Polyadic semigroups

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1. Introduction

A *polyadic semigroup*, specifically an *n-semigroup*, is an associative *n*-ary operation F on a set $\mathfrak{M}$. I. e., F is a function whose domain is the Cartesian power $\mathfrak{M}^n$ and whose range is a subset of $\mathfrak{M}$, such that each of the $n-1$ equations [1])

$$(1) \qquad FFx_1 \ldots x_{2n-1} = Fx_1 \ldots x_k Fx_{k+1} \ldots x_{2n-1} \qquad (1 \leq k \leq n-1)$$

holds for every $(2n-1)$-tuple $x_1 \ldots x_{2n-1}$ of elements of $\mathfrak{M}$. The main purpose of this note is to prove the following result:

Theorem 1. Let F be an *n*-semigroup on a set $\mathfrak{M}$. Assume that F has a *right identity*, i. e., *there exists an* $(n-1)$*-tuple* $b_1 b_2 \ldots b_{n-2} a$ *(briefly, αa) such that:*

$$(3) \qquad Fxb_1 \ldots b_{n-2}a = Fx\alpha a = x$$

for every x in $\mathfrak{M}$. Define a binary operation S_α, and a unary operation f by:

$$(3) \qquad S_\alpha xy = Fx\alpha y,$$

$$(4) \qquad fx = Fax\alpha,$$

for all x,y in $\mathfrak{M}$. Let $c = Fa^n$. Then S_α is a (binary) semigroup with right-identity a, f is an endomorphism of S_α, and we have

$$(5) \qquad Fx_1 x_2 \ldots x_n = S_\alpha^n x_1 fx_2 \ldots f^{n-1}x_n c$$

for all n-tuples $x_1 x_2 \ldots x_n$ in $\mathfrak{M}^n$.

This theorem, which will be proved in §2, is a generalization of a theorem of M. Hosszú ([2]) for polyadic groups. A polyadic *group (n-group)* [2]) is an *n*-semigroup in which, given any *n* of the $n+1$ elements $x_1, \ldots, x_{n+1}$ of $\mathfrak{M}$, a solution of the equation

$$(6) \qquad Fx_1 \ldots x_n = x_{n+1}$$

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[1]) I use the parentheses-free (Lukasiewicz) functional notation. Expressions such as FF, FFF aa, aaaa will be abbreviated to F^2, F^3, a^2, a^4, etc.

[2]) This definition, as well as that of generalized associativity (1), was first introduced by E. Dörnte [1].

for the remaining unknown element of $\mathfrak{M}$ always exists. In other words, each one-place part function [3]) of F is onto $\mathfrak{M}$. This phrasing corresponds to the Huntington definition of a (binary) group as an associative system G where $aG = Ga = G$ ([3]). As in the binary case, it then can be shown that these 1-place part-functions are one-to-one, i.e., the solutions of (6) are unique.

E. L. Post ([4]) in his fundamental Coset Theorem showed that every polyadic group F on a set $\mathfrak{M}$ has a covering (binary) group K. This means that there exists a (unique) group K generated by $\mathfrak{M}$, such that for any n-tuple $x_1 \ldots x_n$ of elements of $\mathfrak{M}$ we have:

$$(7) \qquad\qquad Fx_1 \ldots x_n = K^{n-1} x_1 \ldots x_n,$$

and such that $\mathfrak{M}$ is a generating coset of a cyclic factor group of K.

The use of the covering group K has the disadvantage of introducing elements not in $\mathfrak{M}$. In [2], Hosszú showed that such introduction can be avoided, and that any polyadic group can be realized as a composite of a binary group G on $\mathfrak{M}$ and an automorphism f of G. Specifically, he proved that

$$(8) \qquad\qquad Fx_1 \ldots x_n = G^n x_1 f x_2 \ldots f^{n-2} x_n c,$$

where c is a fixed element of $\mathfrak{M}$, $fc = c$ and $f^{n-1} x = C^2 cxc^{-1}$, i. e., f^{n-1} is an inner automorphism of G.[4])

In § 3, we show that Hosszú's theorem is a special case of Theorem 1. Finally, in § 4 we use Theorem 1 to obtain a generalization to n-semigroups of a theorem of Dörnte on derivability of n-groups.

