On Taylor series absolutely convergent on the circumference of the circle of convergence I.

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P. TURÁN has initiated the following problem: How does the behaviour of a Taylor series change at the periphery of its circle of convergence under conformal mappings of the circle onto itself? To be precise, let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

be regular for $|z|<1$ and let us make the substitution $z=T(w)$:

$$f(T(w)) = \sum_{n=0}^{\infty} b_n w^n$$

where $T(w)$ is a conformal mapping of the unit circle onto itself the most general form which is

$$z = T(w) = c\frac{w-\zeta}{1-w\bar{\zeta}} \quad (|c| = 1, \quad |\zeta| < 1 \text{ are const}).$$

The question is what the relation is between $\sum_{k=0}^{\infty} a_k z^k$ and $\sum_{n=0}^{\infty} b_n w^n$ in terms of convergence, summability, e.t.c. at corresponding points of $|z|=1$ and $|w|=1$. The first theorem in this direction is due to *P.* Turán himself ([1]) and afterwards L. ALPÁR has obtained a great number of interesting results. One of them is that absolute convergence is not always preserved: $\sum_{k=0}^{\infty} |a_k|$ can converge without $\sum_{n=0}^{\infty} |b_n|$ being finite (see [2]). Nevertheless, for all functions $f(z)$ satisfying $\sum_{k=0}^{\infty}|a_k| < +\infty$

$$\sum_{n=0}^{\infty} |b_n|^2 < +\infty$$

by Parseval's inequality since $f(z)$ is then continuous on the closed disk $|z| \leqq 1$. Therefore he raised the question in his same article [2]: Does there exist a function $f_0(z)$ for which

$$\sum_{k=0}^{\infty} |a_k| < +\infty$$

but
$$\sum_{n=0}^{\infty} |b_n|^{2-\varepsilon} = +\infty$$

however small the positive ε should be? In this paper we shall construct such an example $f_0(z)$ and at the same time the best possible one in this connection.

Theorem. *Let* $0<|\zeta|<1$, $|c|=1$, $0<\omega(n)\to+\infty$ *be given in advance. Then there exists a function*
$$f_0(z) = \sum_{k=0}^{\infty} a_k z^k$$
such that
$$\sum_{k=0}^{\infty} |a_k| < +\infty$$
and if
$$f_0(T(w)) = f_0\left(c\,\frac{w-\zeta}{1-w\zeta}\right) = \sum_{n=0}^{\infty} b_n w^n$$
then

(1)
$$\sum_{n=0}^{\infty} |b_n|^{2-\frac{\omega(n)}{\log n}} = +\infty.$$

On the other hand, it is an elementary fact that in case $\omega(n)=O(1)$ the statement is no longer true[1]. There is still a gap between $\omega(n)\to+\infty$ and $\omega(n)=O(1)$, but for the problem in question it is perhaps not very interesting. With slight modification, the proof of footnote [1]) could be applied to the case when instead of $\omega(n)=O(1)$ there exists a d such that
$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{d}{\omega(n)}}}$$
and probably our theorem is valid if such a constant does not exist.

Now, putting the Taylor series of $T(w)$ into that of $f(z)$, we obtain that the relation between $\{a_k\}$ and $\{b_n\}$ is given by a linear transformation
$$b_n = \sum_{k=0}^{\infty} t_{kn} a_k$$

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[1]) Proof. $\sum_{n=0}^{\infty} |b_n|^2 < +\infty$, while $\sum_{n=2}^{\infty} |b_n|^{2-\frac{\text{const}}{\text{og } n}} < +\infty$ is to be proved.

If $|b_n| < \frac{1}{n^2}$ then $|b_n|^{2-\frac{\text{const}}{\log n}} \leqq |b_n| < \frac{1}{n^2}$ for $n \geqq N_0 (\geqq 2)$.

