Some remarks concerning the stable sequences of random variables

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§ 1.

Let us consider a probability space $\{\Omega, \mathscr{A}, P\}$, where $\mathscr{A}$ is a σ-algebra of some subsets of the basic space Ω and P is a measure defined on $\mathscr{A}$ and normed by the condition $P(\Omega)=1$. The elements of Ω will be denoted by ω and the elements of $\mathscr{A}$ will be called events. Random variables are defined as measurable functions defined on Ω. If A and B are events and $P(B)>0$, then $P(A|B)$ denotes the conditional probability of A under the condition B. The sign $\bar{A}$ denotes the event consisting of the non-occurrence of the event A.

Let us recall some definitions.

Definition 1. ([1]) *The sequence $\{\xi_n\}$ of random variables is called strongly mixing within limiting distribution $F(x)$ if the conditional distribution of ξ_n under the condition B with $P(B)>0$, i. e. the probability $P(\xi_n<x|B)$, converges as $n\to+\infty$ to the distribution function $F(x)$.*

Definition 2. ([2]) *The sequence $\{\xi_n\}$ of random variables is called stable if for any event B with $P(B)>0$ the conditional distribution of ξ_n given B tends to a limiting distribution function, i. e.*

$$\lim_{n\to+\infty} P(\xi_n<x|B)=F_B(x)$$

for every x which is a continuity point of the distribution function $F_B(x)$.

It can be easily seen that the set of the discontinuity points of $F_B(x)$ is a subset of the discontinuity points of $F_\Omega(x)$ and hence the set of all discontinuity points for every B is denumerable.

It can be shown [1] that if $\{\xi_n\}$ is a stable sequence of random variables then for every fixed number x the function

$$Q(x, B)=F_B(x)P(B)$$

is a measure in B. $Q(x, B)$ is for fixed x absolutely continuous with respect to then probability measure P. Thus by the Radom—Nykodim theorem

$$Q(x, B)=\int_B \alpha_x(\omega)\, dP(\omega),$$

where $\alpha_x(\omega)$ is the Radon—Nykodim derivative of $Q(x, B)$ with respect to P and hence it is determined uniquely modulo P. The function $\alpha_x(\omega)$ will be called the local density of the stable sequence $\{\xi_n\}$. We have $0 \leqq \alpha_x(\omega) \leqq 1$. Clearly, if $\alpha_x(\omega)$ is constant in ω with probability 1, then the sequence $\{\xi_n\}$ is strongly mixing with limiting distribution α_x.

We remark that $Q(x, B)$ is for fixed B a monotonically non-increasing left-continuous function of x with $\lim_{x \to -\infty} Q(x, B) = 0$ and $\lim_{x \to +\infty} Q(x, B) = P(B)$.

The local density $\alpha_x(\omega)$ is uniquely determined except for a set of P-measure 0. This means that we can change its values on an ω-set of P-measure 0. We say that $\lambda_x(\omega)$ is a variant of $\alpha_x(\omega)$ if they differ for a fixed x only on an ω-set of P-measure 0. Now we prove the following.

LEMMA 1. *Let $\alpha_x(\omega)$ be the local density of the stable sequence $\{\xi_n\}$ of random variables. Then we can define a variant $\lambda_x(\omega)$ of $\alpha_x(\omega)$ which is a distribution function in x with probability* 1.

PROOF. We have evidently

$$P(\alpha_x(\omega) \leqq \alpha_y(\omega)) = 1$$

if $x < y$. Let the sequence of rational numbers be $r_1, r_2, \dots$. We define $\lambda_{r_1}(\omega)$ to be any variant of $\alpha_{r_1}(\omega)$. Now $\lambda_{r_2}(\omega)$ is defined as a variant of $\alpha_{r_2}(\omega)$ such that $\lambda_{r_2}(\omega) \geqq \lambda_{r_1}(\omega)$ for every ω if $r_1 < r_2$ and $\lambda_{r_1}(\omega) \leqq \lambda_{r_1}(\omega)$ for every ω if $r_1 > r_2$. If $\lambda_{r_1}, \lambda_{r_2}, \dots \lambda_{r_k}$ is already defined then we can define $\lambda_{r_{k+1}}(\omega)$ such that $\lambda_{r_{k+1}}(\omega)$ is a variant of $\alpha_{r_{k+1}}(\omega)$ and

$$\lambda_{r_{k+1}}(\omega) \geqq \lambda_{r_j}(\omega), \quad \text{if} \quad j \leqq k \quad \text{and} \quad r_j < r_{k+1},$$

or

$$\lambda_{r_{k+1}}(\omega) \leqq \lambda_{r_j}(\omega) \quad \text{if} \quad j \leqq k \quad \text{and} \quad r_j > r_{k+1}.$$

For an irrational x we define $\lambda_x(\omega)$ to be $\lim_{j \to \infty} \lambda_{r_j}(\omega)$, where $\{r_j\}$ is an increasing sequence of rational numbers tending to x. It is easy to see that the value of $\lambda_x(\omega)$ does not depend on the choice of the increasing sequence $\{r_j\}$ of rational numbers tending to x.

