A three-term relation for the Dedekind-Rademacher sums

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1. For real x put

$$((x)) = \begin{cases} x-[x]-\frac{1}{2} & (x \neq \text{integer}) \\ 0 & (x = \text{integer}) \end{cases}$$

and define the Dedekind sum

$$s(b,a) = \sum_{r \pmod a} \left(\left(\frac{r}{a}\right)\right)\left(\left(\frac{br}{a}\right)\right).$$

RADEMACHER ([3]) has proved the following three-term relation satisfied by $s(b, a)$:

$$s(bc',a)+s(ca',b)+s(ab',c) = -\frac{1}{4}+\frac{1}{12}\left(\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\right), \tag{1.1}$$

where

$$(a, b)=(b, c)=(c, a)=1$$

and a', b', c' are defined by

$$aa' \equiv 1 \pmod{bc}, \quad bb' \equiv 1 \pmod{ca}, \quad cc' \equiv 1 \pmod{ab}.$$

In particular, when $c=c'=1$, (1. 1) reduces to the familiar reciprocity formula

$$s(b,a)+s(a,b) = -\frac{1}{4}+\frac{1}{12}\left(\frac{a}{b}+\frac{1}{ab}+\frac{b}{a}\right). \tag{1.2}$$

In a more recent paper [4], Rademacher has introduced the sum

$$s(h,k;x,y) = \sum_{r \pmod k} \left(\left(h\frac{r+y}{k}+x\right)\right)\left(\left(\frac{r+y}{k}\right)\right) \tag{1.3}$$

and proved the reciprocity formula

$$s(h,k;x,y)+s(k,h;y,x) = \tag{1.4}$$

$$= ((x))((y))+\frac{1}{2}\left\{\frac{h}{k}\Psi_2(y)+\frac{1}{hk}\Psi_2(hy+kx)+\frac{k}{h}\Psi_2(x)\right\},$$

where $(h, k)=1$, x and y are not both integers and $\Psi_2(x)=B_2(x-[x])$, where

$$B_2(x) = x^2-x+\frac{1}{6},$$

the Bernoulli polynomial of degree 2. The writer [1], [2] has proved a generalization of (1. 4).

In the present paper we obtain a three-term realtion satisfied by $s(h, k; x, y)$. It will however be convenient to change the notation defined above. To begin with, we put

$$\Phi(x) = x-[x]-\tfrac{1}{2} \tag{1. 5}$$

for all real x. In the next place we define

$$s(a, b, c; x, y, z) = \sum_{t (\mathrm{mod}\ c)} \Phi\left(a\frac{t+z}{c}-x\right)\Phi\left(y-b\frac{t+z}{c}\right). \tag{1. 6}$$

Despite the presence of the additional parameters, $s(a, b, c; x, y, z)$ is really no more general than $s(h, k; x, y)$ as defined by (1. 3).

We shall prove the following

Theorem. *Let* $(a, b)=(b, c)=(c, a)=1$. *Then we have*

$$s(a, b, c; x, y, z)+s(b, c, a; y, z, x)+s(c, a, b; z, x, y) = \tag{1. 7}$$

$$= \delta-\frac{a}{2bc}\Psi_2(cy-bz)-\frac{b}{2ca}\Psi_2(az-cx)-\frac{c}{2ab}\Psi_2(bx-ay),$$

where $\delta=1$ *if integers* r, s, t *exist such that*

$$\frac{r+x}{a}=\frac{s+y}{b}=\frac{t+z}{c}; \tag{1. 8}$$

$\delta=0$ *otherwise.*

2. We shall need a few preliminary results. Clearly $\Phi(x+1)=\Phi(x)$; also it is familiar that

$$\Phi(-x) = -\Phi(x), \tag{2. 1}$$

provided x is not an integer. We recall also that

$$\sum_{r (\mathrm{mod}\ k)} \Phi\left(x+\frac{r}{k}\right) = \Phi(kx). \tag{2. 2}$$

