On four extensions of the functional equation $|f(x+iy)| = |f(x)+f(iy)|$

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§ 1. Introduction

R. M. ROBINSON solved the following functional equation:

(1) $$|f(x+iy)| = |f(x)+f(iy)|,$$

where $f(z)$ is regular for $|z|<r$ and x, y are real. (See [1].)

In this paper we shall solve four extensions of (1).

Firstly we shall solve the following three functional equations which were presented by J. ACZÉL:

(2) $$|f(x+iy)|=|f(x)+af(iy)|,$$

(3) $$|f(x+iy)|=|af(x)+f(iy)|,$$

(4) $$|f(x+iy)|=|af(x)+bf(iy)|,$$

where $f(z)$ is an entire function of z, and x, y are real, and a, b are complex constants. Here, putting $a=1$, $b=1$, we have (1), and putting $a=1$, $b=-1$, we have $|f(x+iy)|=|f(x)-f(iy)|$ which was solved in [2]. We shall reduce (4) to (2), (3).

Secondly we shall solve the following functional equation which has a geometric meaning:

(5) $$|f(x+iy)+f(0)|=|f(x)+f(iy)|,$$

where $f(z)$ is an entire function of z, and x, y are real. Here, putting $f(0)=0$, we have (1).

§ 2. On the functional equations (2), (3), (4)

Theorem 1. *If $f(z)$ is an entire function of z and satisfies the functional equation (2) for real values of x and y, then the solutions of (2) are the following and only these:*

Case (i) $a=0$. $f(z)=C\exp(\alpha z)$,

where C is an arbitrary complex constant and α is an arbitrary real constant.

Case (ii) $a=1$. $f(z)=Cz$, *or* $f(z)=C\sin\alpha z$, *or* $f(z)=C\sin h\alpha z$, *where C is an arbitrary complex constant and α is an arbitrary real constant.*

Case (iii) $a=-1$. $f(z)=Az+Bz^2$, *or* $f(z)=A\sin\alpha z+B\cos\alpha z-B$, *or* $f(z)= =A\sin h\alpha z+B\cos h\alpha z-B$,

where A, B are arbitrary complex constants and α is an arbitrary real constant.

Case (iv) *other than the cases* (i), (ii), (iii). $f(z)\equiv 0$ *when* $|1+a|\neq 1$, $f(z)\equiv$ $\equiv$ *arbitrary const. when* $|1+a|=1$.

PROOF. Case (A): $f(0)\neq 0$. Putting $x=0, y=0$ in (2), by $f(0)\neq 0$ we have

(6) $$|1+a|=1.$$

We may assume that $f(z)\not\equiv$ const. Putting $g(z)=\dfrac{f(z)}{f(0)}$, we have

$$g(z) = 1+\sum_{n=0}^{+\infty} a_{p+n}z^{p+n}$$

($a_p\neq 0$, where p is a natural number).
By (2) we have

(7) $$g(x+iy)\overline{g(x+iy)} = \big(g(x)+ag(iy)\big)\big(\overline{g(x)+ag(iy)}\big).$$

Equating the coefficients of x^p of both sides in (7), we have

(8) $$\bar{a}a_p+a\bar{a}_p=0.$$

Now, we shall prove that $a=0$. Suppose that $p>1$. Then, equating the coefficients of $x^{p-1}y$ of both sides in (7), we have

(9) $$\bar{a}_p=a_p.$$

Since $a_p\neq 0$, by (8), (9) a is purely imaginary. Hence, by (6) we have $a=0$. Next, suppose that $p=1$. Equating the coefficients of x, iy of both sides in (7), we have

(10) $$\bar{a}a_1+a\bar{a}_1=0,$$

(11) $$a_1-\bar{a}_1=a(1+\bar{a})a_1-\bar{a}(1+a)\bar{a}_1.$$

By (6) we have

(12) $$a=e^{i\theta}-1,$$

where θ is a real constant. Substituting (12) in (10), we have

$$e^{i\theta}=1 \quad\text{or}\quad a_1=e^{i\theta}\bar{a}_1.$$

When $e^{i\theta}=1$, by (12) we have $a=0$. When $a_1=e^{i\theta}\bar{a}_1$, by (11), (12) we have

$$e^{i\theta}\bar{a}_1-\bar{a}_1 = (e^{i\theta}-1)e^{-i\theta}e^{i\theta}\bar{a}_1-(e^{-i\theta}-1)e^{i\theta}\bar{a}_1.$$

Since $\bar{a}_1\neq 0$, we have $e^{i\theta}=1$. Hence, by (12) we have $a=0$.
Thus, by (2) we have