We have to prove that $\lambda_x(\omega)$ is a variant of $\alpha_x(\omega)$. In fact, $Q(r_j, B)$ converges for fixed B to $Q(x, B)$ if r_j converges to x from the left. From this by the Lebesque theorem it follows that

$$\int_\Omega \alpha_x(\omega)\, dP(\omega) = \int_\Omega \lambda_x(\omega)\, dP(\omega).$$

Now evidently

$$P(\alpha_x(\omega) \geqq \lambda_x(\omega)) = 1.$$

From these two relations we obtain $P(\lambda_x(\omega) = \alpha_x(\omega)) = 1$, which means that $\lambda_x(\omega)$ is a variant of the local density.

It follows also that if $x < r$, where x is irrational and r is rational, then $\lambda_x(\omega) \leqq \lambda_r(\omega)$, and so $\lambda_x(\omega) \leqq \lambda_x(\omega)$, if $x < y$, x, y are arbitrary numbers. This mean that $\lambda_x(\omega)$ is for fixed ω a monotonically increasing function.

Let $\{r_j\}$ be an increasing sequence of rational numbers tending to the rational. Then $P(\lim_{j\to\infty} \lambda_{r_j}(\omega) = \lambda_r(\omega)) = 1$. In fact, $Q(x, B)$ being left-continuous at the point r, we have by Lebesque's theorem

$$\int_\Omega (\lambda_r - \lim_{j\to\infty} \lambda_{r_j})\, dP = \lim_{j\to+\infty} \int_\Omega (\lambda_r - \lambda_{r_j})\, dP = 0,$$

which is our assertion. This means that except for an ω-set of P-measure 0 the function $\lambda_x(\omega)$ is left-continuous. It remains to prove that $P(\lim_{x\to+\infty} \lambda_x(\omega) = 1) = P(\lim_{x\to-\infty} \lambda_x(\omega) = 0) = 1$. We have $\lim_{x\to+\infty} Q(x, B) = P(B)$ for every fixed B. This means that

$$\lim_{x\to+\infty} \int_\Omega (1 - \lambda_x(\omega))\, dP(\omega) = 0.$$

By the Lebesgue theorem we have then

$$\int_\Omega (1 - \lim_{x\to+\infty} \lambda_x(\omega))\, dP(\omega) = 0,$$

which is our first assertion. The second can be obtained similarly. In paper [3] of P. Révész a similar lemma is proved for sequences of equivalent random variables. His proof, however, cannot be generalized verbatim to the case of stable sequences of random variables. In fact, his formula (3), which is true for sequences of equivalent random variables, is not true in our case.

On the basis of Lemma 1 we will suppose in what follows, that the local density $\alpha_x(\omega)$ is with probability 1 a distribution function.

In his paper [1] A. Rényi showed that if $\{\xi_n\}$ is a mixing sequence of random variables with limiting distribution $F(x)$ and η an arbitrary random variable having a discrete distribution and $g(x, y)$ a continuous function of two variables,

$$G_n = g(\xi_n, \eta)$$

then the sequence of random variables is stable. We have generalized this result in [4]. Namely, we omitted the supposition that η has a discrete distribution. In this case the local density is

$$\int_{\{g(y,\eta)<x\}} dF(y).$$

A Rényi asked the following problem: supposing $\{\xi_n\}$ to be stable instead of being strongly mixing, is it true that $g(\xi_n, \eta)$ is also stable? The aim of the present paper is to give a positive answer to this problem and to study some related topics.

§ 2.