Applying (2. 2) to (1. 6) we get

$$s(a, b, c; x, y, z) = \sum_{r, s, t} \Phi\left(\frac{t+z}{c}-\frac{r+x}{a}\right)\Phi\left(\frac{s+y}{b}-\frac{t+z}{c}\right), \tag{2. 3}$$

where r, s, t run through complete residue systems, modulo a, b, c respectively. If we put

$$\xi = \frac{r+x}{a},\quad \eta = \frac{s+y}{b},\quad \zeta = \frac{t+z}{c}, \tag{2. 4}$$

we may rewrite (2. 3) compactly as

$$s(a, b, c; x, y, z) = \sum_{r, s, t} \Phi(\zeta-\xi)\Phi(\eta-\zeta). \tag{2. 5}$$

The following lemmas will be used later.

Lemma 1. *We have*

$$\sum_{r(\mathrm{mod}\ a)} \Phi^2\left(\frac{r+x}{a}\right) = \frac{1}{a}\Psi_2(x)+\frac{1}{12}a. \tag{2. 6}$$

Proof. We may assume, without loss of generality, that $0 \leqq x < 1$. Then

$$\sum_{r(\mathrm{mod}\ a)} \Phi^2\left(\frac{r+x}{a}\right) = \sum_{r=0}^{a-1}\left(\frac{r+x}{a}-\frac{1}{2}\right)^2 = \frac{1}{a^2}\sum_{r=0}^{a-1}\left(r+x-\frac{a}{x}\right)^2$$

We recall that

$$\sum_{r=0}^{a-1}(r+y)^2 = \frac{1}{3}\{B_3(y+a)-B_3(y)\},$$

where

$$B_3(y) = y^3-\frac{3}{2}y^2+\frac{1}{2}y.$$

Thus

$$\sum_{r(\mathrm{mod}\ a)} \Phi^2\left(\frac{r+x}{a}\right) = \frac{1}{3a^2}\left\{B_3\left(x+\frac{1}{2}a\right)-B_3\left(x-\frac{1}{2}a\right)\right\},$$

which reduces to (2. 6).

Lemma 2. *Let* $(a, b)=1$. *Then*

$$\sum_{\substack{r(\mathrm{mod}\ a)\\ s(\mathrm{mod}\ b)}} \Phi^2\left(\frac{r+x}{a}-\frac{s+y}{b}\right) = \frac{1}{ab}\Psi_2(bx-ay)+\frac{1}{12}ab. \tag{2. 7}$$

Since

$$\Phi\left(\frac{r+x}{a}-\frac{s+y}{b}\right) = \Phi\left(\frac{br-as}{ab}+\frac{bx-ay}{ab}\right),$$

we have

$$\sum_{r,s} \Phi^2\left(\frac{r+x}{a}-\frac{s+y}{b}\right) = \sum_{t(\mathrm{mod}\ ab)} \Phi^2\left(\frac{t}{ab}+\frac{bx-ay}{ab}\right)$$

and (2. 7) follows at once from (2. 5).

3. We shall now prove the theorem stated in § 1. Let S denote the left hand side of (1. 7). Then by (2. 5) we have

$$S = \sum_{r,s,t}\{\Phi(\xi-\eta)\Phi(\eta-\zeta)+\Phi(\eta-\zeta)\Phi(\zeta-\xi)+\Phi(\zeta-\xi)\Phi(\xi-\eta)\}, \tag{3. 1}$$

where ξ, η, ζ are defined by (2. 4). Now consider the sum

$$T = \sum_{r,s,t}\{\Phi(\xi-\eta)+\Phi(\eta-\zeta)+\Phi(\zeta-\xi)\}^2. \tag{3. 2}$$

