Radius of convexity of a certain class of meromorphically starlike functions.

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1. *Introduction:* Let S denote the class of all univalent functions $f(z)$ given by

$$f(z) = \frac{1}{z} + b_0 + b_1 z + \dots, \tag{1}$$

regular and starlike of order α for $0 < |z| < 1$, that is $f(z)$ satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < -\alpha, \qquad 0 \leqq \alpha < 1.$$

M. S. ROBERTSON ([1]) determined the radius of convexity for the class S for $\alpha = 0, \frac{1}{2}$.

In a recent paper [2] K. S. PADAMNABHAN solved the problem for the case $\frac{1}{2} \leqq \alpha < 1$. His method fails to give a sharp result for $0 \leqq \alpha < \frac{1}{2}$.

In this short paper we shall obtain the radius of convexity for the class S for $0 \leqq \alpha \leqq \frac{1}{2}$.

2. We shall need the following lemma:

Lemma I. *Let $H(z)$ be regular and satisfy* $\operatorname{Re}\{H(z)\} > \alpha$, $0 \leqq \alpha < 1$, *for* $|z| < 1$ *and let* $H(0) = 1$. *Then we have*

$$H(z) = \frac{1 - (2\alpha - 1)w(z)}{1 - w(z)},$$

where $w(z)$ is any regular function satisfying $w(0) = 0$, $|w(z)| \leqq |z|$ for $|z| < 1$, and any function $H(z)$ given by the above formula is regular and satisfies $\operatorname{Re}\{H(z) > \alpha$ *in* $|z| < 1$.

The proof can be found in [4], p. 99.

We prove the following

Theorem 1. *Let*

$$f(z) = \frac{1}{z} + b_0 + b_1 z + \dots .$$

be regular and starlike of order α, $0 \leqq \alpha \leqq \frac{1}{2}$, for $0 < |z| < 1$. Then $f(z)$ maps

$$|z| < \left\{\frac{2\sqrt{1-\alpha^2} - (1+\alpha)}{3 - 5\alpha}\right\}^{\frac{1}{2}}$$

on to a domain the complement of which is convex. The estimate is sharp.

Proof. Since $f(z)$ is starlike of order α, therefore

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < -\alpha, \quad \text{for} \quad 0 < |z| < 1,$$

and hence $f(z)$ does not vanish in $|z|<1$. The function $-\frac{zf'(z)}{f(z)}$ is regular in $|z|<1$ and satisfies the conditions of lemma. Therefore, we can write

$$\frac{zf'(z)}{f(z)} = -\frac{1-(2\alpha-1)w(z)}{1-w(z)}, \tag{2}$$

where $w(z)$ is regular for $|z|<1$ and satisfies $w(0)=0$, $|w(z)| \leqq |z|$.

By differentiating and simplifying (2), we have

$$-\left(1+\frac{zf''(z)}{f'(z)}\right) = \frac{[1+(1-2\alpha)w(z)]^2-2(1-\alpha)zw'(z)}{[1-w(z)][1+(1-2\alpha)w(z)]} = K(z), \quad \text{say}. \tag{3}$$

We wish to find the largest circle with centre at the origin with in which $f(z)$ satisfies the inequality.

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \leqq 0, \tag{4}$$

that is

$$\operatorname{Re}\{K(z)\} \geqq 0, \tag{5}$$

where $K(z)$ is given by (3). In turn, the inequality (5) is equivalent to

$$\left|\frac{K(z)-1}{K(z)+1}\right| = \left|\frac{(1-\alpha)\left[(1-2\alpha)\left(w(z)\right)^2+\left(w(z)-zw'(z)\right)\right]}{1-2\alpha w(z)+(1-\alpha)\left(w(z)-zw'(z)\right)-\alpha(1-2\alpha)\left(w(z)\right)^2}\right| \leqq \tag{6}$$

$$\leqq \frac{(1-\alpha)[(1-2\alpha)|w(z)|^2+|w(z)-zw'(z)|]}{1-2\alpha|w(z)|-(1-\alpha)|w(z)-zw'(z)|-\alpha(1-2\alpha)|w(z)|^2} \leqq 1.$$