(13) $$|f(x+iy)|^2=|f(x)|^2.$$

Putting $f(x+iy)=u+iv$ where u, v are real, by (13) we have $\dfrac{\partial}{\partial y}(u^2+v^2)=0$ in $|z|<+\infty$. Hence we have $uu_y+vv_y=0$ in $|z|<+\infty$. Hence, by the Cauchy—Riemann equations we have in $|z|<+\infty$

(14) $$uv_x-vu_x=0.$$

By our assumption $f(z)\not\equiv 0$. Choosing a vicinity V properly, we have $f(z)\neq 0$ in V. Hence we have in V

$$\frac{f'(z)}{f(z)}=\frac{u_x+iv_x}{u+iv}=\frac{uu_x+vv_x}{u^2+v^2}+i\,\frac{uv_x-vu_x}{u^2+v^2},$$

where $z=x+iy$ (x, y real). Hence, by (14) we have $\operatorname{Im}\left(\frac{f'(z)}{f(z)}\right)=0$ in V. Hence we have $\frac{f'(z)}{f(z)}=\alpha$ where α is a real constant. Solving this differential equation, we have $f(z)=C\exp(\alpha z)$ where C is a complex constant.

Case (B): $f(0)=0$. By (2) we have

$$f(x+iy)\overline{f(x+iy)}=\big(f(x)+af(iy)\big)\big(\overline{f(x)+af(iy)}\big). \tag{15}$$

We may assume that $f(z)\not\equiv$ const. Using the power series $f(z)=\sum_{n=0}^{+\infty}a_{p+n}z^{p+n}$ ($a_p\neq 0$ where p is a natural number) and equating the terms of degree $2p$ with respect to x and y of both sides in (15), we have $p=1$.

Putting $g(z)=\frac{f(z)}{a_1}$, we have in $|z|<+\infty$

$$g(z)=z+b_2z^2+b_3z^3+\ldots+b_nz^n+\ldots.$$

By (2) we have

$$g(x+iy)\overline{g(x+iy)}=\big(g(x)+ag(iy)\big)\big(\overline{g(x)+ag(iy)}\big). \tag{16}$$

Equating the coefficients of xy and y^2, we have

$$\bar a=a, \tag{17}$$

$$|a|=1. \tag{18}$$

By (17), (18) we have $a=1$ or $a=-1$. By [1] and the previous paper [2] the theorem is proved.

Theorem 2. *If $f(z)$ is an entire function of z and satisfies the functional equation (3) for real values of x and y, then the solutions of (3) are the following and only these:*

Case (i) $a=0$. $f(z)=C\exp(i\alpha z)$,
where C is an arbitrary complex constant and α is an arbitrary real constant.

Case (ii) $a=1$. $f(z)=Cz$, *or* $f(z)=C\sin\alpha z$, *or* $f(z)=C\sin h\alpha z$,
where C is an arbitrary complex constant and α is an arbitrary real constant.

Case (iii) $a=-1$. $f(z)=Az+Bz^2$ *or* $f(z)=A\sin\alpha z+B\cos\alpha z-B$, *or* $f(z)=A\sin h\alpha z+B\cos h\alpha z-B$,
where A, B are arbitrary complex constants and α is an arbitrary real constant.

Case (iv) $a\neq 0, 1, -1$. $f(z)\equiv 0$ *when* $|1+a|\neq 1$, $f(z)\equiv$ *arbitrary const. when* $|1+a|=1$.

Proof. Putting $g(z)=\overline{f(i\bar z)}$, $g(z)$ is an entire function of z and by (3) we have

$$|g(x+iy)|=|g(x)+\bar a g(iy)|,$$

where x, y are real. Hence, by Theorem 1 the theorem is proved.

Theorem 3. *If $f(z)$ is an entire function of z and satisfies the functional equation* (4) *for real values of x and y, then the solutions of* (4) *are the following and only these:*

Case (i) $|a|=1$, $b=0$. $f(z)=C\exp(\alpha z)$,
where C is an arbitrary complex constant and α is an arbitrary real constant.

Case (ii) $a=0$, $|b|=1$. $f(z)=C\exp(i\alpha z)$,
where C is an arbitrary complex constant and α is an arbitrary real constant.

Case (iii) $|a|=1$, $a=b$. $f(z)=Cz$, *or* $f(z)=C\sin\alpha z$, *or* $f(z)=C\sin h\alpha z$,
where C is an arbitrary complex constant and α is an arbitrary real constant.