In view of (1. 5) we have

$$T = \sum_{r,s,t}\{[\xi-\eta]+[\eta-\zeta]+[\zeta-\xi]+\tfrac{3}{2}\}^2. \tag{3. 3}$$

Clearly there is no loss in generality in assuming that

$$0 \leqq x < 1, \quad 0 \leqq y < 1, \quad 0 \leqq z < 1 \tag{3. 4}$$

and that

$$0 \leqq r < a, \quad 0 \leqq s < b, \quad 0 \leqq t < c. \tag{3.5}$$

It follows from (3. 4) and (3. 5) that

$$0 \leqq \xi < 1, \quad 0 \leqq \eta < 1, \quad 0 \leqq \zeta < 1$$

and therefore

$$|\xi - \eta| < 1, \quad |\eta - \zeta| < 1, \quad |\zeta - \xi| < 1.$$

Consequently each of $[\xi - \eta]$, $[\eta - \zeta]$, $[\zeta - \xi]$ is equal to 0 or -1.

Two possibilities must be considered:

Case I. Integers r, s, t exist such that

$$\frac{r+x}{a} = \frac{s+y}{b} = \frac{t+z}{c}. \tag{3.6}$$

If such integers exist they are uniquely determined. For assume a second triple r', s', t' such that

$$\frac{r'+x}{a} = \frac{s'+y}{b} = \frac{t'+z}{c}.$$

Then clearly

$$\frac{r-r'}{a} = \frac{s-s'}{b} = \frac{t-t'}{c},$$

which implies

$$r \equiv r' \pmod{a}, \quad s \equiv s' \pmod{b}, \quad t \equiv t' \pmod{c}.$$

Case II. (3. 6) is never satisfied.

If r, s, t satisfy (3. 6) it is evident that

$$[\xi - \eta] + [\eta - \zeta] + [\zeta - \xi] = 0.$$

For all other triples, however, we have

$$[\xi - \eta] + [\eta - \zeta] + [\zeta - \xi] = -1 \quad \text{or} \quad -2.$$

It therefore follows from (3. 3) that

$$T = \begin{cases} \frac{1}{4} abc + 2 & \text{(case I)} \\ \frac{1}{4} abc & \text{(case II).} \end{cases} \tag{3.7}$$

Now, on the other hand, it is clear from (3. 1) and (3. 2) that

$$\begin{aligned} T &= 2S + \sum_{r,s,t} \{\Phi^2(\xi - \eta) + \Phi^2(\eta - \zeta) + \Phi^2(\zeta - \xi)\} \\ &= 2S + a \sum_{s,t} \Phi^2(\eta - \zeta) + b \sum_{t,r} \Phi^2(\zeta - \xi) + c \sum_{r,s} \Phi^2(\xi - \eta). \end{aligned} \tag{3.8}$$

Applying Lemma 2, we get

$$S = \frac{1}{2} T - \frac{abc}{8} - \frac{a}{2bc} \Psi_2(cy - bz) - \frac{b}{2ca} \Psi_2(az - cx) - \frac{c}{2ab} \Psi_2(bx - ay). \tag{3.9}$$

If we put

$$\delta = \begin{cases} 1 & \text{(case I)} \\ 0 & \text{(case II)}, \end{cases}$$

then by (3. 7)

$$\frac{1}{2}T - \frac{abc}{8} = \delta$$

and (3. 9) reduces to (1. 7). This completes the proof of the theorem.

4. We assume in what follows that $0 \leqq x < 1$, $0 \leqq y < 1$, $0 \leqq z < 1$. When $x = y = z = 0$, we have

$$\begin{aligned} s(a, b, c; 0, 0, 0) &= \sum_{t \pmod c} \Phi\left(\frac{at}{c}\right) \Phi\left(-\frac{bt}{c}\right) \\ &= \frac{1}{4} - \sum_{t \pmod c} \left(\left(\frac{at}{c}\right)\right)\left(\left(\frac{bt}{c}\right)\right) \\ &= \frac{1}{4} - \sum_{t \pmod c} \left(\left(\frac{ab't}{c}\right)\right)\left(\left(\frac{t}{c}\right)\right), \end{aligned}$$

so that

$$s(a, b, c;\ 0, 0, 0) = \tfrac{1}{3} - s(ab', c).$$

Thus (1. 7) becomes

$$s(bc', a) + s(ca', b) + s(ab', c) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right),$$

in agreement with (1. 1).

