# On a problem of G. Grätzer

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At a conference on Universal Algebra held a Oberwolfach in July, 1966, Professor GRÄTZER proposed a problem, which can be paraphrased as follows.

Let $\mathfrak{S}$, $\mathfrak{S}_1$ be species of algebras, and let $\mathfrak{S}$ be a subspecies of $\mathfrak{S}_1$ — this means that the operations of $\mathfrak{S}$ are also operations of $\mathfrak{S}_1$. An algebra $A$ in $\mathfrak{S}$ is called the $\mathfrak{S}$-*restriction* of an algebra $A_1$ in $\mathfrak{S}_1$ if, firstly, the carriers (that is: sets of elements) of $A$ and $A_1$ are the same, and, secondly, the effect of the operations of $\mathfrak{S}$ is the same in both $A$ and $A_1$. Thus, for example, if $\mathfrak{S}$ is the species defined by a single binary operation called addition, and $\mathfrak{S}_1$ is defined by addition and another binary operation called multiplication, then the additive group of integers in $\mathfrak{S}$ is the $\mathfrak{S}$-restriction of the ring of integers in $\mathfrak{S}_1$. Next, if $\mathfrak{K}$ is a subclass of the species $\mathfrak{S}$ and $\mathfrak{K}_1$ a subclass of $\mathfrak{S}_1$, then $\mathfrak{K}$ is the $\mathfrak{S}$-restriction of $\mathfrak{K}_1$ if it consists of the $\mathfrak{S}$-restrictions of the algebras in $\mathfrak{K}_1$.

Now G. Grätzer had proved**) that if $\mathfrak{K}$ is a variety (that is: equationally defined class) of algebras in the species $\mathfrak{S}$, and if $\mathfrak{K}$ is defined by a finite set of laws, then there is a species $\mathfrak{S}_1$ obtained from $\mathfrak{S}$ by adding two binary operations, and a variety $\mathfrak{K}_1$ in $\mathfrak{S}_1$, such that, firstly, $\mathfrak{K}$ is the $\mathfrak{S}$-restriction of $\mathfrak{K}_1$, and, secondly $\mathfrak{K}$ is defined by 4 laws. The problem is: can „4" here be replaced by a smaller number?

We shall show here that „4" can be replaced by „1" — which is clearly best possible; this is done by adding the same two binary operations of $\mathfrak{S}$ that Grätzer uses: in fact it is simply done by showing how to replace his 4 laws by a single one (but we do not quote Grätzer's laws). The technique used is a variant of one used in [2]; Professor Grätzer has recently communicated to me a reduction from „4" to „2", also using [2]. Though the present note is self-contained, the reader is referred for the motivation to Professor Grätzer's paper [1], and for a further discussion of the background to [2].

Thus we are given a species $\mathfrak{S}$ with a set $\Omega$ of operations, and a variety $\mathfrak{K}$ in $\mathfrak{S}$ is defined by the laws

(1) $$p_1 = q_1,\ p_2 = q_2,\ \ldots,\ p_m = q_m;$$

*) This work was carried out while the author held a Visiting Professorship at the University of Wisconsin.

**) In [1]. I am indebted to Professor Grätzer for giving me access to a prepublication copy of this paper. In the published version of this paper, Professor Grätzer has already improved „4" to „2".

the $p_i$ and $q_i$ are words in variables $t_1, t_2, \ldots, t_n$ and the operations in $\Omega$. We form the species $\mathfrak{S}_1$ by adding two new binary operations $\varrho$, $\sigma$ to $\Omega$, forming

$$\Omega_1 \rightleftharpoons \Omega \cup \{\varrho, \sigma\}.$$

To avoid brackets, we write $\varrho$, $\sigma$ as right-hand operators. In $\mathfrak{S}_1$ we single out the variety $\mathfrak{R}_1$ by laws to ensure in its algebras the existence of an element $e$ such that, firstly,

(2) $$xy\varrho = e \quad \text{if, and only if,} \quad x = y,$$

and, secondly,

(3) $$xy\sigma = e \quad \text{if, and only if,} \quad x = y = e;$$

and, finally, that

(4) $$p_1 q_1 \varrho p_2 q_2 \varrho \cdots p_m q_m \varrho \sigma^{m-1} = e.$$

Then (2), (3), (4) between them entail the laws (1) in $\mathfrak{R}_1$. Moreover, we have to choose our laws so that they imply no restriction on the carriers of algebras in $\mathfrak{R}_1$ other than that they shall be carriers of algebras in $\mathfrak{R}$; and indeed so that they make the given variety $\mathfrak{R}$ the $\mathfrak{S}$-restriction of the new variety $\mathfrak{R}_1$.