2. Proof of Theorem 1

Lemma 1. *The function S_α defined in (3) is a (binary) semigroup, i.e.,*

$$(9) \qquad\qquad S_\alpha S_\alpha xyz = S_\alpha x S_\alpha yz,$$

for all x, y, z in $\mathfrak{M}$.

PROOF.
$$\begin{aligned}
SSxyz = FSxyaz = & \quad (3) \\
= FFx\alpha y\alpha z = & \quad (3) \\
= Fx\alpha Fy\alpha z = & \quad (1) \\
= Fx\alpha Syz = & \quad (3) \\
= SxSyz. & \quad (3)
\end{aligned}$$

Note that in this proof we did not have to assume the existence of any identity for F. If F has a right-identity αa, then it is immediate from (3) that a is a right-identity for S_α.

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[3]) A k-place part function of F is derived from F by fixing n-k of the arguments.

[4]) As a byproduct, Hosszú was able to immediately obtain a theorem of E. Vincze [5], which itself generalized a theorem of J. Aczél on real solutions of the functional equation of associativity.

Lemma 2. *The function f defined in* (4) *is an endomorphism of* S_α, *i.e.*,

(10)
$$fS_\alpha xy = S_\alpha fxfy,$$

for all x, y *in* $\mathfrak{M}$.

Proof. $S_\alpha fx\, fy = S_\alpha Fax\alpha Fay\alpha =$ (4)

$= F\, Fax\alpha\alpha Fay\alpha =$ (3)

$= F\, Faxb_1 \dots b_{n-2}\alpha Fay\alpha =$

$= Fa\, Fx \dots Fb_{n-2}\alpha a\, y\alpha =$ (1)

$= Fa\, Fxb_1 \dots b_{n-2}\, y\alpha =$ (2)

$= Fa\, Fx\alpha y\alpha =$

$= f\, Fx\alpha y =$ (4)

$= f\, S_\alpha xy.$ (3)

Lemma 3. *For the functions f and* S_α *of Theorem 1, we have:*

(11)
$$S_\alpha^n x_1 fx_2 \dots f^{n-1} x_n c = S_\alpha^2 x_1 fS_\alpha x_2 \dots fS_\alpha x_{n-1} fx_n c.$$

Proof. The proof depends on the fact that if f is an endomorphism of S_α, then so are all the iterates of f. Hence for any positive integer m, we have

(4a)
$$f^m S_\alpha xy = S_\alpha f^m x f^m y.$$

Now starting with the left hand side of (11), we have:

$$S_\alpha^n x_1 fx_2 \dots f^{n-1} x_n c = S_\alpha^2 x_1 S_\alpha fx_2 \dots S_\alpha f^{n-2} x_{n-1} f^{n-1} x_n c = \quad (9)$$

$$= S_\alpha^2 x_1 S_\alpha fx_2 \dots S_\alpha f^{n-2} x_{n-1} f^{n-2} fx_n c =$$

$$= S_\alpha^2 x_1 S_\alpha fx_2 \dots f^{n-2} S_\alpha x_{n-1} fx_n c \quad (4a)$$

Repeating this procedure leads us, after $(n-2)$ more steps, to (11).

Remark: Although f is an endomorphism of S_α, fS_α is in general *not*[5]) a semi-group on $\mathfrak{M}$. For example, let $\mathfrak{M}$ be the set of real numbers, and S ordinary real addition. For any $a \neq 0,1$ let f_a denote the linear function defined by $f_a x = a\cdot x$. Then f_a is an endomorphism of S. But we have:

$$f_a Sf_a Sxyz = a(a(x+y)+z) = a^2x + a^2y + az \neq$$

$$\neq ax + a^2y + a^2z = (a(x+a(y+z))) = f_a Sx f_a Syz.$$

A sufficient condition for fS to be associative is that f be idempotent; in this case, if f is onto $\mathfrak{M}$, then f is the identity function on $\mathfrak{M}$.

