On a class of vector antilattices considered by Fuchs

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Recently L. FUCHS ([2]) has examined a class of partially ordered real linear spaces with the *antilattice property*, viz., two elements having a g. l. b. must be comparable (this term appears to have been coined by KADISON [3]). Our purpose is to show how Fuchs' antilattices may be easily explained within the already extensive framework and literature on ordered linear spaces.

These antilattices exist in great abundance; in fact, they all arise naturally in the following way. Let X be a compact Hausdorff space and let $C(X)$ denote the space of all real continuous functions on X. Let A be any linear subspace of $C(X)$ which contains the constant functions, and call $f \in A$ *positive* if $f(x) > 0$ at *every* point $x \in X$. The positive functions in A, together with the identically zero function, form a cone A^+ which partially orders A.

Lemma 1. *The space A with the ordering induced by A^+ is an antillattice.*

PROOF. Let 1 denote the function constantly equal to one, and suppose $g = \inf(f, 0)$ exists in A, where $f \in A$. It will be enough to show that $g = 0$ or $f = g$, so that $f \geqq 0$ or $f \leqq 0$. Suppose that $g \neq 0$ and $f \neq g$, and let $\alpha = \inf(f(x) - g(x))$ and $-\beta = \sup g(x)$, over all $x \in X$. The compactness of X, continuity of f and g and the assumptions imply $\alpha > 0$ and $\beta > 0$. Put $\varepsilon = \frac{1}{2}\min(\alpha, \beta)$. Then $g < \varepsilon \cdot 1 + g < 0, f$ which contradicts $g = \inf(f, 0)$. The lemma is proved.

Antilattices arising in this way are, of course, non-Archimedean, but they are not too badly so. It is easy to see that the cone A^+ chosen above is just the interior of the „usual" cone (i. e., the cone of functions which are non-negative at each point) in the sup norm topology with zero adjoined. Thus, replacing the non-zero boundary points of the cone will restore the usual Archimedean ordering of functions.

We shall give a number of characterizations of such antilattices below, but to make our account self-contained, we first recall a few basic notions from the general theory. From now on, A will be a partially ordered real linear space, i. e., A is a real linear space together with a subset A^+ such that (i) $A^+ + A^+ \subset A^+$, (ii) $\alpha A^+ \subset A^+$, for $\alpha \geqq 0$, and (iii) $A^+ \cap (-A^+) = \{0\}$. Unlike Fuchs, we shall *not* assume that A possesses the Riesz interpolation property, since this rules out some interesting cases (see examples 1 and 2 below). An element $0 \neq u \in A^+$ is an *order unit* if for any $x \in A$, there is a real $\alpha > 0$ for which $-\alpha u \leqq x \leqq \alpha u$. Following BONSALL ([1]), we say that A is *almost Archimedean* if $x = 0$ whenever there is an $a \in A^+$ with $-\alpha a \leqq x \leqq \alpha a$, for all $\alpha > 0$. This last condition is essentially a restriction on the two-dimensional „slices" of the cone, and the reader can easily verify

Lemma 2. *A is almost Archimedean if and only if each plane through the origin cuts A^+ in an acute planar wedge (possibly a ray) or else in the origin only.*

For any $0 \neq a \in A^+$, let $\|x\|_a = \inf\{\alpha > 0: -\alpha a \leqq x \leqq \alpha a\}$.

Lemma 3. *The function $x \to \|x\|_a$ is a norm if and only if a is an order unit and A is almost Archimedean.*

Proof. The function in question is the Minkowski functional of the set $\{x \in A: -a \leqq x \leqq a\}$, which is clearly convex and circled; it is absorbing if and only if a is an order unit. Finally, A is almost Archimedean if the function is a norm, and the converse is true if a is an order unit, for then the aforementioned set can contain no line, or equivalently, $\|x\|_a = 0$ implies $x = 0$.

If A has an order unit u, then a *state* of A is a real linear functional f such that $f(A^+) \geqq 0$ and $f(u) = 1$. If A is also almost Archimedean, the set of all states is a w^*-compact convex subset of the unit ball of A^*, the dual being taken in the norm $\|\cdot\|_u$; its extreme points will be called *pure states.* A state f is *strictly positive* if $f(a) > 0$, whenever $0 \neq a \in A^+$. An *order ideal* in A is a linear subspace I such that $a \in A$, $b \in I$ and $-b \leqq a \leqq b$ imply $a \in I$,

We can now formulate the main result.