Theorem 1. *Let $g(x)$ be a real-valued continuous function and $\{\xi_n\}$ a stable sequence of random variables with local density $\alpha_x(\omega)$. Then the sequence $g(\xi_n)$ is also stable with local density*

$$\int_{\{g(y)<x\}} d_y \alpha_y(\omega).$$

PROOF. The set $A(x)=\{y: g(y)<x\}$ is an open set on the real axis. Thus $A(x)=\sum_{k=1}^{\infty} I_k(x)$, where $I_k(x)$ are open intervals $(k=1, 2, \ldots)$ and for any event B

$$P\big(g(\xi_n) < x, B\big) = \sum_{k=1}^{\infty} P\big(\xi_n \in I_k(x), B\big).$$

Let us suppose that the endpoints of $I_k(x)$ $(k=1, 2, \ldots)$ are continuity points of $F_B(x)$ where $F_B(x)=\lim_{n\to+\infty} P(\xi_n<x|B)$. Then

$$\lim_{n\to+\infty} P\big(\xi_n \in I_k(x), B\big) = P(B)\big[F_B\big(b_k(x)\big) - F_B\big(a_k(x)\big)\big]$$

exists, where $a_k(x) \leqq b_k(x)$ are the endpoints of the interval $I_k(x)$. Since we have $0=\sum_{k=1}^{\infty}\big(F_B(b_k(x)) - F_B\big(a_k(x)\big)\big) \leqq 1$, we can choose a positive integer $k_0=k_0(\varepsilon)$ such that

$$\sum_{k\geqq k_0} \big(F_B(b_k(x)) - F_B\big(a_k(x)\big)\big) < \varepsilon$$

be satisfied. Let

$$f(y) = \begin{cases} 1, & \text{if} \quad x_k(x) < y < b_k(x), \qquad (k = k_0, k_1, \ldots) \\ 0, & \text{otherwise} \end{cases}$$

and let us put a continuous and bounded function $f_1(y)$ such that $f(y) \leqq f_1(y)$ $(-\infty<y<+\infty)$ be satisfied and at the same time

$$\left| \int_{-\infty}^{+\infty} f_1(y)\, dF_B(y) - \sum_{k=k_0}^{+\infty} \big(F_B\big(b_k(x)\big) - F_B\big(a_k(x)\big)\big) \right| < \varepsilon$$

hold. Here and in what follows integration with respect to a distribution function means always Lebesgue—Stieltjes integration with respect to the measure generated by that distribution function on the real axis. The construction of such a function $f_1(y)$ can be done with standard methods. Then, since

$$\limsup_{n\to+\infty} \int_{-\infty}^{+\infty} f(y)\, dP(\xi_n < y|B) \leqq \int_{-\infty}^{+\infty} f_1(y)\, dF_B(y) \leqq 2\varepsilon,$$

we see that if $n \geqq n_0(\varepsilon)$

$$\sum_{k=k_0}^{\infty} P\big(\xi_n \in I_k(x)|B\big) \leqq 4\varepsilon.$$

If k is an arbitrary fixed positive integer for which $K \geqq k_0$ and if $n \geqq n_0(\varepsilon)$ we have

$$\sum_{k=1}^{K} P\big(\xi_n \in I_k(x), B\big) \leqq P\big(g(\xi_n) < x, B\big) \leqq \sum_{k=1}^{K} P\big(\xi_n \in I_k(x), B\big) + 4\varepsilon.$$

This means that

$$\lim_{n\to+\infty} P\big(g(\xi_n) < x, B\big) = \int_{\{g(y)<x\}} d_y Q(y, B)$$

Now we have for any B of positive probability

$$\int_{\{g(y)<x\}} d_y Q(y, B) = \sum_{k=1}^{\infty} P(B)\left[F_B(b_k(x)) - F_B(a_k(x))\right] =$$

$$= \sum_{k=1}^{\infty} \int_B \Big(\int_{a_k(x)}^{b_k(x)} d_y \alpha_y(\omega) \Big)\, dP(\omega) = \int_B \Big(\int_{\{g(y)<x\}} d_y \alpha_y(\omega) \Big)\, dP(\omega),$$

since by Lemma 1 $\alpha_y(\omega)$ denotes with probability 1 a distribution function of y. It is easy to see that this limit relation is true for every real x with the exception of a denumerable set of values of x. Since the last limit relation holds for every B we conclude that the sequence $g(\xi_n)$ is stable and since the local density is uniquely determined modulo P we see that the local density is with probability 1

$$\int_{\{g(y)<x\}} d_y \alpha_y(\omega)$$

This proves the theorem.

