Sets of radical classes

By W. G. LEAVITT (Lincoln, Neb.)

1. Introduction

A class $\mathscr{C}$ of rings is said to be *homomorphically closed* if $R \in \mathscr{C}$ implies $R\Phi \in \mathscr{C}$ for every homomorphism Φ of R, and is said to be *hereditary* if $R \in \mathscr{C}$ implies $I \in \mathscr{C}$ for every ideal I of R. A class which is both homomorphically closed and hereditary is said to be a *universal* class. In the following all rings considered are from an arbitrary but fixed class $\mathscr{C}$ of (not necessarily associative) rings.

For a class $\mathscr{P} \subseteqq \mathscr{C}$ define

(1) $$\mathscr{S}\mathscr{P} = \{R \in \mathscr{C} \mid I \notin \mathscr{P} \text{ for every non-zero ideal } I \text{ of } R\}$$

and

(2) $$\mathscr{U}\mathscr{P} = \{R \in \mathscr{C} \mid R\Phi \notin \mathscr{P} \text{ for every non-zero homomorphism } \Phi \text{ of } R\}$$

Remark 1. It is well-known that if $\mathscr{P}$ is a radical class [1, 3] then $\mathscr{S}\mathscr{P}$ is its semisimple class and $\mathscr{U}\mathscr{S}\mathscr{P} = \mathscr{P}$ ([4] Theorem 1, p. 4). (Note that the proofs of [4] Chapter I apply equally well to classes of possibly non-associative rings.)

Remark 2. On the other hand, if $\mathscr{P}$ is a semisimple class then $\mathscr{U}\mathscr{P}$ is a radical class (called the *upper* radical defined by the class $\mathscr{P}$) and $\mathscr{S}\mathscr{U}\mathscr{P} = \mathscr{P}$ ([4] Theorem 2, p. 5).

Remark 3. If $\mathscr{P}$ is a radical class we will write $\mathscr{P}(R)$ for the $\mathscr{P}$-radical of the ring R. Note that if H is an ideal of R such that $R/H \in \mathscr{S}\mathscr{P}$ then $\mathscr{P}(R) \subseteqq H$. This is clear since otherwise $(\mathscr{P}(R)+H)/H \cong \mathscr{P}(R)/\mathscr{P}(R) \cap H$ would be a non-zero $\mathscr{P}$-ideal of R/H contradicting the definition (1).

Remark 4. In the Kurosh construction of the *lower* radical for a homomorphically closed class of rings $\mathscr{H}$ (as modified by ANDERSON, DIVINSKY and SULINSKI ([4] footnote p. 12) a class $\mathscr{H}_\alpha$ is of degree α over $\mathscr{H} = \mathscr{H}_1$ (for an arbitrary ordinal $\alpha > 1$) if $\mathscr{H}_\alpha = \{R \in \mathscr{C} \mid$ every non-zero homomorphic image $R\Phi$ has an ideal $I \in \mathscr{H}_\beta$ for some $\beta < \alpha\}$. We will write $\mathscr{L}\mathscr{H}$ for the lower radical class defined by $\mathscr{H}$, namely $\mathscr{L}\mathscr{H} = \bigcup_\alpha \mathscr{H}_\alpha$. Note that from the definitions (1) and (2) it follows that $\mathscr{H}_2 = \mathscr{U}\mathscr{S}\mathscr{H}$, for if every $R\Phi$ has an ideal $I \in \mathscr{H}_1 = \mathscr{H}$ then $R\Phi \notin \mathscr{S}\mathscr{H}$, so that $R \in \mathscr{U}\mathscr{S}\mathscr{H}$. The converse is also clear.

2. *The intersection radical*

Let $\{\mathscr{P}_j\}_{j\in J}$ be a set of radical subclasses of $\mathscr{C}$.