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[5]) Occasionally one finds the statement such as: If f is an endomorphism of the semigroup S then the image fS is a (sub-) semigroup. This is the result of using the same symbol „S" to denote the set $\mathfrak{M}$ and the semigroup, i.e., the pair $(\mathfrak{M}, S)$. What is meant is: The pair $(f\mathfrak{M}, S')$, where S' is the restriction of S to the set $f\mathfrak{M}'$, is a subsemigroup of $(\mathfrak{M}, S)$.

Lemma 4. *If* $c = Fa^n$, *then*

(12) $$Fx_1 \ldots x_n = S_\alpha^2 x_1 f S_\alpha x_2 \ldots f S_\alpha x_{n-1} f x_n c.$$

PROOF.

$$Fx_1 \ldots x_n = FFx_1 \alpha a\ Fx_2 \alpha a \ldots Fx_{n-1} \alpha a\ Fx_n \alpha a = \qquad (2)$$

$$= FFx_1 \alpha Fax_2 \alpha \ldots Fax_{n-1} \alpha Fax_n \alpha a = \qquad (1)$$

$$= FFx_1 \alpha Fax_2 \alpha \ldots Fax_{n-1} \alpha f x_n a = \qquad (4)$$

$$= FFx_1 \alpha Fax_2 \alpha \ldots Fax_{n-1} \alpha Ffx_n \alpha aa = \qquad (2)$$

$$= FFx_1 \alpha Fax_2 \alpha \ldots FaFx_{n-1} \alpha f x_n \alpha a^2 = \qquad (1)$$

$$= FFx_1 \alpha Fax_2 \alpha \ldots FaS_\alpha x_{n-1} f x_n \alpha a^2 = \qquad (3)$$

$$= FFx_1 \alpha Fax_2 \alpha \ldots f\ S_\alpha x_{n-1} f x_n a^2. \qquad (4)$$

Applying the procedure in the last 4 steps $n-2$ times in succession we obtain:

$$Fx_1 \ldots x_n = FFx_1 \alpha f S_\alpha x_2 \ldots f\ S_\alpha x_{n-1} f x_n a^{n-1} = \qquad (2)$$

$$= FFx_1 \alpha FfS_\alpha x_2 \ldots f\ S_\alpha x_{n-1} f x_n \alpha a a^{n-1} = \quad (2)$$

$$= FFx_1 \alpha f S_\alpha x_2 \ldots f\ S_\alpha x_{n-1} f x_n \alpha Fa^n = \qquad (1)$$

$$= F S_\alpha x_1 f S_\alpha x_2 \ldots f S_\alpha x_{n-1} f x_n \alpha c = \qquad (2)$$

$$= S_\alpha^2 x_1 f S_\alpha x_2 \ldots f S_\alpha x_{n-1} f x_n c. \qquad (2)$$

Equations (11) and (12) together yield (5), and so *the proof of Theorem* 1 *is complete.*

N. B. Similar arguments will show that a semigroup with a *left* identity $a\alpha$ can be realized in the form:

$$Fx_1 \ldots x_n = S_\alpha^n c g^{n-1} x \ldots g x_{n-1} x_n,$$

where $S_\alpha xy = Fx\alpha y$, $gx = Fa xa$, and $c = Fa^n$.

3. Hosszú's Theorem

Lemma 5. *If* F *is an n-group,* α *is any fixed* $(n-2)$ *tuple, and* S_α *is defined via* (3), *then* S_α *is a group.*

PROOF. By Lemma 1, S_α is a semigroup. And since F is an n-group, any equation of the form

$$S_\alpha x_1 x_2 = x_3$$

has a solution whenever two of the x's are given. But this is one of the standard definitions of a group.

In dealing with groups, we shall generally write G_α instead of S_α.

Lemma 6. *For every* $(n-2)$-*tuple* $\alpha \in \mathfrak{M}^{n-2}$, *there exists an* $a \in \mathfrak{M}$ *such that* αa *and* $a\alpha$ *are each (two-sided) identities of the n-group* F.