If $|b_n| \geqq \frac{1}{n^2}$ then $|b_n|^{2-\frac{\text{const}}{\log n}} \leqq |b_n|^2 \left(\frac{1}{n^2}\right)^{-\frac{\text{const}}{\log n}} = |b_n|^2 e^{2\,\text{const}}$.

Hence
$$\sum_{n\geqq N_0} |b|_n^{2-\frac{\text{const}}{\log n}} \leqq \sum_{n=1}^{\infty} \frac{1}{n^2} + e^{2\,\text{const}} \sum_{n=0}^{\infty} |b_n|^2 < +\infty. \quad \text{Q. e. d.}$$

where t_{kn} is the nth coefficient of the kth power of $T(w)$. First we give a sufficient condition for a matrix $\|u_{kn}\|$ to turn an absolutely convergent series $\sum a_k$ into another $\sum b_n$ fulfilling (1) and then verify this condition in our case $u_{kn}=t_{kn}$.

Lemma. *Let $0<\lambda_n\leqq\lambda$, $|u_{kn}|\leqq M$ where λ and M are independent of n, and k and n respectively. Assume further that*

$$U_k = \sum_{n=0}^{\infty} |u_{kn}|^{\lambda_n} \neq O(1) \qquad (k\to\infty). \tag{2}$$

In this case there exists an absolutely convergent series for which the transformed sequence

$$b_n = \sum_{k=0}^{\infty} u_{kn} a_k$$

satisfies

$$\sum_{n=0}^{\infty} |b_n|^{\lambda_n} = +\infty.$$

Here we strove but for giving a condition easily verifiable in our special case. To find the necessary and sufficient condition may turn out difficult.

PROOF OF THE LEMMA. We can assume U_k finite for each k, otherwise we could choose $a_k=0$ except for a single k with infinite U_k.

We successively construct integers k_m, n_m and positive numbers A_m in the following way. Let $k_0=n_0=0$, $A_0=1$. Assume that they are already defined for $m<m'$. We choose $A_{m'}$ subject only to the conditions

$$0 < A_{m'} < \frac{1}{2} A_{m'-1}, \quad \sum_{n=0}^{n_{m'-1}} A_{m'}^{\lambda_n} (2M)^{\lambda_n} \leqq 1. \tag{3}$$

With $A_{m'}$ so fixed, the expression

$$\frac{(m'+2)^{\lambda}}{A_{m'}^{\lambda}} \left(\sum_{m=0}^{m'-1} U_{k_m} + m' \right) \tag{4}$$

has a well-determined finite value. U_k is not bounded and therefore we can find a $k_{m'}>k_{m'-1}$ such that $U_{k_{m'}}$ exceeds this value. A partial sum of $U_{k_{m'}}$ of large enough index also does this and as a final step of this definition by induction we determine $n_{m'}>n_{m'-1}$ to be such an index.

Now we put $a_{k_m}=A_m$, $a_k=0$ if $k\neq k_m$ $(m=0, 1, \ldots)$ and prove that for the transformed series

$$\sum_{n=0}^{n_{m'}} |b_n|^{\lambda_n} \to +\infty \quad \text{as} \quad m'\to+\infty.$$

From the first part of (3) $\sum_{k=0}^{\infty} |a_k| = \sum_{m=0}^{\infty} A_m < +\infty$ trivially follows.

We shall use the elementary inequality

$$\left(\sum_{i=1}^{l} x_i\right)^\alpha \leqq l^\alpha \sum_{i=1}^{l} x_i^\alpha \qquad (\alpha > 0,\ x_i \geqq 0).$$

For $\alpha \leqq 1$ even with 1, for $\alpha \geqq 1$ with $l^{\alpha-1}$ instead of l^α so that it holds in any case.