Theorem 1. *If $\mathscr{P}=\bigcap_{j\in J}\mathscr{P}_j$ then $\mathscr{P}$ is a radical class such that for any ring R*

$$\mathscr{P}(R)\subseteqq\bigcap_{j\in J}\mathscr{P}_j(R). \tag{3}$$

If all the classes $\mathscr{P}_j$ are hereditary, then so is $\mathscr{P}$, and the relation (3) *becomes an equality.*

Proof. Since each $\mathscr{P}_j$ is homomorphically closed, the same is true of $\mathscr{P}$. Also if $R\notin\mathscr{P}$ then $R\notin\mathscr{P}_j$ for some j, so there exists a non-zero $R\Phi\in\mathscr{S}\mathscr{P}_j\subseteqq\mathscr{S}\mathscr{P}$. By [4] (Theorem 1, p. 4) $\mathscr{P}$ is thus a radical class. Relation (3) is clear from the definition of the $\mathscr{P}$-radical. It is also clear that when all $\mathscr{P}_j$ are hereditary then so is $\mathscr{P}$, and if $H=\bigcap_{j\in J}\mathscr{P}_j(R)$ then H is an ideal of each $\mathscr{P}_j(R)$. By the hereditary property, $H\in\mathscr{P}_j$ whence $H\in\mathscr{P}$ so that $H\subseteqq\mathscr{P}(R)$.

Call $\mathscr{P}(R)$ the *intersection* radical of the set $\{\mathscr{P}_j\}$. We now proceed to construct for an arbitrary ring R a collection of subrings which, when R is associative, all equal $\mathscr{P}(R)$.

Define $V_1=R$. For a given ordinal β assume that subrings V_α have been defined for all ordinals $\alpha<\beta$ and define:

$$V_\beta=\begin{cases}\bigcap_{\alpha<\beta}V_\alpha & \text{if } \beta \text{ is a limit ordinal. Otherwise,}\\ \mathscr{P}_j(V_{\beta-1}) & \text{for some } j\in J \text{ (if such exists) for which } \mathscr{P}_j(V_{\beta-1})\neq V_{\beta-1}.\end{cases} \tag{4}$$

Since $\{V_\alpha\}$ is a set, it cannot be inductive for the class of all ordinals. Hence there must exist an ordinal γ such that $\mathscr{P}_j(V_\gamma)=V_\gamma$ for all $j\in J$.

Note that (4) may permit considerable arbitrariness at each non-limit ordinal. We have thus in general defined a collection of such subrings V_γ. As the following theorem shows, however, these coincide in the case R is associative.

Theorem 2. *For an arbitrary ring R let V_β be defined by* (4). *Then there exists an ordinal γ such that $\mathscr{P}(V_\gamma)=V_\gamma$. In all cases $\mathscr{P}(R)\subseteqq V_\gamma$, and if R is associative this becomes an equality.*

Proof. As in the above remarks there exists γ such that $\mathscr{P}_j(V_\gamma)=V_\gamma$ for all $j\in J$. But then $V_\gamma\in\bigcap_{j\in J}\mathscr{P}_j$ so that $V_\gamma\subseteqq\mathscr{P}(V_\gamma)$ and we have equality.

Since $V_1=R$, $\mathscr{P}(R)\subseteqq V_1$. Thus assume for induction that for a given ordinal β, $\mathscr{P}(R)\subseteqq V_\alpha$ for all ordinals $\alpha<\beta$. Then clearly by definition $\mathscr{P}(R)\subseteqq V_\beta$ when β is a limit ordinal. If $\beta-1$ exists then by the induction hypothesis, $\mathscr{P}(R)\subseteqq V_{\beta-1}$ and since $\mathscr{P}(R)\in\mathscr{P}_j$ it is a $\mathscr{P}_j$-ideal of $V_{\beta-1}$. Thus $\mathscr{P}(R)\subseteqq\mathscr{P}_j(V_{\beta-1})=V_\beta$. By induction $\mathscr{P}(R)\subseteqq B_\beta$ for all β and hence $\mathscr{P}(R)\subseteqq V_\gamma$. But we already have that V_γ is a $\mathscr{P}$-ring, so the reverse inequality would follow if V_γ were an ideal of R. Now when R is associative, the radical of an ideal is an ideal of R [4; Theorem 47, p. 124], and so by induction V_γ is an ideal of R.

