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Equivariant embeddings of smooth G-pseudomanifolds

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Abstract. Given a compact Lie group G acting smoothly (C^∞) on a compact manifold M, then by a well known result of Mostow, there is an equivariant smooth embedding of M into some Euclidean space with an orthogonal G-action.

In this work we extend Mostow's result to smooth G-pseudomanifolds (in the sense of Goresky and Mac Pherson).

§0. Introduction

Given a compact Lie group G acting smoothly on a compact manifold M, then a well known result of Mostow [4] states the following.

Theorem. *There is a Euclidean space $\Re^n$ with an orthogonal G-action, together with an equivariant smooth embedding $\theta : M \to \Re^n$.*

The objective of this work is to extend Mostow's equivariant embedding theorem to smooth G-pseudomanifolds.

A smooth G-pseudomanifold is a certain G-space which is a topological pseudomanifold with the orbit type filtration [2,3], and whose strata are smooth manifolds.

In §1 we define smooth G-pseudomanifolds and give some examples.

In §2 we prove a theorem on equivariant embeddings into Euclidean space, for smooth G-pseudomanifolds.

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§1. Smooth G-pseudomanifolds

First we recall some definitions given in [5], which will be used in this paper.

Let G be a compact Lie group. By a *G-space* we mean a completely regular space X, together with a continuous action $G \times X \to X$.

Let $G_x = \{g \in G : g \cdot x = x\}$ be the isotropy subgroup of G at $x \in X$. Denote by X/G the corresponding orbit space with the quotient topology induced by the canonical projection $\pi : X \to X/G$.

Also let $cX = X \times [0,1)/(x,0) \sim (x',0)$ be the open cone of X. Then there is a canonical action on cX, given as follows $g \cdot [x,r] = [g \cdot x, r]$ where $g \in G,\ x \in X,\ r \in [0,1)$. Here $[x,r]$ denotes the equivalence class of (x,r) in cX. Notice that the distinguished point $* = [x,0]$ of cX is G-fixed.

Now let X be a G-space.

Definition 1.1. Given an orbit P in X, then a slice S_x in X at $x \in P$ is called a *conical slice* of P, if it satisfies the following condition.

There is a compact H-space L (possibly empty), without fixed points, called a *link* of P, where $H = G_x$, together with an H-equivariant homeomorphism $\phi : S_x \to \Re^{i_0} \times cL$, for an integer $i_0 \geq 0$, where H acts trivially on $\Re^{i_0}$.

For the definition of slices, and their existence in any G-space, see [1] pp. 82–86.

We now define G-pseudomanifolds.

Definition 1.2. A (-1)-*dimensional G-pseudomanifold* is the empty set.

A *n-dimensional G-pseudomanifold* $(n \geq 0)$ is a G-space X which satisfies the following conditions.

(C1) Each orbit P in X has a conical slice $S_x \simeq^{\phi} \Re^{i_0} \times cL$, such that L is an $n-i-1$-dimensional H-pseudomanifold, where $i = i_0 + dim G/H \neq n-1$.

(C2) There is a family of orbits with an empty link in X, containing an orbit over each connected component of X/G, such that all the orbits in this family have the same type.

For the definition of orbit type, see [1] p. 42.

If X is a G-pseudomanifold, define on X the following equivalence relation $x \sim y \iff G_x \sim G_y$ i.e. the corresponding isotropy subgroups are conjugate. The equivalence classes of this relation are denoted by $X_{(H)}$, which is the union of all orbits of type G/H in X.

Given an orbit P of type G/H in X and S_x a conical slice at $x \in P$, let $\Phi^{-1} : G \times_H S_x \to X$ be the G-equivariant embedding onto an open set in X, given by $[g,s] \mapsto g \cdot s$ for $g \in G, s \in S_x$. The image Γ of Φ^{-1} is called [1, p. 82] a *tubular neighborhood* of P in X. Clearly we have

$$\Gamma \cap X_{(H)} \simeq^{\Phi} \{G \times_H S_x\}_{(H)} \simeq^{[1,\phi]} \{G \times_H (\Re^{i_0} \times cL)\}_{(H)} \simeq G/H \times \Re^{i_0},$$

since H acts on L without fixed points. Therefore the connected components of the sets $X_{(H)}$, called *strata* of X, are topological manifolds embedded in X, with dimension $i = i_0 + \dim G/H \leq n, \quad (i \neq n-1)$.