Case (iv) $|a|=1$, $a=-b$. $f(z)=Az+Bz^2$, *or* $f(z)=A\sin\alpha z+B\cos\alpha z-B$, *or* $f(z)=A\sin h\alpha z+B\cos h\alpha z-B$,
where A, B are arbitrary complex constants and α is an arbitrary real constant.

Case (v) *other than the cases* (i), (ii), (iii), (iv). $f(z)\equiv 0$ *when* $|a+b|\neq 1$, $f(z)\equiv$ *arbitrary const. when* $|a+b|=1$.

PROOF. Case (A) $|a|<1$, $|b|<1$. By (4) we have in $|x|<+\infty$

$$|f(x)| \leqq |a|\,|f(x)|+|b|\,|f(0)|. \tag{19}$$

Since $|a|<1$, by (19) we have in $|x|<+\infty$

$$|f(x)| \leqq \frac{|b|\,|f(0)|}{1-|a|}. \tag{20}$$

By (4) we have in $|y|<+\infty$

$$|f(iy)| \leqq |a|\,|f(0)|+|b|\,|f(iy)|. \tag{21}$$

Since $|b|<1$, by (21) we have in $|y|<+\infty$

$$|f(iy)| \leqq \frac{|a|\,|f(0)|}{1-|b|}. \tag{22}$$

By (4), (20), (22) we have in $|x+iy|<+\infty$

$$|f(x+iy)| \leqq \frac{|a|\,|b|\,|f(0)|}{1-|a|}+\frac{|a|\,|b|\,|f(0)|}{1-|b|}. \tag{23}$$

By (23) and Liouville's theorem we have $f(z)\equiv$const.

Case (B) $|a|<1$, $|b|>1$.
Since $|a|<1$, by (20) we have in $|x|<+\infty$

$$|f(x)| \leqq \frac{|b|\,|f(0)|}{1-|a|}. \tag{24}$$

By (4) we have in $|x+iy|<+\infty$ $|f(x+iy)| \geqq |b|\,|f(iy)|-|a|\,|f(x)|$.
Hence we have in $|y|<+\infty$

$$|f(iy)| \leqq \frac{|a|\,|f(0)|}{|b|-1}. \tag{25}$$

By (4), (24), (25) we have in $|x+iy|<+\infty$

$$|f(x+iy)| \leqq \frac{|a|\,|b|\,|f(0)|}{1-|a|}+\frac{|b|\,|a|\,|f(0)|}{|b|-1}. \tag{26}$$

By (26) and Liouville's theorem we have $f(z)\equiv\text{const.}$

Case (C) $|a|>1$, $|b|<1$. Putting $g(z)=\overline{f(i\bar{z})}$, $g(z)$ is an entire function of z and by (4) we have $|g(x+iy)| = |\bar{b}g(x)+\bar{a}g(iy)|$, where x, y are real. Hence, by the result of Case (B) we have $f(z)\equiv\text{const.}$

Case (D) $|a|>1$, $|b|>1$.

By (4) we have in $|x+iy|<+\infty$

$$|f(x+iy)| \geqq |a|\,|f(x)|-|b|\,|f(iy)|.$$

Hence we have in $|x|<+\infty$

$$|f(x)| \leqq \frac{|b|\,|f(0)|}{|a|-1}. \tag{27}$$

Since $|b|>1$, by (25) we have in $|y|<+\infty$

$$|f(iy)| \leqq \frac{|a|\,|f(0)|}{|b|-1}. \tag{28}$$

By (4), (27), (28) we have in $|x+iy|<+\infty$

$$|f(x+iy)| \leqq \frac{|a|\,|b|\,|f(0)|}{|a|-1}+\frac{|a|\,|b|\,|f(0)|}{|b|-1}. \tag{29}$$

By (29) and Liouville's theorem we have $f(z)\equiv\text{const.}$

Case (E) other than the cases (A), (B), (C), (D).

When $|a|=1$, by (4) we have

$$|f(x+iy)| = \left|f(x)+\frac{b}{a}f(iy)\right|.$$

Thus the solution of (4) reduces to that of (2). Next, when $|b|=1$, by (4) we have

$$|f(x+iy)| = \left|\frac{a}{b}f(x)+f(iy)\right|.$$

Thus the solution of (4) reduces to that of (3). Thus the theorem is proved.