3. *The join radical*

Again let $\{\mathscr{P}_j\}_{j\in J}$ be a set of radical classes and write $\mathscr{H}=\bigcup_{j\in J}\mathscr{P}_j$. In general $\mathscr{H}$ is not a radical class, but in the associative case turns out to be only one degree from a radical. First we have

Lemma 1. *$\mathscr{S}\mathscr{H}=\bigcap_{j\in J}\mathscr{S}\mathscr{P}_j$ and $\mathscr{H}\subseteqq\mathscr{U}\mathscr{S}\mathscr{H}$.*

Proof. The first statement is clear since a ring has no non-zero ideals in any $\mathscr{P}_j$ if and only if it is a member of every $\mathscr{S}\mathscr{P}_j$. For the second relation it suffices, by remark 4, to observe that $\mathscr{H}$ is homomorphically closed.

We will write $\mathscr{J}(R)$ for the sum of all ideals of a ring R which are members of $\mathscr{U}\mathscr{S}\mathscr{H}$. From lemma 1 it follows that $\Sigma\mathscr{P}_j(R)\subseteqq\mathscr{J}(R)$.

Let $\mathscr{A}$ be the class of all associative rings and write $\bar{\mathscr{S}}\mathscr{H}=\mathscr{S}\mathscr{H}\cap\mathscr{A}$. (Note that this is equivalent to $\mathscr{S}\mathscr{H}$ defined by (1) relative to the universal class $\bar{\mathscr{C}}=\mathscr{C}\cap\mathscr{A}$.)

Theorem 3. *$\bar{\mathscr{S}}\mathscr{H}$ is a semisimple class in $\bar{\mathscr{C}}$ and hence for R an associative ring $\mathscr{J}(R)=\mathscr{L}\mathscr{H}(R)$.*

Proof. First suppose $R\in\bar{\mathscr{S}}\mathscr{P}_j$ for all $j\in J$. Since $\bar{\mathscr{S}}\mathscr{P}_j$ is a semisimple class for a radical class $(\mathscr{P}_j\cap\mathscr{A})$ in an associative universal class $\bar{\mathscr{C}}$, it is a hereditary class [4; Corollary 2, p. 125]. Thus every ideal of R is a member of $\bigcap_{j\in J}\bar{\mathscr{S}}\mathscr{P}_j$, so by Lemma 1 it follows that $\bar{\mathscr{S}}\mathscr{H}$ is hereditary. On the other hand, let $R\in\bar{\mathscr{C}}$ such that $R\notin\bar{\mathscr{S}}\mathscr{P}_j$ for some j. Then ([4] Theorem 2, p. 5) R has a non-zero ideal I all of whose non-zero images $I\Phi\notin\bar{\mathscr{S}}\mathscr{P}_j$. Thus $I\Phi\notin\bigcap_{j\in J}\bar{\mathscr{S}}\mathscr{P}_j=\bar{\mathscr{S}}\mathscr{H}$ for all homomorphisms Φ. Hence, again by [4; Theorem 2], $\bar{\mathscr{S}}\mathscr{H}$ is a semisimple subclass of $\bar{\mathscr{C}}$.

Thus the upper radical $\bar{\mathscr{U}}\bar{\mathscr{S}}\mathscr{H}$ defined in $\bar{\mathscr{C}}$ is a radical class with radical $\mathscr{J}(R)$ in any $R\in\bar{\mathscr{C}}$. But (relative to $\bar{\mathscr{C}}$) $\mathscr{H}_2=\bar{\mathscr{U}}\bar{\mathscr{S}}\mathscr{H}\subseteqq\mathscr{L}\mathscr{H}$ and by the minimality of the lower radical ([4] lemma 5, p. 13) this becomes an equality. Thus for any $R\in\bar{\mathscr{C}}$ we have $\mathscr{J}(R)=\mathscr{L}\mathscr{H}(R)$. We will call $\mathscr{J}(R)$ the *join* radical for the set $\{\mathscr{P}_j\}$.

