Generalizations of two theorems of Meier concerning boundary behavior of meromorphic functions

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In this paper*) we shall prove two theorems concerning the boundary behavior of functions meromorphic in the open unit disk. These theorems may be regarded as modifications, made possible by application of the ambiguous-point theorem ([1]), of two theorems of Meier ([5], p. 329, Theorems 3 and 4). We replace a hypothesis of Meier's concerning chords by a similar hypothesis concerning more general curves, thereby weakening Meier's assumptions. Our conclusions are weaker than Meier's in that we are able to make assertions concerning only the global range at a point instead of the angular range; whether this is so of necessity or not, we do not know (see the Remark at the conclusion of the paper).

MEIER has also proved two theorems about holomorphic functions ([5], p. 330, Theorems 7 and 8) which are special cases of the two theorems of his mentioned above. Noshiro ([6], p. 74) has generalized one of these, and I have generalized both of them ([2], p. 423; [3], Theorem 8). The two theorems established in the present paper contain all these generalizations as special cases.

Denote the open unit disk by D, the unit circle by Γ, and the Riemann sphere by Ω. By an arc at a point $\zeta \in \Gamma$ we mean a simple continuous curve $\Lambda: z = z(t)$ $(0 \leqq t < 1)$ such that $|z(t)| < 1$ for $0 \leqq t < 1$ and $z(t) \to \zeta$ as $t \to 1$. In particular, if Λ is rectilinear, we speak of a chord at ζ; if Λ is a subarc of a circle that is internally tangent to Γ at ζ, we speak of a horocyclic arc at ζ. A terminal subarc of an arc Λ at ζ is a subarc of Λ of the form $z = z(t)$ $(t_0 \leqq t < 1)$, where $0 \leqq t_0 < 1$. If $\Lambda_1, \Lambda_2, \Lambda_3$ are three arcs at a point $\zeta \in \Gamma$, we say that Λ_1 and Λ_3 are separated by Λ_2 provided that there exist terminal subarcs $\Lambda_1', \Lambda_2', \Lambda_3'$ of $\Lambda_1, \Lambda_2, \Lambda_3$, respectively, such that Λ_2' lies between Λ_1' and Λ_3'. Finally, if $\mathscr{E}$ is an at most enumerable set of arcs at the point 1, then for every $\zeta \in \Gamma$ we let $\mathscr{E}_\zeta$ denote the set of arcs at ζ obtained by rotating each arc in $\mathscr{E}$ about the origin through the angle $\arg \zeta$.

Let $\zeta \in \Gamma$ and suppose that $0 < r < 1$. Then the circle of radius r internally tangent to Γ at ζ is called a horocycle at ζ. If $0 < r_1 < r_2 < 1$, $0 < r_3 < 1$, and if r_3 is so large that the circle $|z| = r_3$ intersects both horocycles at ζ with radii r_1 and r_2, then each of the two regions lying between the two horocycles at ζ as well as in the exterior of the circle $|z| = r_3$ will be termed a horocyclic angle at ζ.

Now suppose that $f(z)$ is a single-valued function defined in D whose values belong to Ω, and let $\zeta \in \Gamma$. Then as is customary (see [6]) we shall denote the cluster

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set of f at ζ by $C(f, \zeta)$ and the range of f at ζ by $R(f, \zeta)$. If Λ is an arc at ζ, and if Δ is a Stolz angle or a horocyclic angle at ζ, then $C_\Lambda(f, \zeta)$ and $C_\Delta(f, \zeta)$ stand for the cluster set of f at ζ relative to Λ, Δ, respectively. The angular range of f at ζ is defined by Meier [5, p. 328] to be the set $A(f, \zeta)$ of all values $\omega \in \Omega$ with the property that f assumes the value ω in every Stolz angle at ζ arbitrarily close to ζ.

The set of all Fatou points of ζ will be denoted as usual by $F(f)$. We call a point $\zeta \in \Gamma$ a generalized Plessner point of f, provided that for every Stolz angle and every horocyclic angle Δ at ζ, we have $C_\Delta(f, \zeta) = \Omega$. The set of all generalized Plessner points of f will be denoted by $I^*(f)$. We call a point $\zeta \in \Gamma$ a generalized Meier point of f, provided that $C(f, \zeta)$ is a proper subset of Ω and for every chord and every horocyclic arc Λ at ζ, we have $C_\Lambda(f, \zeta) = C(f, \zeta)$. The set of all generalized Meier points of f will be denoted by $M^*(f)$.