Remark 1. *If the sequence* $\{\xi_n\}$ *is strongly mixing with limiting distribution* $F(x)$, *then the sequence* $g(\xi_n)$ *is also strongly mixing with limiting distribution*

$$\int_{\{g(y)<x\}} dF(y).$$

In fact, in this case

$$Q(y, B) = F(y)P(B),$$

and thus

$$\lim_{n\to+\infty} P\big(g(\xi_n) < x, B\big) = P(B) \int_{\{g(y)<x\}} dF(y).$$

Let in what follows $\underline{\eta}$ and $\underline{\eta}_n$ denote random vector variables of l dimensions $(n=1, 2, \ldots)$. We say that $\underline{\eta}_n$ converges in probability measure to $\underline{\eta}$ if the i-th component of $\underline{\eta}_n$ $(i=1, 2, \ldots, l)$ converges in probability measure to the i-th component of $\underline{\eta}$. Let further $g(x, y_1, y_2, \ldots, y_l)$ denote a continuous real-valued function of $l+1$ variables. We shall denote it in the sequel briefly by $g(x,\underline{y})$, where $\underline{y}$ is a vector of l variables.

Theorem 2. *Let* $\{\xi_n\}$ *be a stable sequence of random variables with local density* $\alpha_x(\omega)$ *and* $\underline{\eta}_n$ *a sequence of random vector variables of* l *dimensions. We suppose that* $\underline{\eta}_n$ *converges in probability measure to the random vector variable* $\underline{\eta}$. *Let further* $g(x, \underline{y})$ *be a continuous function of* $l+1$ *variables. Then the sequence* $X_n = g(\xi_n, \underline{\eta}_n)$ *is also stable with local density*

$$\int_{\{g(x,\underline{\eta}(\omega))<z\}} d_x \alpha_x(\omega).$$

To prove this theorem we state first some lemmas.

Lemma 2. *Let us suppose that η_n converges in probability measure to the vector variable η. Let D denote the domain, determined by the intervals $[a_i, b_i)$ $(i=1, 2, \ldots, l)$. We suppose that a_i and b_i are continuity points of the distribution function of the i-th component of η. Let A and A_n denote the events $\{\eta \in D\}$ and $\{\eta_n \in \Phi\}$ respectively. Then $P(A \circ A_n) \to 0$ as $n \to +\infty$. Here $A \circ A_n$ denotes the symetric difference of A and A_n.*

The proof of Lemma 2 is almost trivial.

Lemma 3. *Let $\{\xi_n\}$ be a sequence of random variables which has a limiting distribution as $n \to +\infty$ and $\underline{\eta}_n$ a sequence of random vector variables converging in probability measure to the vector variable $\underline{\eta}$. If $g(x,y)$ is a continuous function of $l+1$ variables, then the random variables*

$$g(\xi_n, \underline{\eta}_n) - g(\xi_n, \underline{\eta})$$

converge in probability to zero as $n \to +\infty$.

Proof. Since $\lim_{n \to +\infty} P(\xi_n < x)$ exist we can choose the numbers a and b and the domain D defined in the same manner as in Lemma 2 such that the inequality

$$P(a \leqq \xi_n < b, \underline{\eta}_n \in D) > 1 - \varepsilon$$

be satisfied if $n \geqq n_0(\varepsilon)$. Let us consider the bounded domain of $l+1$ dimensions: $\{a \leqq x < b, \underline{y} \in D\}$. The function $g(x, \underline{y})$, being continuous, is uniformly continuous in this domain. Thus the domain D can be splitted with the aid of the non-overlapping subdomains $D_1, D_2, \ldots, D_k$, such that if $\underline{y}_1$ and $\underline{y}_2$ are points of a subdomain, say of D_j, then the inequality $|g(x,\underline{y}_1) - g(x,\underline{y}_2)| < \delta$ holds. We have then

$$P(|g(\xi_n, \eta_n) - g(\xi_n, \underline{\eta})| > \delta) \leqq$$

$$\leqq \sum_{i=1}^{k} P(|g(\xi_n, \underline{\eta}_n) - g(\xi_n, \underline{\eta})| > \delta, \underline{\eta} \in D_i, \underline{\eta}_n \in D_i, a \leqq \xi_n < b) +$$

$$+ \sum_{i=1}^{k} P(|g(\xi_n, \underline{\eta}_n) - g(\xi_n, \eta)| > \delta, \eta \in D_i, \underline{\eta}_n \overline{\in} D_i, a \leqq \xi_n < b) + \varepsilon.$$

The first sum on the right-hand side of this inequality is zero because of the choice of the domains D_i. The second sum is smaller than

$$\sum_{i=1}^{k} P(\underline{\eta} \in D_i, \underline{\eta}_n \overline{\in} D_i).$$

Since k is fixed, this sum converges to zero by means of Lemma 2, if the domains D_i are defined accordingly to Lemma 2. (and this can be done without any difficulties.)