We now proceed, by a construction analogous to that of Section 2, to construct a collection of ideals which, when R is associative, will all coincide with $\mathscr{J}(R)$.

Let $W_1=0$ and for a given ordinal β assume that ideals W_α have been defined for all ordinals $\alpha<\beta$. Then define W_β as follows:

(5) $W_\beta=\bigcup_{\alpha<\beta}W_\alpha$ when β is a limit ordinal. Otherwise by

$W_\beta/W_{\beta-1}=\mathscr{P}_j(R/W_{\beta-1})$ for some $j\in J$ (if such exists) for which $\mathscr{P}_j(R/W_{\beta-1})\neq 0$.

This means, again, that to avoid $1-1$ correspondence between the class of all ordinals and a subset of the set of all ideals of R, there must exist some ordinal γ such that

$$\mathscr{P}_j(R/W_\gamma)=0 \quad \text{for all} \quad j\in J. \tag{6}$$

Note that here also the arbitrariness of the choice of j at each non-limit ordinal means that we have in general defined a collection of such W_γ.

Theorem 4. *For a ring R let W_γ be an ideal defined by* (5), *satisfying condition* (6). *Then $W_\gamma \subseteq \mathcal{J}(R)$ and if R is associative this becomes an equality.*

Proof. Suppose for induction we assume for a given ordinal β that $W_\alpha \in \mathcal{U}\mathcal{S}\mathcal{H}$ for all $\alpha < \beta$. If β is a limit ordinal, W_β also satisfies this condition. So suppose $W_\beta / W_{\beta-1} = \mathcal{P}_j(R/W_{\beta-1})$ for some $j \in J$. Let $W_\beta / K \neq 0$ be an arbitrary homomorphic image of W_β. Since $W_1 = 0 \subseteq K$ while $W_\beta \subsetneqq K$ we may assume there exists an ordinal α such that $W_\alpha \subseteq K$ but $W_{\alpha+1} \subsetneqq K$. Thus $(W_{\alpha+1}+K)/K \cong W_{\alpha+1}/W_{\alpha+1} \cap K \neq 0$. But $W_\alpha \subseteq K$ so from the natural homomorphism of $W_{\alpha+1}/W_\alpha$ onto $W_{\alpha+1}/W_{\alpha+1} \cap K$ and the fact that by (5) $W_{\alpha+1}/W_\alpha \in$ some $\mathcal{P}_j$, it follows that $(W_{\alpha+1}+K)/K \in \mathcal{P}_j$. Thus W_β/K contains a non-zero ideal from $\mathcal{H}$. Hence $W_\beta/K \notin \mathcal{S}\mathcal{H}$ and since K was arbitrary it follows that $W_\beta \in \mathcal{U}\mathcal{S}\mathcal{H}$. By induction all W_β and hence $W_\gamma \in \mathcal{U}\mathcal{S}\mathcal{H}$ so that $W_\gamma \subseteq \mathcal{J}(R)$.

Now by condition (6) we have $R/W_\gamma \in \bigcap_{j \in J} \mathcal{S}\mathcal{P}_j = \mathcal{S}\mathcal{H}$. By Theorem 3 the class $\bar{\mathcal{S}}\mathcal{H}$ is semisimple in $\bar{\mathcal{C}}$ and is thus the semisimple class of the radical class $\bar{\mathcal{U}}\bar{\mathcal{S}}\mathcal{H}$. From Remark 3 it follows that when R is associative $\mathcal{J}(R) \subseteq W_\gamma$ and equality follows.

References

[1] A. G. Kurosh, Radicals in rings and algebras, Mat. Sb. (N. S.) 33 (75) (1953), 13—26.

[2] S. A. Amitsur, A general theory of radicals. I: Radicals in complete lattices, *Amer J. Math.* **74** (1952), 774—786.

[3] S. A. Amitsur, *A general theory of radicals.* II: *Radicals in rings and bicategories, Amer J. Math.* **76** (1954), 100—125.

[4] N. J. Divinsky, Rings and Radicals, *Toronto* (1965).

(Received August 30, 1966.)