Definition 1. *The set $P(f, \zeta)$ is the set of all points $\alpha \in \Omega$ with the property that there exist arcs Λ_1, Λ_2 at ζ, separated by a Stolz angle or a horocyclic angle at ζ, for which*

$$\alpha \notin C_{\Lambda_1}(f, \zeta) \cup C_{\Lambda_2}(f, \zeta).$$

Definition 2. *Suppose that $\mathscr{E}$ is an at most enumerable set of arcs at unity. The set $Q_{\mathscr{E}}(f, \zeta)$ is the set of all points $\beta \in \Omega$ with the property that there exist arcs Λ_1, Λ_2 at ζ, separated by some arc in $\mathscr{E}_\zeta$, for which*

$$\beta \notin C_{\Lambda_1}(f, \zeta) \cup C_{\Lambda_2}(f, \zeta).$$

Theorem 1. *Let $f(z)$ be a meromorphic function in D. Then there exists a subset Z of Γ of measure zero such that for every $\zeta \in \Gamma - Z$ either $\zeta \in F(f)$ or $P(f, \zeta) \subseteqq R(f, \zeta)$.*

PROOF. According to [3], Corollary 1,

$$\Gamma = F(f) \cup I^*(f) \cup Z',$$

where Z' is a subset of Γ of measure zero. Let S denote the set of points of $I^*(f)$ for which $P(f, \zeta) \nsubseteq R(f, \zeta)$. We shall show that S is an at most enumerable set. If we then define Z to be the set $S \cup Z'$, the conclusion of Theorem 1 evidently holds.

Consider any point $\zeta \in S$. Since $P(f, \zeta) \nsubseteq R(f, \zeta)$, there exists a value $\alpha \in P(f, \zeta)$ such that $\alpha \notin R(f, \zeta)$. The fact that $\alpha \in P(f, \zeta)$ implies the existence of arcs Λ_1, Λ_2 at ζ, separated by a Stolz angle or a horocyclic angle Δ at ζ, with the property that

$$\alpha \notin C_{\Lambda_1}(f, \zeta) \cup C_{\Lambda_2}(f, \zeta).$$

Now $\zeta \in I^*(f)$, and therefore $C_\Delta(f, \zeta) = \Omega$; in particular, $\alpha \in C_\Delta(f, \zeta)$. In view of the fact that $\alpha \notin R(f, \zeta)$, Iversen's theorem enables us to conclude that α is an asymptotic value of f at the point ζ. But then ζ is an ambiguous point f. The ambiguous-point theorem ([1], p. 380, Theorem 2) shows that S is at most enumerable.

Theorem 2. *Let $f(z)$ be a meromorphic function in D and $\mathscr{E}$ be an at most enumerable set of arcs at unity. Then there exists a subset Y of Γ of first category such that for every $\zeta \in \Gamma - Y$ either $\zeta \in M^*(f)$ or $Q_{\mathscr{E}}(f, \zeta) \subseteqq R(f, \zeta)$.*

PROOF. According to [3], Corollary 5,

$$\Gamma = M^*(f) \cup I^*(f) \cup Y',$$

where Y' is a subset of Γ of first category. Let T denote the set of points of $I^*(f)$ for which $Q_{\mathscr{E}}(f,\zeta) \subsetneqq R(f,\zeta)$. We shall show that T is a set of first category. If we then define Y to be the set $T \cup Y'$, the conclusion of Theorem 2 evidently holds.

Assume, to the contrary, that T is a set of second category. Since $T \subseteqq I^*(f)$, we have $C(f,\zeta) = \Omega$ for every $\zeta \in T$. By a theorem of Collingwood ([4], p. 381, Corollary 1), there exists a subset T' of T of second category with the property that, for every $\zeta \in T'$, the cluster set of f at ζ along each arc belonging to $\mathscr{E}_\zeta$ is Ω, and hence, in particular, contains β. Consider any point $\zeta \in T'$. Since $Q_{\mathscr{E}}(f,\zeta) \subsetneqq R(f,\zeta)$, there exists a value $\beta \in Q_{\mathscr{E}}(f,\zeta)$ such that $\beta \notin R(f,\zeta)$. The fact that $\beta \in Q_{\mathscr{E}}(f,\zeta)$ implies the existence of arcs Λ_1, Λ_2 at ζ, separated by some arc in $\mathscr{E}_\zeta$, for which

$$\beta \notin C_{\Lambda_1}(f,\zeta) \cup C_{\Lambda_2}(f,\zeta).$$

In view of the relation $\beta \notin R(f,\zeta)$, Iversen's theorem enables us to conclude that β is an asymptotic value of f at the point ζ, which makes ζ an ambiguous point of f. This implies that T' is at most enumerable, which contradicts the fact that T' is of second category. Thus our assumption is untenable.

Remark. It would be interesting to ascertain whether or not it is possible in Theorems 1 and 2 to replace $R(f,\zeta)$ by $\Lambda(f,\zeta)$.

Bibliography

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