Lemma 4. *Let us suppose that $\{X_n\}$ and $\{Y_n\}$ are sequences of random variables $X_n - Y_n$ converges in probability measure to zero as $n \to +\infty$. If one of them is stable then the other is also stable with the same local density.*

Proof. Suppose that the sequence X_n is stable. Then for any event B of positive probability we have

$$P(Y_n < x | B) = P(Y_n < x, A_n | B) + P(Y_n < x, \bar{A}_n | B),$$

where $A_n=\{|X_n-Y_n|<\varepsilon\}$. Obviously

$$P(X_n<x-\varepsilon|B)-P(\bar{A}_n|B)\leqq P(Y_n<x|B)\leqq P(X_n<x+\varepsilon|B)+P(\bar{A}_n|B).$$

If x is a continuity point of the limiting distribution $\lim_{n\to+\infty} P(X_n<x|B)$ we obtain

$$\lim_{n\to+\infty} P(Y_n<x|B)=\lim_{n\to+\infty} P(X_n<x|B), \quad \text{since} \quad \lim_{n\to+\infty} P(\bar{A}_n|B)=0.$$

This proves the lemma, since $\varepsilon>0$ was chosen arbitrarily.

Now the proof of Theorem 2. is as follows. On the basis of Lemmas 3. and 4. it is enough to prove that the sequence $g(\xi_n, \underline{\eta})$ is stable with the local density

$$\beta_z(\omega)=\int\limits_{\{(g(x,\underline{\eta}(\omega))<z\}} d_x\alpha_x(\omega).$$

Let for this purpose B be any event and let us consider the integral

$$\int\limits_B \beta_z(\omega)\,dP(\omega).$$

Since we have $0\leqq\beta_z(\omega)\leqq 1$, we can divide the interval [0, 1] by the splitting points $\beta_z^{(k)}$ $(k=0, 1, \dots, m)$ such that $\beta_z^{(k)}<\beta_z^{(k+1)}$ $(k=0, 1, \dots, m-1)$, $\beta_z^{(0)}=0$, $\beta_z^{(m)}=1$ and that the inequalities

$$\left|\int\limits_B \beta_z(\omega)\,dP(\omega)-\sum_{k=0}^{m-1}\beta_z^{(k)}P\big(\beta_z^{(k)}\leqq\beta_z(\omega)<\beta_z^{(k+1)}, B\big)\right|<\varepsilon,$$

and

$$\left|\int\limits_B \beta_z(\omega)\,dP(\omega)-\sum_{k=0}^{m-1}\beta_z^{(k+1)}P(\beta_z^{(k)}\leqq\beta_z(\omega)<\beta_z^{(k+1)}, B)\right|<\varepsilon$$

be satisfied. We denote by A_k the event $\{\beta_z^{(k)}\leqq\beta_z(\omega)<\beta_z^{(k+1)}\}$. Since the sequence $\{\xi_n\}$ is stable, there can be found the real numbers a and b such that $a<b$ and for $n\geqq n_0(\varepsilon)$ the inequality

$$P(a\leqq\xi_n<b)>1-\frac{\varepsilon}{2}$$

hold. Further there can be found a domain $D=\{a_i\leqq y_i<b_i;\ i=1, 2, \dots, l\}$ such that

$$P(\underline{\eta}(\omega)\in D)>1-\frac{\varepsilon}{2}$$

be satisfied. It follows from these that

$$P\big(a\leqq\xi_n<b,\ \eta(\omega)\in D\big)>1-\varepsilon$$

holds if $n\geqq n_0(\varepsilon)$.