In the next place, if we take $c = 1$, $z = 0$ and replace y by $-y$, (1. 6) implies

$$s(a, b, 1;\ x, -y, 0) = \Phi(-x)\Phi(-y),$$

$$s(b, 1, a; -y, 0, x) = \sum_{r \pmod a} \Phi\left(b\frac{r+x}{a} + y\right) \Phi\left(-\frac{r+x}{a}\right),$$

$$s(1, a, b; 0, x, -y) = \sum_{s \pmod b} \Phi\left(-\frac{s+y}{b}\right) \Phi\left(x + a\frac{s+y}{b}\right).$$

Thus (1. 7) becomes

$$\begin{aligned} &\sum_{r \pmod a} \Phi\left(b\frac{r+x}{a} + y\right) \Phi\left(-\frac{r+x}{a}\right) + \sum_{s \pmod b} \Phi\left(-\frac{s+y}{b}\right) \Phi\left(x + a\frac{s+y}{b}\right) \\ &= \delta - \Phi(-x)\Phi(-y) - \frac{a}{2b}\Psi_2(y) - \frac{b}{2a}\Psi_2(x) - \frac{1}{2ab}\Psi_2(bx + ay). \end{aligned} \tag{4. 1}$$

To show that (4. 1) is equivalent to (1. 4), we remark first that in the present case ($c = 1$, $z = 0$), $\delta = 1$ if and only if $x = y = 0$. If $x = y = 0$, then since

$$\Phi(-x) = -\Phi(x) \qquad (x \neq \text{integer}),$$

(4. 1) reduces to

$$\sum_{r=1}^{a-1} \Phi\left(\frac{br}{a}\right) \Phi\left(\frac{r}{a}\right) + \sum_{s=1}^{b-1} \Phi\left(\frac{s}{b}\right) \Phi\left(\frac{as}{b}\right) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right),$$

which is correct.

If $x=0$, $y \neq 0$, (4. 1) becomes

$$\sum_{r=1}^{a-1} \Phi\left(\frac{br}{a} + y\right) \Phi\left(\frac{r+x}{a} + \sum_{s=0}^{b-1} \Phi\left(\frac{s+y}{b}\right) \Phi\left(a\frac{s+y}{b}\right) = \right.$$

(4. 2)

$$= \frac{a}{2b} \Psi_2(y) + \frac{1}{2ab} \Psi_2(xay) + \frac{b}{2a} \Psi_2(0).$$

If for some integer r_0,

$$\frac{br_0}{a} + y = s_0, \tag{4. 3}$$

where s_0 is an integer, it follows that

$$\frac{a(y-s_0)}{b} = -r_0, \quad \frac{r_0}{a} = -\frac{y-s_0}{b}.$$

Thus (4. 2) is in agreement with (1. 4). If (4. 3) is not satisfied there is of course no difficulty. The case $x \neq 0$, $y=0$ is handled in exactly the same way.

Finally let $xy \neq 0$. Then if for some integer r_0, we have

$$b\frac{r_0+x}{a} + y = s_0, \tag{4. 4}$$

where s_0 is an integer, it follows that

$$x + a\frac{y-s_0}{b} = -r_0, \quad \frac{r_0+x}{a} = -\frac{y-s_0}{b}.$$

Thus (4. 1) agrees with (1. 4). If (4. 4) is not satisfied there is no difficulty.

Therefore, in all cases, (4. 1) agrees with (1. 4).

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References

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