Theorem. *Let A be a partially ordered real linear space with positive cone A^+. Then the following conditions are equivalent:*

1) A *is linearly order-isomorphic to a subspace of $C(X)$, X compact Hausdorff, which contains constants and separates points, such that each $0 \neq a \in A^+$ corresponds to a function on X which is strictly positive at every point.*

2) *Each $0 \neq a \in A^+$ is an order unit and A is almost Archimedean.*

3) *For each $0 \neq a \in A^+$, the function $x \to \|x\|_a$ is a norm.*

4) *A has an order unit, is almost Archimedean, and each (pure) state of A is strictly positive.*

5) *The only order ideals which are spanned by their positive elements are $\{0\}$ and A, and the intersection of all maximal order ideals is $\{0\}$.*

6) *The „blunted cone" $A^+ - \{0\}$ is open in the finest locally convex topology of A, and its closure C in this topology satisfies: $C \cap (-C) = \{0\}$.*

7) *For any two linearly independent elements $a, b \in A^+$, the „blunted planar wedge" $(P \cap A^+) - \{0\}$ is open in P, and acute, where P is the plane through the origin spanned by a and b.*

8) *For each $0 \neq a \in A^+$, the order interval $(-a, a) = \{x \in A: -a < x < a\}$ is open in the finest locally convex topology of A, and contains no line.*

Finally, a space A satisfying one, and hence all of the above conditions is an antilattice.

Proof. Clearly 1) implies 2). The implication 2) implies 3) is half of Lemma 3.

To see that 3) implies 4), note first that any $0 \neq a \in A^+$ is an order unit by Lemma 3, and A is almost Archimedean. It is clear, however, that no state can vanish on an order unit.

Now 4) implies 5), for by a general result of Kadison ([1], Theorem 3, p. 405), any order ideal $I \neq A$ is contained in the null space of some state. Since the set of states vanishing on I is w^*-compact and convex, the Krein—Milman Theorem yields a pure state f vanishing on I. By 4), f cannot vanish on any non-zero element of $I \cap A^+$, so the latter is $\{0\}$; and if I is spanned by its positive elements, then I must be $\{0\}$ too. By Corollary 2 of [1] (p. 406), and the comments preceding Theorem 4 of [1] (p. 405), the maximal order ideals correspond one-to-one to the null spaces of states, and the states (or pure states, by a simple Krein—Milman argument) separate elements of A, proving 5).

To show that 5) implies 6), first note that any $0 \neq a \in A^+$ is an order unit, since $I = \{x \in A: -\alpha a \leqq x \leqq \alpha a$, for some $\alpha \geqq 0\}$ is a non-zero order ideal spanned by its positive elements. In fact, if $x \in I$, $\alpha a \pm x \in I \cap A^+$ for a suitable $\alpha > 0$, so that $x = \frac{1}{2}((\alpha a + x) - (\alpha a - x))$. Thus $I = A$, and a is an order unit. In the topology described, the open neighborhoods of a point must contain an open segment about the point in every direction. Thus an order unit cannot be on the boundary of A^+ in this topology and if zero (a boundary point) is deleted from A^+,each remaining point is interior. We have already remarked on the correspondence between maximal order ideals and states. Our assumption amounts to assuming that states separate the elements of A. But C is the intersection of all closed positive half-spaces of states, so if $a \in C \cap (-C)$, we have $f(a) = 0$, for each state, so that $a = 0$, and 6) holds.

Now 7) is clearly the two-dimensional localization of 6) and hence is implied by it.

Assuming 7), we now prove that 8) holds. First observe that if $(-a, a)$ contained a line, then so would some planar section through the origin, which is clearly impossible by 7). Finally, a boundary point of $(-a, a)$ in the topology described would also be a boundary point in some planar section through the origin, again an impossibility by 7).

To complete the chain of implications, assume that 8) obtains. The assumptions imply that the Minkowski functional of $\{x \in A: -a \leqq x \leqq a\}$ for $0 \neq a \in A^+$, is a norm, that A is almost Archimedean and, further, that any $0 \neq a \in A^+$ is an order unit (see the proof of Lemma 3). We remarked above that this last piece of information shows every state to be strictly positive. Let X be the w^*-closure of the set of pure states, and let $\bar{a}$ be the affine w^*-continuous function on X defined by $\bar{a}(f) = f(a)$, for $f \in X$. The fact that A is almost Archimedean makes the linear imbedding $a \to \bar{a}$ of A into $C(X)$ one-to-one ([1], p. 406, Corollary 2). This is enough to make $a \geqq 0$ imply $\bar{a} \geqq 0$, but the converse is generally false for almost Archimedean spaces. In our case, however, $A^+ - \{0\}$ has no boundary points (these cause all the trouble) and if $\bar{a} \geqq 0$, then either $a = 0$ or else a is not annihilated by any state and hence is interior to A^+ in the finest locally convex topology of A. This proves 1).