We have $\sum_{k=0}^{m-1} A_k=\Omega$ and $A_iA_k=\emptyset$ if $i\neq k$. Since D is a bounded domain and thus $g(x, \underline{y})$ is uniformly continuous in the domain $\{a\leqq x<b, \underline{y}\in D\}$ of $l+1$ dimensions, we can divide D into disjoint subdomains $D_1, D_2, \dots, D_p$ such that if $\underline{y}_1$ and $\underline{y}_2$ are points of one of them, say of D_j, $(j=1, 2, \dots, p)$, then $|g(x, \underline{y}_1)-$

$-g(x, y_2)|<\delta$ hold. Let us denote by A_{kj} $(k=0, 1, \ldots, n-1; j=1, 2, \ldots, p)$ the event $A_k\{\omega: \underline{\eta}(\omega)\in D_j\}$. Then $A_{kj}A_{\nu\mu}=\emptyset$, if $k\neq\nu$ or $j\neq\mu$ and $\sum_{j=0}^{m-1}\sum_{k=1}^{p} A_{kj} =$ $=\{\omega: \underline{\eta}(\omega)\in D\}$. Let $\underline{y}_j\in D_j$ $(j=1, 2, \ldots, p)$ be fixed points. Then as it can be easily seen,

$$-\varepsilon+\sum_{k=0}^{m-1}\sum_{j=1}^{p} P\big(g(\xi_n, \underline{y}_j) < z-\delta, A_{kj}, B\big) \leqq P\big(g(\xi_n, \underline{\eta}) < z, B\big) \leqq$$

$$\leqq \sum_{k=0}^{m-1}\sum_{j=1}^{p} P\big(g(\xi_n, \underline{y}_j) < z+\delta, A_{kj}, B\big)+\varepsilon.$$

Since the sequence $\{\xi_n\}$ is stable and $\underline{y}_j$ is fixed, we conclude by Theorem 1. that the sequence $g(\xi_n, \underline{y}_j)$ is stable. Thus, if z is a continuity point of the limiting distribution of this sequence and $\delta>0$ is chosen suitably, we obtain as $n\to+\infty$

$$-\varepsilon+\sum_{k=0}^{m-1}\sum_{j=1}^{p}\int_{A_{kj}B}\Big(\int_{\{g(u, y_j)<z-\delta\}} d_u\alpha_u(\omega)\Big)\, dP(\omega) \leqq \liminf_{n\to\infty} P\big(g(\xi_n, \underline{\eta}) < z, B\big) \leqq$$

$$\leqq \limsup_{n\to+\infty} P\big(g(\xi_n, \underline{\eta}) < z, B\big) \leqq \sum_{k=0}^{m-1}\sum_{j=1}^{p}\int_{A_{kj}B}\Big(\int_{\{g(u y_j)<z+\delta\}} d_u\alpha_u(\omega)\Big)\, dP(\omega)+\varepsilon.$$

By the choice of a, b and of the domain D for $n\geqq n_0(\varepsilon)$ we conclude

$$P(a\leqq\xi_n<b,\ \eta(\omega)\in D,\ B)\geqq P(B)-2\varepsilon.$$

Thus letting $n\to+\infty$ and taking into account that $\alpha_n(\omega)$ is with probability 1 a distribution function in $\underline{u}$, we obtain

$$P(B)-2\varepsilon \leqq \sum_{k=0}^{m-1}\sum_{j=1}^{p}\int_{A_{kj}B}\Big(\int_{\{a\leqq u<b\}} d_u\alpha_u(\omega)\Big)\, dP(\omega) \leqq P(B).$$

This means that

$$\sum_{k=0}^{m-1}\sum_{j=1}^{p}\int_{A_{kj}B}\Big(\int_{\substack{\{g(u, y_j)<z\pm\delta\}\\ u<a;\ u\geqq b}} d_u\alpha_u(\omega)\Big)\, dP(\omega) \leqq 2\varepsilon.$$

We have thus from the above limiting relation

$$-3\varepsilon+\sum_{k=0}^{m-1}\sum_{j=1}^{p}\int_{A_{kj}B}\Big(\int_{\substack{\{g(u, \underline{y}_j)<z-\delta\}\\ (a\leqq u<b)}} d_u\alpha_u(\omega)\Big)\, dP(\omega) \leqq \liminf_{n\to+\infty} P\big((\xi_n, \underline{\eta}) < z, B\big) \leqq$$

$$\leqq \limsup_{n\to+\infty} P\big(g(\xi_n, \underline{\eta}) < z, B\big) \leqq \sum_{k=0}^{m-1}\sum_{j=1}^{p}\int_{A_{kj}B}\Big(\int_{\substack{\{g(u, \underline{y}_j)<z+\delta\}\\ (a\leqq u<b)}} d_u\alpha_u(\omega)\Big)\, dP(\omega)+3\varepsilon.$$