Let m' be fixed and $n \leqq n_{m'}$. We have

$$b_n = \sum_{k=0}^{\infty} u_{kn} a_k = \sum_{m=0}^{\infty} u_{k_m n} A_m = \sum_{m=0}^{m'-1} u_{k_m n} A_m + u_{k_{m'} n} A_{m'} + \sum_{m=m'+1}^{\infty} u_{k_m n} A_m.$$

Using the inequality with $\alpha = \lambda_n$, $l = m'+2$ and the fact that $A_{m+1} < \frac{1}{2} A_m \leqq 1$, $|u_{kn}| \leqq M$, we obtain

$$|u_{k_{m'} n} A_{m'}|^{\lambda_n} = \left| b_n - \sum_{m=0}^{m'-1} u_{k_m n} A_m - \sum_{m=m'+1}^{\infty} u_{k_m n} A_m \right|^{\lambda_n} \leqq$$

$$\leqq \left(|b_n| + \sum_{m=0}^{m'-1} |u_{k_m n}| + 2MA_{m'+1} \right)^{\lambda_n}$$

$$\leqq (m'+2)^\lambda \left[|b_n|^{\lambda_n} + \sum_{m=0}^{m'-1} |u_{k_m n}|^{\lambda_n} + (2M)^{\lambda_n} A_{m'+1}^{\lambda_n} \right],$$

$$\sum_{n=0}^{n_{m'}} |b_n|^{\lambda_n} \geqq \frac{A_{m'}^{\lambda}}{(m'+2)^\lambda} \sum_{n=0}^{n_{m'}} |u_{k_{m'} n}|^{\lambda_n} - \sum_{m=0}^{m'-1} U_{k_m} - \sum_{n=0}^{n_{m'}} (A_{m'+1})^{\lambda_n} (2M)^{\lambda_n}.$$

The sum in the first term is greater than (4) so that the first term exceeds the second one by at least m' while the third term is less than 1 in view of (3) applied for $m'+1$ in place of m'. Hence

$$\sum_{n=0}^{n_{m'}} |b_n|^{\lambda_n} \geqq m' - 1 \to +\infty$$

and our lemma is proved.

Proof of Theorem. [2]) Let us put in the lemma $u_{kn} = t_{kn}$, $\lambda_n = 2 - \frac{\omega(n)}{\log n}$ where we recall t_{kn} is the nth coefficient of $T^k(w)$:

$$t_{kn} = \frac{1}{2\pi} \int_0^{2\pi} \frac{T^k(e^{i\vartheta})}{e^{in\vartheta}}\, d\vartheta. \tag{5}$$

$0 < \lambda_n \leqq 2$ since we can assume $\omega(n) < \log n$. Also $|t_{kn}| \leqq 1$ since $|T(e^{i\vartheta})| = 1$. Hence all suppositions of our lemma are trivially fulfilled except for (2). To prove this, first of all we show that

$$t_{kn} = O\left(\frac{1}{n^{1/3}}\right)$$

uniformly in k.

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[2]) The proof follows that of Bajšanski (see [4], Theorem 3).

Let us introduce the notation

$$T(e^{i\vartheta}) = e^{iF(\vartheta)}$$

where $F(\vartheta)$ is monotonically increasing with derivative

$$F'(\vartheta) = \frac{1-r^2}{1-2r\cos(\vartheta-\varphi)+r^2} \quad (r = |\xi|, \quad \varphi = \arg\xi)$$

and maps $(0, 2\pi)$ onto an interval $(\psi, \psi+2\pi)$. Denoting the inverse function of $F(\vartheta)$ by $G(t)$ we get

$$t_{kn} = \frac{1}{2\pi}\int_0^{2\pi} e^{ikF(\vartheta)-in\vartheta}\,d\vartheta = \frac{1}{2\pi}\int_{\psi}^{\psi+2\pi} e^{ikt-inG(t)}G'(t)\,dt. \tag{6}$$