We achieve all this by means of a single law in variables $x, y, z, x_1, y_1, x_2, y_2, z_2, x_3, y_3, z_3, t_1, t_2, \ldots, t_n$, namely

(5) $$xw\varrho\; yz\varrho\; yx\varrho^3 = z,$$

where we have used the abbreviations

$$w = v_1 v_2 v_3 v_4 \varrho^3,$$

$$v_1 = x_1 y_1 \sigma y_1 x_1 \sigma \varrho,$$

$$v_2 = x_2 y_2 \sigma z_2 \sigma x_2 y_2 z_2 \sigma^2 \varrho,$$

$$v_3 = x_3 y_3 y_3 \varrho \sigma x_3 \varrho z_3 z_3 \sigma z_3 \varrho^2,$$

$$v_4 = p_1 q_1 \varrho p_2 q_2 \varrho \cdots p_m q_m \varrho \sigma^{m-1}.$$

[Thus $v_4$ is the left-hand side of (4), and involves the variables $t_1, t_2, \ldots, t_n$ only.]

If $A$ is an algebra in $\mathfrak{R}$, we make it the $\mathfrak{S}$-restriction of an algebra $A_1$ in $\mathfrak{R}_1$ by defining on its carrier an arbitrary abelian group structure with

(6) $$xy\varrho = xy^{-1},$$

and also a semilattice structure with the neutral element $e$ of the abelian group as least element, and with

(7) $$xy\sigma = x \cup y.$$

It is then easy to verify that $v_1, v_2, v_3, v_4$, and thus also $w$, are constant and equal to $e$ for all choices of the variables in them; and that the law (5) is satisfied.

We now assume, conversely, that the law (5) is satisfied, and show that (2), (3), (4), and thus the laws (1), are a consequence of it. To this end we introduce the following notation. The „right $\varrho$-multiplication" $R_y$, the „left $\varrho$-multiplication"

$L_x$, and the „right $\sigma$-multiplication" $S_y$ are defined, as mappings of the carrier of an algebra in $\mathfrak{S}_1$ into itself, by

$$xR_y = yL_x = xy\varrho,$$

$$xS_y = xy\sigma;$$

the identity mapping is I.

With this notation, (5) can be expressed in the form

$$L_y R_{yx\varrho} L_{xw\varrho} = I. \tag{8}$$

It follows that every $L_y$ has a right inverse, and the left $\varrho$-multiplications of the form $L_{xw\varrho}$ have, moreover, a left inverse, and thus are permutations. Thus also $R_{yx\varrho}$ is, for all $x$ and $y$, a permutation. It follows then that $yx\varrho$ ranges over the whole carrier of our algebra and so all right $\varrho$-multiplications are permutations. Now notice that $x$ does not occur among the variables in $w$: hence $xw\varrho$ ranges with $x$ over the whole carrier, and so also all left $\varrho$-multiplications are permutations; in other words, our algebra is a quasigroup with respect to $\varrho$.

Next we notice that

$$L_y R_{yx\varrho} = L_{xw\varrho}^{-1}$$

does not depend on $y$; hence

$$yz\varrho\, yx\varrho^2 = y'z\varrho\, y'x\varrho^2. \tag{9}$$

Here we put $z = x$ and $y = uR_x^{-1}$, $y' = u'R_x^{-1}$, with arbitrary $u, u'$. Then $yz\varrho = yx\varrho = u$, $y'z\varrho = y'x\varrho = u'$, and

$$uu\varrho = u'u'\varrho.$$

We denote this constant element by $e$, and have then

$$xx\varrho = ee\varrho = e. \tag{10}$$

As we are dealing with a quasigroup with respect to $\varrho$, this implies the validity of (2).