Consider the following canonical filtration of the G-pseudomanifold X, called *the orbit type filtration*,

$$X = X_n \supset X_{n-1} = X_{n-2} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset,$$

where each X_i is the union of the strata of X with dimension less than, or equal to i. Note that $x \in X - X_{n-2}$ if and only if $G(x)$ has an empty link. It is shown in [5] that the connected components of each $X_i - X_{i-1}$ coincide with the i-dimensional strata of X, for $i = 0, \cdots, n$.

Now let P be an orbit in a stratum $X_i - X_{i-1}$, with a given conical slice $S_x \simeq^{\phi} \Re^{i_0} \times cL$, where $G_x = H$.

Let $\sigma : W \to G$ be a local section of $\pi_0 : G \to G/H$, with W a chart of G/H, $eH \in W$ and $\sigma(eH) = e$.

Then $N = \pi_0^{-1}(W) \cdot S_x$ is called a *distinguished neighborhood* of x.

It is shown in [5] that there is a stratum-preseving homeomorphism $\varphi : N \to \Re^i \times cL$, where N has the relative filtration, and $\Re^i \times cL$ the canonical filtration induced by the H-orbit type filtration of L, given explicitly as follows $\varphi(g \cdot z) = (gH, p_1\phi(z), \sigma(gH)^{-1} g \cdot p_2\phi(z))$, for $g \in \pi_0^{-1}(W)$ and $z \in S_x$. Clearly $\varphi | S_x = \phi$.

Also $g \cdot N$ is a *distinguished neighborhood* of $g \cdot x$ for $g \in G$, with a similar homeomorphism.

It is proved in [5] that a G-pseudomanifold X, with the orbit type filtration, is a topological pseudomanifold as defined by Goresky and Mac Pherson [2,3].

Now let X,Y be two non-empty filtered spaces, whose strata are smooth (C^∞) manifolds.

Definition 1.3. Let $\operatorname{len} X = \sup\{p - q : X_p - X_{p-1} \neq \emptyset \neq X_q - X_{q-1}\}$ be the *length of* X. Put $\operatorname{len} \emptyset = -1$.

Definition 1.4. A stratum-preserving map $f : X \to Y$ is said to be *smooth*, or a *smooth embedding*, if f is smooth, or a smooth embedding, on the strata of X.

Definition 1.5. A stratum-preserving homeomorphism $h : X \to Y$ is said to be a *diffeomorphism*, if h and h^{-1} are smooth.

We now define the concept of a smooth G-pseudomanifold.

Definition 1.6. Let X be a G-pseudomanifold.
For $\operatorname{len} X = -1$, $X = \emptyset$ is a *smooth G-pseudomanifold*.
For $\operatorname{len} X \geq 0$, X is a *smooth G-pseudomanifold*, if it satisfies the following conditions.

(C1) Each $X_i - X_{i-1}$ is a smooth manifold, with a smooth restricted G-action.

(C2) Each orbit P in any $X_i - X_{i-1}$ has a conical slice $S_x \simeq^{\phi} \Re^{i_0} \times cL$, with L a smooth H-pseudomanifold, such that $\varphi : N \to \Re^i \times cL$ is a diffeomorphism for some distinguished neighborhood $N = \pi_0^{-1}(W) \cdot S_x$.

Since $\phi : S_x \to \Re^{i_0} \times cL$ is an H-equivariant homeomorphism, then S_x is a smooth H-pseudomanifold (see 2.7 in [5]), which is smoothly embedded into the tubular neighborhood $\Gamma \simeq G \times_H S_x$ of the orbit P.

We now give some examples.

(1) Let X be a torus in $\Re^3$ which is pinched at the points $q_{\pm 1} = (0, \pm 1, 0)$ and intersects the xz plane in $L = S_1^1 \cup S_2^1$, the union of two circles centered at the points $(0, \pm 2)$.

There is a canonical Z_2-action on X given by reflection through the xy plane. Clearly T is a smooth Z_2-pseudomanifold embedded in $\Re^3$.

Notice in particular the embedding in $\Re^2$ of the smooth Z_2-pseudomanifold L, which is a link of the orbits $q_{\pm 1}$, and the corresponding embeddings of the smooth Z_2-pseudomanifolds $c_{q_{\pm 1}} L$ in $\Re^3$.