Now, the second derivative of $F(\vartheta)$ only vanishes for $\vartheta=\varphi$ and $\vartheta=\varphi\pm\pi$ and therefore that of its inverse only for $F(\varphi)$ and $F(\varphi\pm\pi)$. Also $F'''(\vartheta)$ does not vanish for φ and $\varphi\pm\pi$ and neither does $G'''(t)$ for $F(\varphi)$ and $F(\varphi\pm\pi)$. Omitting from $(\psi, \psi+2\pi)$ the intervals $\left[F(\varphi)-\frac{1}{n^{1/3}},\ F(\varphi)+\frac{1}{n^{1/3}}\right]$, $\left[F(\varphi\pm\pi)-\frac{1}{n^{1/3}},\ F(\varphi\pm\pi)+\frac{1}{n^{1/3}}\right]$, at the points of the remaining at most three intervals $|G''(t)|>\frac{\text{const}}{n^{1/3}}$. To these intervals we can apply the following lemma of VAN DER CORPUT (see [3], p. 116—117.):

If $u''(t)$ is continuous, $|u''(t)|>\varrho$ in (a, b) then

$$\left|\int_a^b e^{iu(t)}dt\right| \leqq \frac{8}{\varrho^{1/2}}.$$

Let $u(t)=kt-nG(t)$ where $|u''(t)|=n|G''(t)|\geqq \text{const}\ \frac{n}{n^{1/3}}$ independently of k on the remaining intervals. Hence

$$\left|\int e^{ikt-inG(t)}dt\right| \leqq \frac{\text{const}}{n^{1/3}},$$

the integration is over the remaining intervals. But for the intervals omitted of total length at most $4/n^{1/3}$ the same estimation is trivially satisfied and so

$$|c_{kn}| \overset{\text{def}}{=} \left|\frac{1}{2\pi}\int_{\psi}^{\psi+2\pi} e^{ikt-inG(t)}dt\right| = O\left(\frac{1}{n^{1/3}}\right)$$

even for negative k. c_{kn} is the $(-k)$th Fourier coefficient of $e^{-inG(t)}$ and denoting that of $G'(t)$ by d_k where $\sum_{k=-\infty}^{\infty}|d_k|<+\infty$ since $G'(t)$ is twice continuously differentiable, we get finally for the Fourier coefficients of the product $e^{-inG(t)}G'(t)$

$$|t_{kn}| = \left|\sum_{l=-\infty}^{\infty} c_{k-l,n}d_l\right| = O\left(\frac{1}{n^{1/3}}\right)\sum_{l=-\infty}^{\infty}|d_l| = O\left(\frac{1}{n^{1/3}}\right)$$

what we wanted to prove.

Using this bound in the series to be estimated from below

$$\sum_{n=0}^{\infty} |t_{kn}|^{2-\frac{\omega(n)}{\log n}} \geqq \text{const} \sum_{n=1}^{\infty} |t_{kn}|^2 e^{\frac{\omega(n)}{\log n} \frac{\log n}{3}} = \text{const} \sum_{n=1}^{\infty} |t_{kn}|^2 e^{\frac{\omega(n)}{3}} \geqq$$

$$\geqq \text{const}\, e^{\frac{K}{3}} \sum_{\substack{n \geqq 1 \\ \omega(n) \geqq K}} |t_{kn}|^2.$$

Here K can be any number. Since $\omega(n) \to +\infty$ the number of terms missing in this last sum is finite for each K. But for all fixed n $t_{kn} \to 0$ as we learn from (6) by Riemann's lemma, while by Parseval's equality

$$\sum_{n=0}^{\infty} |t_{kn}|^2 = \frac{1}{2\pi} \int_0^{2\pi} |T^k(e^{i\vartheta})|^2 \, d\vartheta = 1$$

from which we conclude

$$\varliminf_{k \to \infty} \sum_{n=0}^{\infty} |t_{kn}|^{2-\frac{\omega(n)}{\log n}} > \text{const}\, e^{\frac{K}{3}}$$

for all K and letting $K \to +\infty$, condition (2) of the lemma in a slightly stronger form than required is also verified. Q.e.d.

Reference

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[4] Bajšanski, B. M.: Sur une classe générale de procédés de sommation du typ Euler—Borel, *Publications de l'Institut Mathématique de l'Académie Serbe des Sciences*, **10** (1956), 131—152.

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