PROOF. The (binary) group G_α has an identity, say a. Then $G_\alpha ax = Fa\alpha x = x = Gxa = Fx\alpha a$. Thus αa is a right identity and $a\alpha$ a left identity of F. Now, to show that $a\alpha$ is also a right identity we have:

$$Fxa\alpha = Fxab_1...b_{n-2} = Fxab_1...Fb_{n-2}\alpha a = FxFab_1...b_{n-2}\alpha a =$$

$$= FxFa\alpha b_1...b_{n-2}a = Fxb_1...b_{n-2}a = Fx\alpha a = x.$$

Similarly, we can show, that αa is also a left identity.

This lemma is a special case of a more general theorem of POST ([4]): Given any $n-2$ elements $a_1...a_{i-1}a_{i+1}...a_{n-1}$ of $\mathfrak{M}$ $(i=1, 2, ..., n-1)$, there exists an element a_i in $\mathfrak{M}$ such that $a_1...a_{n-1}$ is an identity of the n-group; furthermore, any cyclic permutation of $a_1...a_{n-1}$ is also an identity.

Lemma 7. *Let* $a\alpha$ *be a right identity of* F *and* $c = Fa^n$. *If* f *is as defined in* (4), *then* c *is a fixed point of* f.

PROOF. $fc = Fac\alpha = FaFa^n\alpha = FFaa^n\alpha = FFa^na\alpha = Fca\alpha = c$.

Note that the proof does not require F to be an n-group.

Lemma 8. *If* F *is an n-group, then* f *is an automorphism of* G_α *and* f^{n-1} *an inner automorphism of* G_α: *Specifically* $f^{n-1}x = G_\alpha^2 cxc^{-1}$ *where* c *is as in Lemma* 7.

PROOF. 1. By Lemma 2, f is an endomorphism of G_α. Since F is an n-group, f is one-to-one onto $\mathfrak{M}$, i. e., an automorphism.

2. $f^{n-1}x = F^{n-1}a^{n-1}x\alpha^{n-1} = F^{n-1}a^{n-2}ax\alpha^{n-1} = F^{n-1}a^{n-2}Fa\,a\alpha x\alpha^{n-1} =$

$$= FFFa^n\alpha x\alpha F^{n-3}\alpha^{n-2} = FFc\alpha x\alpha F^{n-3}\alpha^{n-2} = FG_\alpha cx\alpha F^{n-3}\alpha^{n-2} = G_\alpha^n cxF^{n-3}\alpha^{n-2}$$

To complete the proof we need only show that $F^{n-3}\alpha^{n-2} = c^{-1}$, i.e., that $G_\alpha cF^{n-3}\alpha^{n-2} = a$. Now

$$GcF^{n-3}\alpha^{n-2} = G\,Fa^n F^{n-3}\alpha^{n-2} = FFa^n\alpha F^{n-3}\alpha^{n-2} =$$

$$= F^{n-1}a^n\alpha^{n-1} = F^{n-2}a^{n-1}Fa\,\alpha b_1 ... b_{n-2}\alpha^{n-3},$$

where $b_1...b_{n-2} = \alpha$. Since $a\alpha$ is a left identity, we have

$$GcF^{n-3}\alpha^{n-2} = F^{n-2}a^{n-1}b_1 ... b_{n-2}\alpha^{n-3} = F^{n-2}a^{n-1}\alpha^{n-2}.$$

Repeating the procedure, we reduce the right hand side to $Faa\alpha = a$.

Lemmas 5, 6, 7 and 8 together show that Hosszú's Theorem is a special case of Theorem 1.

4. Derivable n-semigroups

If $Fa^n = a$, we call a an *idempotent* element of F.

Lemma 9. *Let* F *be an n-semigroup with an idempotent element* a, *and let a function* f *be defined by*

(13)
$$fx = Faxa^{n-2}.$$

Then $f^{n-1}x = F^2 a^{n-1} x a^{n-1}$, $f^n = f$ and f^{n-1} is an idempotent function, i.e., $f^{n-1} f^{n-1} = f^{n-1}$.