If $\omega\in A_{kj}$ and $a\leqq u<b$ we have

$$|g(u, \underline{y}_j)-g(u, \underline{\eta}(\omega))|<\delta.$$

We obtain from two last inequalities

$$-3\varepsilon+\sum_{k=0}^{m-1}\sum_{j=1}^{p}\int_{A_{kj}B}\Big(\int_{\substack{\{g(u,\underline{\eta})<z-2\delta\}\\(a\leqq u<b)}} d_u\alpha_u(\omega)\Big)\,dP(\omega)\leqq \liminf_{n\to+\infty} P\big(g(\xi_n,\underline{\eta})<z,B\big)\leqq$$

$$\leqq \limsup_{n\to+\infty} P\big(g(\xi_n,\underline{\eta})<z,B\big)\leqq \sum_{k=0}^{m-1}\sum_{j=1}^{p}\int_{A_{kj}B}\Big(\int_{\substack{\{g(u,\eta)<z+2\delta)\\(a\leqq u<b)}} d_u\alpha_u\Big)\,dP(\omega)+\varepsilon.$$

Thus omitting the restriction $a\leqq u<b$ and summing with respect to j one has

$$-5\varepsilon+\sum_{k=0}^{m-1}\int_{A_kB}\Big(\int_{\{g(u,\underline{\eta})<z-2\delta\}} d_u\alpha_u(\omega)\Big)\,dP(\omega)\leqq \liminf_{n\to+\infty} P\big(g(\xi_n,\underline{\eta})<z,B\big)\leqq$$

$$\leqq \limsup_{n\to+\infty} P\big(g(\xi_n,\underline{\eta})<z,B\big)\leqq \sum_{k=0}^{m-1}\int_{A_kB}\Big(\int_{\{g(u,\underline{\eta})<z+2\delta\}} d_u\alpha_u(\omega)\Big)\,dP(\omega)+3\varepsilon.$$

Now if B is fixed, the function

$$\int_B\Big(\int_{\{g(u,\underline{\eta})<z} d_u\alpha_u(\omega)\Big)\,dP(\omega)$$

is monotonically increasing in z. It is also left continuous. In fact, if z_0 is an arbitrary point of this function and $\{z_n\}$ is a monotonical sequence of numbers tending from the left to z_0, then by our Lemma 1 and by Lebesgue's theorem

$$\lim_{z_n\to z_0}\int_B\Big(\int_{\{g(u,\underline{\eta})<z\}} d_u\alpha_u(\omega)\Big)\,dP(\omega)=\int_B\Big(\lim_{z_n\to z_0}\int_{\{g(u,\underline{\eta})<z\}} d_u\alpha_u(\omega)\Big)\,dP(\omega).$$

Then the inequality

$$\int_B\Big(\int_{\{g(u,\underline{\eta})<z_0\}} d_u\alpha_u(\omega)\Big)\,dP(\omega)>\int_B\Big(\lim_{z_n\to z_0}\int_{\{g(u,\underline{\eta})<z\}} d_u\alpha_u(\omega)\Big)\,dP(\omega)$$

is a contradiction, since

$$\int_{\{g(u,\underline{\eta})<z\}} d_u\alpha_u(\omega)$$

is with probability 1 a distribution function in z and so is left continuous with probability 1.

Thus if z is a continuity point of this function and if $\delta>0$ is chosen according to $\varepsilon>0$, we conclude

$$-6\varepsilon+\sum_{k=0}^{m-1}\int_{A_kB}\Big(\int_{\{g(u,\underline{\eta})<z\}} d_u\alpha_u(\omega)\Big)\,dP(\omega)\leqq \liminf_{n\to+\infty} P\big(g(\xi_n,\underline{\eta})<z,B\big)\leqq$$

$$\leqq \limsup_{n\to+\infty} P\big(g(\xi_n,\underline{\eta})<z,B\big)\leqq \sum_{k=0}^{m-1}\int_{A_kB}\Big(\int_{\{g(u,\eta)<z\}} d_u\alpha_u(\omega)\Big)\,dP(\omega)+4\varepsilon.$$

Taking into account that on the set A_k we have

$$\beta_z^{(k)} \leqq \int_{\{g(u,\eta)<z\}} d_u \alpha_u(\omega) \leqq \beta_z^{(k+1)},$$

we obtain finally

$$-6\varepsilon + \sum_{k=0}^{m-1} \beta_z^{(k)} P(A_k B) \leqq \liminf_{n\to+\infty} P(g(\xi_n, \underline{\eta}) < z, B) \leqq$$

$$\leqq \limsup_{n\to+\infty} P(g(\xi_n, \underline{\eta}) < z, B) \leqq \sum_{k=0}^{m-1} \beta_z^{(k+1)} P(A_k B) + 4\varepsilon.$$

This proves our theorem.