The last statement is a consequence of 1) and Lemma 1, and the theorem follows.

Examples. 1) Let A be any subspace of the real linear space S of all bounded self-adjoint operators on a complex Hilbert space and suppose that A contains the identity operator 1. Let A^+ be the cone consisting of all positive definite (invertible) operators together with the zero operator. It is not hard to see that $A^+ - \{0\}$ is open in the norm topology for operators. In fact, if a is positive definite, the norm open ball $\{a^{\frac{1}{2}} x a^{\frac{1}{2}} : \|a - a^{\frac{1}{2}} x a^{\frac{1}{2}}\| < \|a^{-1}\|^{-1}, x \in A\}$ consists entirely of positive definite operators, because x must satisfy $\|1 - x\| < 1$. Hence this set is open in the finest locally convex topology of A. The closure C of A^+ in the latter topology is contained in the norm closure P of A^+ and it is clear that $P \cap (-P) = \{0\}$ (the only operator which is both positive and negative semi-definite is the zero operator). Thus A satisfies 6). If $A = S$, then A, ordered in the usual way by P, is also an antilattice ([3], p. 507, Theorem 6). The interior of P in the norm-topology, of course, is just $A^+ - \{0\}$.

2) Let H be any real pre-Hilbert space (of dimension $\geqq 2$) and let $A=R\oplus H$ ($R=$ the reals) be ordered by taking $A^+=\{(\alpha, x): \|x\|<\alpha\}$. Then A satisfies 7). If $P=\{(\alpha, x): \|x\|\leqq\alpha\}$, then it can easily be shown that P is the norm closure of A^+ in the norm $\|\cdot\|_u$, where $u=(1, 0)$. Moreover, when ordered by the cone P, A is also an antilattice ([4], p. 43, Proposition 21; if H were one-dimensional, P would give a lattice ordering of the plane A). In fact, this example is really a special case of example 1) above [5], Theorem 2. It would be of some interest to characterize those antilattices A (described in the theorem) for which the norm closure (in any of the equivalent norm topologies given by condition 3) of A^+ also provides an antilattice ordering. It is known ([4], p. 44, bottom) that JW-factors A have this property when A^+ is chosen as in example 1) above.

3) Let A be the space of all polynomials without constant term (zero included) on $[1, 2]$, and let A^+ be the set of all polynomials $p\in A$ such that $p(t)>0$, for $1\leqq t\leqq 2$, together with zero. Although A is a space of continuous functions, it does not contain non-zero constants, so Lemma 1 does not immediately apply. But if $0\neq a\in A^+$, then the linear order isomorphism $p\to pa^{-1}$ gives a representation of the type considered in 1) of the theorem. This shows that to satisfy the conditions of the theorem, a subspace A of $C(X)$ need not contain constants if it contains a function which is strictly positive at every point.

It should perhaps be mentioned that the space S of all selfadjoint operators on a complex Hilbert space (of dimension at least two) in example 1) fails to have the Riesz interpolation property in either of the two orderings described. The same is true for the space $A=R\oplus H$ (assuming H is a real Hilbert space) in example 2), again, with either ordering. For suppose to the contrary, that $A(=S$ or $R\oplus H)$ had the Riesz property. Then as Fuchs shows ([2], Theorem 9), the conjugate space A^* is an abstract (L)-space, and in particular, a vector lattice. This would imply, in the first instance, that S^{**} is a vector lattice. But S^{**} is linearly order isomorphic to the set of all self-adjoint operators in some von Neumann algebra, and a result of S. SHERMAN (see e.g., [3], Corollary 5) would imply that this algebra is abelian, and also that all operators in S commute. Since this happens only when the underlying Hilbert space of S is one dimensional, we have a contradiction, so our assumption that S had the Riesz property was erroneous. In the second example, it is not difficult to see that A^*, ordered by the cone of positive linear functionals, is linearly order isomorphic to A (here we need completeness of H) ordered by the cone P, so A^* cannot be a lattice and consequently A cannot satisfy the Riesz condition.

References

[1] F. F. BONSALL, Sublinear functionals and ideals in partially ordered vector spaces, *Proc. London Math. Soc.* **4** (1954), 402—418.

[2] L. Fuchs, On partially ordered vector spaces with the Riesz interpolation property, *Publ. Math. Debrecen* **12** (1965), 335—343.

[3] R. V. KADISON, Order properties of bounded self-adjoint operators, *Proc. Amer. Math. Soc.* **2** (1951). 505—510.

[4] D. TOPPING, Jordan algebras of self-adjoint operators, *Amer. Math. Soc. Memoir* **53** (1965).

[5] D. TOPPING, An isomorphism invariant for spin factors, *J. Math. Mech.* **15** (1966), 1055—1064.

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