Next we put $x = z = w$ in (5) and obtain, by repeated application of (10),

$$w = e, \tag{11}$$

independently of the values of the variables in $w$. Substituting this in (5), we get the law

$$xe\varrho\, yz\varrho\, yx\varrho^3 = z. \tag{12}$$

Here we put $z = x$ and apply (10) to obtain

$$xe\varrho\, e\varrho = x. \tag{13}$$

Now put $z = e$, $y' = x$ in (9); then the right-hand side becomes $xe\varrho e\varrho$, which by (13) is $x$: thus we get

$$ye\varrho\, yx\varrho^2 = x,$$

which, with $y = e$, gives

$$eex\varrho^2 = x,$$

or

$$L_e^2 = I.$$

On the other hand, putting $x=y=e$ in (8) and recalling that $w=e$, we also get

$$L_eR_eL_e=I:$$

Thus finally

$$R_e=I$$

or

(14) $$xe\varrho=x,$$

which supersedes (13), and allows us to simplify (12) to

$$xyz\varrho yx\varrho^3=z.$$

This is the law (4. 11) of [2], which entails that our algebra is an abelian group with the interpretation (6); but this fact is not needed for our argument.

We now analyse $w$. One easily sees that each of $v_1, v_2, v_3, v_4$ is constant, as they each involve distinct variables. The constant value in each case is $e$, as we now show:

In $v_1$ we put $x_1=y_1$ and obtain

$$v_1=x_1x_1\sigma x_1x_1\sigma\varrho=e.$$

It follows that $\sigma$ is commutative:

(15) $$xy\sigma=yx\sigma.$$

This implies in particular that

$$x_2y_2\sigma z_2\sigma=z_2x_2y_2\sigma^2;$$

hence, putting $x_2=y_2=z_2$ in $v_2$, we obtain

$$v_2=x_2x_2x_2\sigma^2x_2x_2x_2\sigma^2\varrho=e.$$

It follows that $\sigma$ is associative:

(16) $$xy\sigma z\sigma=xyz\sigma^2.$$

Next we put $x_3=z_3=e$ in $v_3$, and notice that $y_3y_3\varrho=e$, too:

$$v_3=ee\sigma e\varrho ee\sigma e\varrho^2=e.$$

Thus — as $v_1=v_2=v_3=e$ and

$$w=v_1v_2v_3v_4\varrho^3=e$$

— we also have $v_4=e$, that is we have verified (4).

We return to $v_3=e$. This means that

(17) $$xe\sigma x\varrho=yy\sigma y\varrho,$$

and this in turn implies that this can not depend on either $x$ or $y$, but is constant, say

(18) $$xe\sigma x\varrho=c.$$

Now (18) allows us to conclude that right $\sigma$-multiplication by $e$, that is $S_e$, is a permutation; for the equation

$$xe\sigma=y$$

has for each $y$ precisely one solution $x$, namely

$$x = cL_y^{-1}.$$

We return to the law (17) and put $y = x$. Then

$$xe\sigma x\varrho = xx\sigma x\varrho,$$

and as $R_x$ is a permutation, this implies

(19) $$xe\sigma = xx\sigma.$$

In this wc put $x = ee\sigma$, to get

$$eS_e^2 = ee\sigma e\sigma = ee\sigma ee\sigma\sigma = ee\sigma e\sigma e\sigma = ee\sigma S_e^2$$

[using the definition of $S_e$, (19), (16), and the definition of $S_e$ again]. As $S_e$ is a permutation, this implies

$$ee\sigma = e.$$

Putting $x = e$ in (18), we that see that $c = e$, and (18) itself then gives

(20) $$xe\sigma = x,$$

or $S_e = I$.

To complete the proof, we assume that

(21) $$xy\sigma = e,$$

and show that then $x = y = e$. Now we have

$$x = xe\sigma = xxy\sigma^2 = xx\sigma y\sigma = xe\sigma y\sigma = xy\sigma = e,$$

by (20), (21), (16), (19), (20), (21); and by the commutativity of $\sigma$, also $y = e$. Thus we have established also (3), and the result follows.

## References

[1] G. GRÄTZER, On the spectra of classes of algebras, *Proc. Amer. Math. Soc.* **18**, (1967), 729-735.

[2] GRAHAM HIGMAN and B. H. NEUMANN: Groups as groupoids with one law, *Publ. Math. Debrecen* **2**, (1952), 215—221.

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