(2) Let G be a compact Lie group acting smoothly on a paracompact manifold M. Asume that the orbit type filtration of M has no strata of codimension one, and M/G is connected. Claim that M is a smooth G-pseudomanifold.

The proof is by induction on $\operatorname{len} M$. For $\operatorname{len} M = 0$ it is trivial. Assume that the claim is true for smooth G-manifolds of length strictly smaller than $\operatorname{len} M$.

Given an orbit P in M, choose a point $x \in P$ with $G_x = H$. Consider a linear slice [1, p. 171], $S_x \simeq E$ at x in M, where H acts orthogonally on an Euclidean space E. Let $V = (E^H)^{\perp}$ denote the orthogonal complement of E^H with respect to an H-invariant inner product on E. Put L to be the unit sphere of V with respect to the associated H-invariant metric.

Since $L \simeq S^l$ canonically, where $l+1 = \dim V$, we can put a smooth structure on L. Clearly $H \times L \to L$ is smooth, because H acts orthogonally on V. Since $\operatorname{len} L < \operatorname{len} M$, we have by the inductive hypothesis that L, with the H-orbit type filtration, is a smooth H-pseudomanifold.

Now by [1, p. 308] we have a diffeomorphism $\Gamma \simeq G \times_H E$ between smooth G-manifolds, where Γ is the tubular neighborhood of P corresponding to S_x.

Hence for a chart W of G/H at eH, we have a diffeomorphism of filtered spaces

$$\pi_0^{-1}(W) \cdot S_x \simeq W \times \{E^H \oplus (E^H)^\perp\} \simeq \Re^i \times cL,$$

for $i = \dim G/H + \dim E^H$, since $V - \{0\} \simeq L \times (0,1)$ is a diffeomorphism between smooth H-manifolds, and the strata of M are smooth submanifolds [1, p. 309]. Therefore M is a smooth G-pseudomanifold.

§2. The embedding theorem

In this section we extend to smooth G-pseudomanifolds the equivariant embedding theorem of Mostow.

First we prove the following.

Lemma 2.1. *Let X be a smooth G-pseudomanifold and $\Gamma \simeq G \times_H S_x$ a tubular neighborhood of the orbit $G(x)$, corresponding to a conical slice S_x given in Definition 1.6 (C2). Then Γ with the restricted G-action is also a smooth G-pseudomanifold.*

Proof. Let $S_x \simeq^\phi \Re^{i_0} \times cL$ be a conical slice at x in X, for a smooth H-pseudomanifold L. Then given $y = \phi^{-1}(p_1\phi(y), [l,r]) \in S_x$, $r > 0$ and S_l a conical K-slice at l in L, where $K = G_y = H_y = H_l$, we have that

$$S_y \simeq^\phi \Re^{i_0} \times S_l \times (0,1) \simeq^{1\times\phi'} \Re^{i_0+k_0+1} \times cQ,$$

is a conical K-slice at y in the H-space S_x, for some $k_0 \geq 0$. Here Q is a link of $H(l)$, which is a $n-i-k-2$-dimensional smooth K-pseudomanifold, where $k = k_0 + \dim H/K$. Therefore [1, p. 84], S_y is a conical K-slice at y in X.

Let $\pi_0 : G \to G/H, \quad \pi_1 : H \to H/K, \quad$ and $\quad \pi_2 : G \to G/K.$

Consider local sections $\sigma_0 : W_0 \to G$ of π_0, and $\sigma_1 : W_1 \to H$ of π_1, where $W_0 = \{gH : g \in G^0\}$ and $W_1 = \{\sigma_0(gH)^{-1}gK : g \in G^0\}$, for some open neighborhood $G^0 \subset G$ of e, with $\sigma_0(eH) = \sigma_1(eK) = e$.

Then we have a local section $\sigma_2 : W_2 \to G$ of π_2, given by

$$\sigma_2(gK) = \sigma_0(gH) \cdot \sigma_1(\sigma_0(gH)^{-1}gK), \quad \text{where } W_2 = \{gK : g \in G^0\}.$$

Since $G^0 = \sigma_0(W_0) \cdot \sigma_1(W_1) \cdot K$, we obtain the following diffeomorphism

$$W_2 \simeq \sigma_2(W_2) = \sigma_0(W_0) \cdot \sigma_1(W_1) \simeq \sigma_0(W_0) \times \sigma_1(W_1) \simeq W_0 \times W_1.$$

Now by the explicit formulation of φ (see p. 3), the following composition is a diffeomorphism

$$\sigma_2(W_2) \cdot S_y \simeq^{\varphi} \sigma_0(W_0) \cdot \Re^{i_0} \times (0,1) \times \sigma_1(W_1) \cdot S_l \simeq^{1\times\varphi'}$$
$$W_0 \times W_1 \times \Re^{i_0+k_0+1} \times cQ \simeq \sigma_2(W_2) \cdot \Re^{i_0+k_0+1} \times cQ \simeq \Re^{i+k+1} \times cQ.$$

□

We now obtain the following result.

