=Y\Gamma$ for some $Y\in I_n$, say $Y=(i_{jk})$. Now there exist matrices $P=(p_{jk})$, $N=(n_{jk})$ in M_n such that $(x)^{11}+P=Y+N$, whence $x+p_{11}=i_{11}+n_{11}$. Thus $xv+p_{11}v=i_{11}v+n_{11}v$, and, since $p_{11}v=n_{11}v=M$ and M is the zero of S/M, we have $xv=i_{11}v\in Iv$. So Iv is an h-ideal of S/M and S is of type (H).

The following theorem and its corollary give conditions that insure that the hypotheses of Theorem 5. 8 hold. The corresponding theorem for rings may be found in [6]. Before giving the theorem, however, we need the following definition.

Definition 5. 9. If S is a semiring and $x\in S$ then $\& x \&$ denotes the set of all finite sums of the form $\sum_{k=1}^{n} a_kxb_k$ where $a_k, b_k\in S$.

Theorem 5. 10. *Let S be any hemiring such that x is in $\& x \&$ for each $x \in S$, and let n be a positive integer. If $\mathscr{L}$ is a semi-ideal [k-ideal, h-ideal] of S_n, and M is the set of all elements in S that appear as entries in at least one matrix in $\mathscr{L}$, then $\mathscr{L} = M_n$ and M is a semi-ideal [k-ideal, h-ideal] of S.*

PROOF. Let $(m_{ij}) \in M_n$, and let i, j be any pair of positive integers with $1 \leqq i, j \leqq n$. Since $m_j \in \& m_{ij} \&$ we have, say, $m_{ij} = \sum_{k=1}^{l} a_k m_{ij} b_k$. By lemma 5. 1, $(a_k m_{ij} b_k)^{ij} \in \mathscr{L}$ for each $k (1 \leqq k \leqq l)$. Thus from $(m_{ij})^0 = \sum_{k=1}^{l} (a_k m_{ij} b_k)^{ij}$ it follows that $(m_{ij})^0 \in \mathscr{L}$. Since $(m_{ij}) = \sum_{i,j=1}^{n} (m_{ij})^0$, we have $(m_{ij}) \in \mathscr{L}$, whence $M_n \subseteqq \mathscr{L}$. Since $\mathscr{L} \subseteqq M_n$ we have $M_n = \mathscr{L}$. Now M is a semi-ideal; for suppose $m_1, m_2 \in M$. Now $m_1 \in \& m_1 \&$, say $m_1 = \sum_{k=1}^{l} a_k m_1 b_k$. As above we have $(m_1)^{11} \in \mathscr{L}$. Similarly $(m_2)^{11} \in \mathscr{L}$, whence $(m_1 + m_2)^{11} = (m_1)^{11} + (m_2)^{11} \in \mathscr{L}$, which implies that $m_1 + m_2 \in M$. Since $m_1 \in M$, $m_1 = u_{pq}$ for some matrix $U = (u_{ij}) \in \mathscr{L}$. Let $a \in S$. Now $(a)^{1p} U \in \mathscr{L}$ since $\mathscr{L}$ is a semi-ideal, and the $(1, q)$ entry in $(a)^{1p} U$ is $a u_{pq} = a m_1$. Thus $a m_1 \in M$ so that M is a left semi-ideal of S. Similarly, M is a right semi-ideal of S. Now suppose $\mathscr{L}$ is a k-ideal of S_n, and $x + m_1 = m_2$ with $m_1, m_2 \in M$. As in the foregoing argument, $(m_1)^{11}$, $(m_2)^{11} \in \mathscr{L}$. Since $(x)^{11} + (m_1)^{11} = (m_2)^{11}$, it follows from the fact that $\mathscr{L}$ is a k-ideal that $(x)^{11} \in \mathscr{L}$, whence $x \in M$. In a similar manner we see that M is an h-ideal of S if $\mathscr{L}$ is an h-ideal of S_n.

Corollary 5. 11. *Let S be an arbitrary hemiring with identity element, and let n be any positive integer. Then if $\mathscr{L}$ is a semi-ideal [k-ideal, h-ideal] of S_n, $\mathscr{L} = M_n$ for some semi-ideal [k-ideal, h-ideal] M of S.*

Combining Corollary 5. 11, Theorem 5. 7, and Theorem 5. 8 we have

Theorem 5. 12. *If N is the H-radical of a hemiring S of type (H) with identity element, and n is a positive integer, then the H-radical of S_n is N_n.*

6. *Other Radicals.* We recall that the results of Section 2 were based upon mapping each additively commutative semiring S into its collection of non-empty subsets under a mapping F_S subject to a single condition. It was not until section 3 that we restricted F_S to map into the h-ideals of S and required that it meet a second condition. In view of the three concepts of ideal in a semiring and the two congruence relations (Bourne and Iizuka), it is natural to inquire whether a radical with some useful theory can be obtained by mapping S into its semi-ideals or k-ideals only, and changing condition (ii) of Section 3 in some way. In this section we indicate briefly our results along these lines.

First of all, it does not appear that much information can be obtained in addition to that of Section 2 by requiring only that F_S map S into its collection of semi-ideals subject to condition (i), even with some condition similar to condition (ii). For with such a mapping we have been unable to obtain anything like Theorem 3. 2, a theorem which proved essential to our later work. The crux of the proof of Theorem 3. 2 is that M^* is the zero element of S/M^*, a fact which depends upon M^* being a k-ideal and not just a semi-ideal. In the proof, however, M^* is actually an h-ideal, and perhaps something more than we really need. Thus we might try mapping S into its collection of k-ideals, and, since this proves useful, we fix the ideas more precisely.