PROOF 1. $f^{n-1}x = F^{n-1} a^{n-1} x a^{(n-2)(n-1)} = F^{n-1} a^{n-1} x a^n a^{(n-2)(n-1)-n} =$

$$= F^{n-2} a^{n-1} x F a^n a^{(n-2)(n-1)-n} = F^{n-2} a^{n-1} x a a^{(n-2)(n-1)-n} =$$

$$= F^{n-2} a^{n-1} x a^{(n-3)(n-1)}.$$

Repeating the above steps $n-2$ more times we get:

$$f^{n-1}x = F^2 a^{n-1} x a^{n-1}.$$

2. $f^n x = f^{n-1} fx = F^2 a^{n-1} fx\, a^{n-1} = F^2 a^{n-1} Fa\, x a^{n-2} a^{n-1} =$

$$= F^2 a^n x F a^n a^{n-3} = Fa\, x a a^{n-3} = Fa\, x a^{n-2} = fx.$$

3. $f^{n-1} f^{n-1} = f^{2n-2} = f^n f^{n-2} = f f^{n-2} = f^{n-1}.$

Lemma 10. *Let F be an n-semigroup on a set $\mathfrak{M}$. Let a be an element of $\mathfrak{M}$, and let f be as defined in (13). Then the following statements are equivalent:* (i) *a is an idempotent element of F, and f is onto $\mathfrak{M}$;* (ii) *a^{n-1} is a (2-sided) identity of F.*

PROOF. (i) implies (ii).
Since f is onto $\mathfrak{M}$, f^{n-1} is onto $\mathfrak{M}$. But, by Lemma 9, f^{n-1} is idempotent, and an idempotent function onto a set is the identity function. We therefore have:

$$x = f^{n-1} x = F^2 a^{n-1} x a^{n-1} = F^2 a a^{n-2} x a^{n-1} = F^2 Fa^n a^{n-2} x a^{n-1} =$$

$$= F^2 a^{n-1}(Fa^{n-1}x) a^{n-1} = f^{n-1} Fa^{n-1} x = Fa^{n-1} x,$$

i.e., a^{n-1} is a left identity of F. Similarly we can show that a^{n-1} is a right identity.
(ii) implies (i).

1. $Fa^n = Faa^{n-1} = a$ i.e., a is idempotent.

2. $f^{n-1} x = F^2 a^{n-1} x a^{n-1} = Fa^{n-1} Fx\, a^{n-1} = Fa^{n-1} x = x,$

i.e., f^{n-1} is onto $\mathfrak{M}$. Since the range of f^{n-1} is a subset of the range of f, the proof is complete.

DÖRNTE ([1]) already distinguished between genuine (echte) and derivable (ableitbare) n-groups. An n-semigroup F is derivable from a binary semigroup S if $F = S^{n-1}$.

Theorem 2. *Let F be an n-semigroup on a set $\mathfrak{M}$. Then the following statements are equivalent:* (i) *F is derivable from a (binary) semigroup S on $\mathfrak{M}$ with identity a;* (ii) *F has an idempotent element a such that $Faxa^{n-2} = x$ for all $x \in \mathfrak{M}$;* (iii) *there is an element $a \in \mathfrak{M}$ such that a^{n-1} is a right identity of F, and such that $Faxa^{n-2} = x$ for all $x \in \mathfrak{M}$.*

PROOF. (i) implies (ii). $Fa^n = S^{n-1} a^n = a$, i.e., a is an idempotent element of F. Furthermore

$$Fa\, x a^{n-2} = S^{n-1} a x a^{n-2} = x.$$

(ii) implies (iii). Since $fx = Faxa^{n-2} = x$ means that f is onto $\mathfrak{M}$, this implication follows from Lemma 10.

(iii) implies (i). Let $\alpha = a^{n-2}$. By hypothesis αa is a (right) identity. We can therefore apply Theorem 1 and obtain

$$Fx_1 \ldots x_n = S_\alpha^n x_1 fx_2 \ldots f^{n-1} x_n Fa^n = S_\alpha^n x_1 fx_2 \ldots f^{n-1} x_n a.$$

Now $fx = Fax\alpha = Faxa^{n-2} = x$ by hypothesis, and a is the identity of S_α. Hence we have

$$Fx_1 \ldots x_n = S_\alpha^{n-1} x_1 \ldots x_n.$$

Since any group has an identity, Theorem 2 includes as a special case the necessary and sufficient criterion for n-groups to be derivable from binary groups (Dörnte, [1], §2, Theorem 6).

Acknowledgment

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