§ 3. On the functional equation (5)

Theorem 4. *If $f(z)$ is an entire function of z and satisfies the functional equation (5) for real values of x and y, then the solutions of (5) are the following and only these:*

$$f(z)=Az+B, \quad \text{or} \quad f(z)=A\sin\alpha z+B\cos\alpha z, \quad \text{or} \quad f(z)=A\sin h\alpha z+B\cos h\alpha z,$$

where A, B are arbitrary complex constants and α is an arbitrary real constant.

PROOF. We may assume that $f(0)\neq 0$. Using the power series $f(z)=\sum_{n=0}^{+\infty} a_n z^n$ and putting $g(z)=\frac{f(z)}{a_0}$ in $|z|<+\infty$, we have in $|z|<+\infty$

$$g(z) = 1+b_1 z+b_2 z^2+b_3 z^3+\dots+b_n z^n+\dots . \tag{30}$$

By (5) we have in $|z|<+\infty$

$$(g(z)+1)(\overline{g(z)}+1) = \left(g\left(\frac{z+\bar z}{2}\right)+g\left(\frac{z-\bar z}{2}\right)\right)\overline{\left(g\left(\frac{z+\bar z}{2}\right)+g\left(\frac{z-\bar z}{2}\right)\right)}. \tag{31}$$

Substituting (30) in (31) and equating the coefficients of z^2 of both sides, we have $2b_2=b_2+\bar b_2$. Hence we have $\bar b_2=b_2$. Hence b_2 is real. Substituting (30) in (31) and equating the coefficients of z^n of both sides for $n>2$, we have

$$\frac{2^{n-1}-1}{2^{n-1}}\, b_n-\frac{1}{2^{n-1}}\,\bar b_n = P(b_1, b_2, b_3, \dots, b_{n-1}, \bar b_1, \bar b_2, \bar b_3, \dots \bar b_{n-1}),$$

where $n(>2)$ is even and P is a polynomial in the earlier coefficients $b_1, b_2, b_3, \dots, \dots, b_{n-1}, \bar b_1, \bar b_2, \bar b_3, \dots, \bar b_{n-1}$, and

$$\frac{2^{n-1}-1}{2^{n-1}}\, b_n = P(b_1, b_2, b_3, \dots, b_{n-1}, \bar b_1, \bar b_2, \bar b_3, \dots, \bar b_{n-1}),$$

where $n(>2)$ is odd, and P is a polynomial in the earlier coefficients $b_1, b_2, b_3, \dots, \dots, b_{n-1}, \bar b_1, \bar b_2, \bar b_3, \dots, \bar b_{n-1}$.

Since $2^{n-1}-2\neq 0$ (>0) for $n>2$, the remaining coefficients b_n $(n>2)$ are uniquely determined in terms of b_1, b_2 where b_2 is real. On the other hand

$$g(z) = \frac{b_1}{\sqrt{-2b_2}}\sin\sqrt{-2b_2}\, z+\cos\sqrt{-2b_2}\, z = 1+b_1 z+b_2 z^2+\dots,$$

or $$g(z) = \frac{b_1}{\sqrt{2b_2}}\sin h\sqrt{2b_2}\, z+\cos h\sqrt{2b_2}\, z = 1+b_1 z+b_2 z^2+\dots,$$

or $$g(z) = 1+b_1 z,$$

respectively, are solutions of the functional equation $|g(x+iy)+1| = |g(x)+g(iy)|$, if b_2 is negative or positive or 0.

Since the remaining coefficients b_n $(n>2)$ are uniquely determined in terms of b_1, b_2, there can be no other normalized solutions. Thus the theorem is proved.

Example. (See [3].) By the above theorem we can solve the following functional equation under the hypothesis that $f(x)$ is an entire function of x (x complex):

$$|f(x+y)+f(x-y)| = |f(x+\bar y)+f(x-\bar y)|, \tag{32}$$

where x, y are complex.

Solution. Putting $x=y=\frac{s+it}{2}$ in (32) where s, t are real, we have

$$|f(s+it)+f(0)| = |f(s)+f(it)|. \tag{33}$$

By the above theorem the solutions of (32) are the following and only these:

$$f(z)=A+Bz, \text{ or } f(z)=A\sin\alpha z+B\cos\alpha z, \text{ or } f(z)=A\sin h\alpha z+B\cos h\alpha z,$$

where A, B are arbitrary complex constants and α is an arbitrary real constant.

Remark. The sufficiency of this example gives the following theorem:

For a family of confocal ellipses and hyperbolas, let M, N be the middle points of the two diagonals of a curvilinear rectangle formed by any two ellipses and two hyperbolas and let O be the center of this family. Then we have $\overline{OM}=\overline{ON}$ (and the above statement on (32) gives all curves which have this property).

References

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