Let C be the class of all additively commutative semirings, and with each S in C associate a fixed mapping F_S of S into the collection of its k-ideals subject to the following two conditions:

(i) Same as in Section 2;

(ii)′ Same as in Section 3 except to change h-ideal to k-ideal.

The F-, FK-, and FH-radicals are defined as before. With this mapping an analogue to each result in Section 3 can be obtained in which the FH-radical is replaced by the FK-radical and h-ideals by k-ideals. Also the concept of subdirect h-irreducibility is replaced by that of subdirect irreducibility, the latter being in some way preferable (as indicated in [8]). However, type (H) is no longer a useful condition and must be replaced whenever it occurs by a condition we call *type* (K). Semirings of type (K) are defined by Definition 2. 8 modified by replacing the word h-ideal by k-ideal and the letter H by K. Although additively periodic hemirings are of type (H), even finite hemirings need not be of type (K). Nor need additively regular hemirings be of type (K). However, we remark that requiring that a hemiring S be of type (K) is actually weaker than requiring that under an arbitrary homomorphism of S its k-ideals be preserved; the corresponding question concerning type (H) is unanswered.

The results of section 4 may also be modified to give a theory of a *K-radical* of a hemiring of type (K). We begin by letting C be the class of all hemirings of type (K), and defining $F_S(a)=(I_a)_k$, that is, the k-ideal generated by I_a. Definition 4. 1 is modified in the obvious way and we speak of the LK-radical.

Again, if S is a hemiring of type (K) that is actually a ring, the LK-radical is just the Brown—McCoy radical of S. By replacing the *h-concepts* by the corresponding *k-concepts*, that is, by replacing type (H), h-ideals, subdirect h-irreducibility, and LH-radical by type (K), k-ideals, subdirect irreducibility, and LK-radical, respectively, the proofs of Theorems 4. 2 and 4. 3 carry over without difficulty. The remarks following Theorem 4. 3 also cary over so that we may speak of the *K-radical*. It is perhaps more interesting to note that the K-radical of a hemiring of type (K) is the intersection of all h-ideals M such that S/M is a simple ring with an identity element, and hence is an h-ideal of S.

Theorems 4. 4, 4. 5, 4. 6, and 4. 8 remain valid with *k-concepts* replacing *h-concepts*, except that we must add to the hypotheses of Theorems 4. 6 and 4. 8 that S is of type (K). Similarly, Theorem 4. 7 carries over, but here we can require the minimum condition either for k-ideals or h-ideals in view of our earlier statement that the K-radical is the intersection of a class of h-ideals.

Finally, all results in Section 5 carry over readily with k-concepts.

Now with both the K-radical and the H-radical we have used the Bourne congruence relation and factor system exclusively. Can anything of interest be obtained by using the Iizuka congruence relation and factor system? The answer appears to be *no* if we merely map a hemiring S into its collection of semi-ideals or k-ideals, because unless the ideal M of S is an h-ideal it is not the zero element of $S[/]M$. It is for precisely this reason that we have again been unable to obtain anything like Theorem 3. 2 using the Iizuka relation. However, by mapping into the h-ideals, the Iizuka relation can be used to obtain a theory parallel to that given in sections 2-5. We begin by returning to Section 2 and introducing the concept that is to replace that of type (H).

Definition 6. 1. An additively commutative semiring S is said to be of *type* (Q) provided that if I is an h-ideal of S, and v is the natural homomorphism of S onto $S[/]I$, then the image, under v, of any h-ideal of S is an h-ideal of $S[/]I$.

Every additively regular or additively periodic hemiring is of type (Q), and, since type (Q) coincides with type (H) in the presence of additive cancellation, both I^+ and E^+ are of type (Q) and an example shows that not all hemirings are of type (Q).

To obtain analogues of the results in Section 3 of [7], we replace type (H) by type (Q), the Bourne factor system by the Iizuka factor system, and k-ideals by h-ideals.

In Section 3 of the present paper, we restrict the mapping F_S to map S into its collection of h-ideals subject to the now familiar condition (i) and condition (ii)″, condition (ii)″ being condition (ii) with S/I replaced by $S[/]I$. All results in this section now carry over by merely replacing the Bourne factor system by the Iizuka system wherever the former occurs, and likewise replacing type (H) by type (Q).

To modify Section 4, we let C be the class of all hemirings of type (Q), and define F_S for $S \in C$ exactly as in Section 4. It is routine to verify that F_S meets conditions (i) and (ii)″. We leave Definition 4. 1 unchanged except for replacing type (H) by type (Q), and, as before, if a hemiring of type (Q) is actually a ring then the LH-radical is just the Brown—McCoy radical. Continuing to replace type (H) by type (Q) and the Bourne factor system by the Iizuka system, all results in this section carry over, with one exception to be discussed below. In all cases, the proofs are modified very little. The one exception occurs in the assertion immediately following Theorem 4. 3, *i.e.*, it is *not* ture that if S is any hemiring then the set of all h-ideals M of S such that $S[/]M$ is a simple ring coincides with the set of all k-ideals M of S such that $S[/]M$ is a simple ring.

Finally, replacing type (H) by type (Q), the Bourne factor system by the Iizuka system, and k-ideals by h-ideals, all results in Section 5 carry over.

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