Theorem 2.2. *Let X be a compact, smooth G-pseudomanifold. Then there is a Euclidean space $\Re^n$ with an orthogonal G-action, together with an equivariant smooth embedding $\theta : X \to \Re^n$.*

Proof. By induction on the length of X. For $\operatorname{len} X = 0$ the statement follows from the smooth embedding theorem of Mostow. Assume that the statement is valid for G-pseudomanifolds of length strictly smaller than $\operatorname{len} X$.

Let P be an orbit in X, with a conical slice S_x in X at $x \in P$ given by Definition 1.6 (C2). Hence, there is an H-equivariant homeomorphism $\phi : S_x \to D(\Re^{i_0}, 1) \times cL$, with $G_x = H$, where L is a compact smooth H-pseudomanifold, and H acts trivially on $D(\Re^{i_0}, 1)$ the open unit disk of $\Re^{i_0}$. Assume that $L \neq \emptyset$.

Now if Γ is the tubular neighborhood of P corresponding to the slice S_x, then Γ is a smooth G-pseudomanifold by Lemma 2.1. Hence, since the map $\Phi : \Gamma \to G \times_H S_x$ is a G-equivariant homeomorphism, we can put on $G \times_H S_x$ a smooth G-pseudomanifold structure, such that Φ is a diffeomorphism.

Since $\operatorname{len} L < \operatorname{len} X$, by the inductive hypothesis there is a Euclidean space $\Re^m$ with an orthogonal H-action, together with an equivariant smooth embedding $\theta_1 : L \to \Re^m$.

Therefore there is an H-equivariant smooth embedding $\theta_2 : S_x \to V$, where $\theta_2 = (1 \times c\theta_1)\phi$ and $V = \Re^{i_0} \oplus \Re \oplus \Re^m$, which has an orthogonal

H-action given by the sum of these representations, with H acting trivially on $\Re^{i_0}$ and $\Re$. Here $c\theta_1 : cL \to c\Re^m \subset \Re \oplus \Re^m$.

Let $D(V, r) = \{v \in V : \|v\| < r\}$, and $c(L, r) = L \times [0, r)/(l, 0) \sim (l', 0)$ for $0 < r \leq 1$. Using a suitable homothety, we may assume that there is an H-equivariant smooth embedding of S_x into $D(V, \sqrt{3})$.

By symmetry there is an H-equivariant smooth embedding of $S_x(r) = \phi^{-1}\{D(\Re^{i_0}, r) \times c(L, r)\}$ into $D(V, r\sqrt{3})$. It can easily be shown using an equivariant retraction, that $S_x(r)$ is a conical slice in X at $x \in P$.

Hence the following composition is a G-equivariant smooth embedding

$$\Gamma \xrightarrow{\Phi} G \times_H S_x \xrightarrow{[1\times\theta_2]} G \times_H D(V, \sqrt{3}) \xrightarrow{[1\times i]} G \times_H V.$$

For $L = \emptyset$ the above is trivially satisfied.

To conclude the proof, we shall give an equivariant smooth embedding of $G \times_H V$ into Euclidean space, as in [1] p. 315.

By [1] p. 24, there exists an orthogonal representation of G on some Euclidean space V_0 and a point $v_0 \in V_0$ with $G_{v_0} = H$. Now by [1] p. 18, the orthogonal representation of H on the Euclidean space V given above, may be extended to an orthogonal representation of G on some Euclidean space $V' \supset V$, which extends the H-action on V. Then, G acts orthogonally on $W = V_0 \oplus V'$ via the sum of these two representations (i.e. diagonally).