This will be satisfied if

$$(1-2\alpha)|w(z)|^2+2(1-\alpha)|w(z)-zw'(z)|+2\alpha|w(z)|-1 \leqq 0. \tag{7}$$

Let

$$\varphi(z) = \frac{w(z)}{z}, \tag{8}$$

then $\varphi(z)$ is regular and satisfies $|\varphi(z)| \leqq 1$, for $|z|<1$. Also for such functions $\varphi(z)$ we have ([3], p. 18),

$$|\varphi'(z)| \leqq \frac{1-|\varphi(z)|^2}{1-|z|^2}. \tag{9}$$

We shall show that the estimate (9) is sharp.

Take

$$\varphi(z) = -\frac{\gamma - z}{1-\bar{\gamma}z}, \qquad |\gamma| < 1.$$

Then

$$\varphi'(z) = \frac{1-|\gamma|^2}{(1-\bar{\gamma}z)^2}.$$

(10) $$|\varphi'(z)| = \frac{|1-\bar{\gamma}z|^2 - |\gamma - z|^2}{(1-|z|^2)|1-\bar{\gamma}z|^2} = \frac{1-|\varphi(z)|^2}{1-|z|^2}.$$

Differentiating (8) we get

(11) $$\varphi'(z) = \frac{zw'(z)-w(z)}{z^2}$$

From (10) and (11) we obtain

(12) $$|zw'(z)-w(z)| = \frac{|z|^2-|w(z)|^2}{1-|z|^2}.$$

Consequently (7) becomes

$$(1-2\alpha)|\omega(z)|^2 + 2(1-\alpha)\frac{r^2-|\omega(z)|^2}{1-r^2} + 2\alpha|\omega'(z)| - 1 \leqq 0,$$

here $r=|z|$, or

$$|w(z)|^2[1+(1-2\alpha)r^2] - 2\alpha(1-r^2)|w(z)| + (2\alpha-3)r^2+1 \geqq 0,$$

or

$$[1+(1-2\alpha)r^2]\left[|\omega(z)| - \frac{\alpha(1-r^2}{1+r^2(1-2\alpha)}\right]^2 + \left[((2\alpha-3)r^2+1) - \frac{\alpha^2(1-r^2)^2}{1+(1-2\alpha)r^2}\right] \geqq 0$$

This last inequality is satisfied if

$$(2\alpha-3)r^2+1-\frac{\alpha^2(1-r^2)^2}{1+(1-2\alpha)r^2} \geqq 0,$$

or if

$$(3-5\alpha)r4 + 2(1+\alpha)r^2 - (1+\alpha) \leqq 0,$$

which holds for

(13) $$r^2 < \frac{2\sqrt{1-\alpha^2}-(1+\alpha)}{3-5\alpha} = b^2, \quad \text{say.}$$

Therefore, $f(z)$ is convex in $0<|z|<b$.

Now we shall show that the estimate (13) is sharp. Let

$$w(z) = -\frac{z(z-\gamma)}{1-\gamma z},$$

where γ is defined by

(14) $$\gamma = \frac{1-b^2(4\alpha-1)}{4b(1-\alpha)} = \frac{-7b^6-5b^4+3b^2-1}{8b(b^2-1)^2}.$$

It is easy to show that $|\gamma| < 1$. Then the function $w(z) = -\dfrac{z(z-\gamma)}{1-\gamma z}$ satisfies (12) and makes

$$[1+(1-2\alpha)w(z)]^2 - 2(1-\alpha)zw'(z) = 0, \quad \text{for} \quad z = b,$$

which implies that $K(b)=0$. It follows that the function $f(z)$ defined by

$$\frac{zf'(z)}{f(z)} = -\frac{1-(2\alpha-1)\,w(z)}{1-w(z)}, \quad w(z) = \frac{-z(z-\gamma)}{1-\gamma z}$$

is not convex in $0<|z|<R$ if R exceeds b. Therefore the bound b is sharp.

Acknowledgement: I wish to thank Professor VIKRAMADITYA SINGH for his kind help and guidance.

References

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(Received August 22, 1966.)