§ 3.

In this section we consider some questions concerning the moments of a stable sequence of random variables.

Theorem 3. *Let $\{\xi_n\}$ be a stable sequence of random variables with local density $\alpha_x(\omega)$ and let us suppose that the sequence $\{\xi_n\}$ is uniformly integrable with respect to P. Let P^* be an arbitrary probability measure, which is absolutely continuous with respect to P and let us suppose that dP^*/dP — the Radon—Nykodim derivative of P^* with respect to P — is bounded with probability* 1. *Then*

$$\lim_{n\to+\infty} \int_\Omega \xi_n \, dP^* = \int_{-\infty}^{+\infty} x \, dF_\Omega^*(x),$$

where

$$F_\Omega^*(x) = \lim_{n\to+\infty} P^*(\xi_n < x).$$

Proof. The uniform integrability of $\{\xi_n\}$ with respect to P implies (cf. e.g. [2], p. 196.).

$$\lim_{n\to+\infty} \xi_n \, dP = \int_{-\infty}^{+\infty} x \, dF_\Omega(x).$$

It is known [4] that the sequence $\{\xi_n\}$ is also stable with respect to P^* and its local density in the probability space $\{\Omega, \mathscr{A}, P^*\}$ is the same as in $\{\Omega, \mathscr{A}, P\}$. It turns out particularly that $F_\Omega^*(x) = \lim_{n\to+\infty} P^*(\xi_n < x)$ exists. By the supposition that dP^*/dP is bounded with probability 1, we deduce that the sequence $\{\xi_n\}$ is uniformly integrable with respect to P^* too. It follows from this that

$$\lim_{n\to+\infty} \int_\Omega \xi_n \, dP^* = \int_{-\infty}^{+\infty} x \, dF_\Omega^*(x).$$

This is our assertion.

Remark 2. *Putting especially in Theorem* 3. $P^*(A) = P(A|B)$ *where* $P(B) > 0$, *we obtain*

$$\lim_{n\to+\infty} M(\xi_n|B) = \int_{-\infty}^{+\infty} x \, dF_B(x).$$

Remark 3. *Let us suppose that the sequence* $\{\xi_n\}$ *is strongly mixing with limiting distribution* $F(x)$ *and that the other conditions of Theorem* 3. *hold. In this case* $\alpha_x(\omega) = F(x)$ *with probability* 1, *thus we have* $F^*_\Omega(x) = F(x)$. *This means that*

$$\lim_{n\to+\infty}\int_\Omega \xi_n\,dP = \lim_{n\to+\infty}\int_\Omega \xi_n\,dP^* = \int_{-\infty}^{+\infty} x\,dF(x).$$

It follows particularly that putting $P^*(A) = P(A|B)$ *where* B *is a fixed event of positive probability we obtain*

$$\lim_{n\to+\infty} M(\xi_n|B) = \lim_{n\to+\infty} M(\xi_n) = \int_{-\infty}^{+\infty} x\,dF(x).$$

(cf. [5].)

Remark 4. It is easy to verify that one cannot omit in Theorem 3. the assumption that dP^*/dP is bounded with probability 1. In fact, let $\{\xi_n\}$ be a sequence of independent and identically distributed random variables with mean-value 0 and variance 1. Let us suppose that the fourth moment of ξ_n does not exist. By the central limit theorem we have

$$\lim_{n\to+\infty} P\left(\frac{\xi_1+\ldots+\xi_n}{\sqrt{n}} < x\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt = \Phi(x)$$

and the sequence

$$\eta_n = \frac{\xi_1+\ldots+\xi_n}{\sqrt{n}}$$

is strongly mixing with limiting distribution $\Phi(x)$. Thus it is a fortiori stable. Let us consider the measure

$$P^*(A) = \int_A \xi_1^2\,dP.$$

Obviously this is absolutely continuous with respect to P and it is a probability measure. We have $dP^*/dP = \xi_1^2$ with probability 1, and dP^*/dP is not bounded. The sequence η_n^2 by Theorem 1. is also strongly mixing with limiting distribution $2\Phi(\sqrt{x})-1$ and uniformly integrable with respect to P. Then

$$\lim_{n\to+1}\int_\Omega \eta_n^2 dP^*$$

does not exist, since the fourth moment of ξ_1 is not finite.

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