Consider the map $\alpha : G \times_H V \to V_0 \oplus V' = W$, defined by $\alpha[g, v] = g(v_0 + v)$. If $\alpha[g, v] = \alpha[g', v']$ then $g(v_0 + v) = g'(v_0 + v')$, so that $g^{-1}g'(v_0) = v_0$ and $g^{-1}g'(v') = v$. Thus $h = g^{-1}g' \in H$ and $h(v') = v$. Therefore $[g, v] = [gh, h^{-1}v] = [g', v']$ and hence α is injective. Since $G \times_H V$ has the differentiable structure induced from that of $G \times V$ and since the action map $G \times V \to W$ is smooth, it follows that α is smooth.

Now the isotropy group at $[e, v]$ is H_v, and this is also the isotropy group at $v_0 + v \in W$. Thus α takes the orbit of $[e, v]$ diffeomorphically onto $G(v_0 + v)$. The differential of α is thus injective on the tangent space to the orbit at $[e, v]$. However, the normal space to the orbit of $[e, v]$ is V and α maps this one-one affinely into W. Hence α_* is injective on the whole tangent space to $G \times_H V$ at $[e, v]$. By equivariance, α_* is everywhere injective, so that α is an injective immersion. Since α is obviously proper, it is a smooth embedding.

(It is only important that α be an embedding near the 0-section G/H.)

Hence the following map β is a G-equivariant smooth embedding

$$\Gamma \xrightarrow{\Phi} G \times_H S_x \xrightarrow{[1\times\theta_2]} G \times_H D(V, \sqrt{3}) \xrightarrow{[1\times i]} G \times_H V \xrightarrow{\alpha} W.$$

Now given $1 > s > t > 0$, let $f : \Re \to \Re$ be a smooth function such that

$$\begin{cases} f(r) = 1 & \text{for } r \leq t \\ f(r) \neq 0 & \text{for } r < s \\ f(r) = 0 & \text{for } r \geq s. \end{cases}$$

If $\rho : \Gamma \to [0,1)$ is the smooth invariant function obtained from the ratio of cL, we can define a smooth equivariant map $\psi : \Gamma \to W$ by $y \mapsto f(\rho(y)) \cdot \beta(y)$ for $y \in \Gamma$. Since $\rho^{-1}([0,s])$ is closed [1, p. 34], ψ extends to a smooth equivariant map on X.

Also the smooth invariant function $\gamma : \Gamma \to \Re$, given by $y \mapsto f(\rho(y) \cdot s/t)$ for $y \in \Gamma$, extends to a smooth invariant function on X.

Thus for each orbit P in X we have an orthogonal representation of G on an Euclidean space W_x, and a smooth equivariant map $\psi_x : X \to W_x$, which is a smooth embedding on the tubular neighborhood Γ_x of P corresponding to the conical slice $S_x(t)$ at $x \in P$.

Additionally, we have a smooth invariant function $\gamma_x : X \to \Re$, which is nonzero exactly on Γ_x.

Since X is compact, it can be covered by finitely many tubular neighborhoods $\Gamma_{x_1}, \dots, \Gamma_{x_k}$. Let $\theta : X \to W_{x_1} \oplus \cdots \oplus W_{x_k} \oplus \Re^k \simeq \Re^n$ be given as follows

$$\theta(x) = (\psi_{x_1}(x), \dots, \psi_{x_k}(x), \gamma_{x_1}(x), \dots, \gamma_{x_k}(x)) \quad \text{for } x \in X,$$

this map is clearly smooth and equivariant.

If $x, y \in \bigcup \Gamma_{x_p}$ and $\theta(x) = \theta(y)$, then we have that for some $p = 1, \dots, k$, $\gamma_{x_p}(x) = \gamma_{x_p}(y) \neq 0$, which implies that $x, y \in \Gamma_{x_p}$ and hence that $x = y$, since ψ_{x_p} is injective on Γ_{x_p}. Because ψ_{x_p} is a smooth embedding on Γ_{x_p} for all p, it follows that θ is a smooth embedding. □

References

[1] G. Bredon, Introduction to compact transformation groups, *Academic Press*, *New York*, 1972.

[2] M. Goresky and R. Mac Pherson, Intersection homology theory, *Topology* **19** (1980), 135–162.

[3] M. Goresky and R. Mac Pherson, Intersection homology II, *Invent. Math.* **71** (1983), 77–129.

[4] G. Mostow, Equivariant embeddings in Euclidean space, *Ann. Math.* **65** no. 3 (1957), 432–446.
[5] R. Popper, G-pseudomanifolds, *Acta Mathematica Hungarica* (*to appear*).

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