Publ. Math. Debrecen
49 / 1-2 (1996), 7–16

BV-solutions of some systems of nonlinear functional equations

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Abstract. In the paper the solutions of bounded variation (BV-solutions) of the systems of functional equations $\varphi_i(x) = h_i(x, \varphi_1[f_{i,1}(x)], \dots, \varphi_m[f_{i,m}(x)])$ and $\varphi_i[f(x)] = g_i(x, \varphi_1(x), \dots, \varphi_m(x))$, $i = 1, 2, \dots, m$, are considered. Under suitable hypotheses about given functions h_i, g_i and $f_{i,j}$ it is proved that he first system has in an interval $\langle a, b\rangle$ a unique BV-solutions and that BV-solution of the second system in an interval $(a, b\rangle$ depends on an arbitrary function.

Introduction. In this paper we consider the solutions of bounded variation (BV-solutions) of the systems of functional equations

$$\varphi_i(x) = h_i(x, \varphi_i[f_{i,1}(x)], \dots, \varphi_m[f_{i,m}(x)]), \quad i = 1, 2, \dots, m \tag{1}$$

and

$$\varphi_i[f(x)] = g_i(x, \varphi_1(x), \dots, \varphi_m(x)), \quad i = 1, 2, \dots, m, \tag{2}$$

where φ_i are any unknown functions.

In our previous papers [1], [2] (with J. Matkowski) and [3] we have considered BV-solutions of the single functional equations (linear, nonlinear of first order and nonlinear of higher order). This is the first paper in which the BV-solutions of the systems of functional equations are considered. The special role in our considerations play some Lemmas and Theorems proved by Professor J. Matkowski in his habilitation work [4].

Let (X, d) be a metric space and $I \subset R$ an interval. By $P(I)$ we denote the set of all finite partitions $p : x_0 < x_1 < \dots < x_s$ of the

Mathematics Subject Classification: Primary 39 B 30.
Key words and phrases: Functional equations, BV-solutions.

interval I, where $x_i \in I$, $i = 0, 1, \dots, s$. Similarly as in the papers [2] and [3], for the function $\varphi : I \to M$ we denote by

$$\operatorname*{Var}_I \varphi := \sup_{P(I)} \sum_{i=1}^{s} d(\varphi(x_i),\ \varphi(x_{i-1})). \tag{3}$$

By BV(I) we denote the space of all functions $\varphi : I \to X$ such that $\operatorname*{Var}_I \varphi < \infty$.

Also similarly as in [4] let us define the numbers $c_{i,k}^{(r)}$ as follows:

$$c_{i,k}^{(r+1)} = \begin{cases} c_{1,1}^{(r)}\ c_{i+1,k+1}^{(r)} - c_{i+1,1}^{(r)}\ c_{1,k+1}^{(r)} & \text{for } i = k \\ c_{1,1}^{(r)}\ c_{i+1,k+1}^{(r)} + c_{i+1,1}^{(r)}\ c_{1,k+1}^{(r)} & \text{for } i \neq k \end{cases} \tag{4}$$

$i, k = 1, 2, \dots, m - r - 1; \quad r = 0, 1, \dots, m - 2.$

1. In this section we assume the following hypotheses:

(i) (X, d) is a complete metric space,

(ii) $f_{i,k} : \langle a, b\rangle \to \langle a, b\rangle$ are continuous and strictly increasing in $\langle a, b\rangle$, $i, k = 1, 2, \dots, m$,

(iii) $h_i : \langle a, b\rangle \times X^m \to X$, $i = 1, 2, \dots, m$,

(iv) There are $H_i \in$ BV $\langle a, b\rangle$ and $\ell_{i,k} \in \mathbb{R}$ such that $0 < \ell_{i,k} < 1$, $i, k = 1, 2, \dots, m$, and

$$\begin{gathered} d(h_i(x, y_1, \dots, y_m), h_i(\bar{x}, \bar{y} - 1, \dots, \bar{y}_m)) \\ \leq \sum_{k=1}^{m} \ell_{i,k} d(y_k, \bar{y}_k) + d(H_i(x), H_i(\bar{x})), \end{gathered} \tag{5}$$

$i = 1, 2, \dots, m,\ (x, y_1, \dots, y_m), (\bar{x}, \bar{y}_1, \dots, \bar{y}_m) \in \langle a, b\rangle \times X^m$

and the numbers

$$c_{i,k}^{(0)} = \begin{cases} 1 - \ell_{i,k} & \text{for } i = k \\ \ell_{i,k} & \text{for } i \neq k \end{cases} \quad i, k = 1, 2, \dots, m \tag{6}$$

fulfil the conditions

$$c_{i,i}^{(r)} > 0, \quad i = 1, 2, \dots, m - r; \quad r = 0, 1, \dots, m - 1, \tag{7}$$

where $c_{i,k}^{(r)}$ are defined by (4).

Now we have the following:

Theorem 1. *If hypotheses* (i)-(iv) *are fulfilled then the system of equations* (1) *has in the interval* $\langle a,b\rangle$ *a unique solution* $\varphi = (\varphi_1,\dots,\varphi_m)$, *such that* $\varphi_i \in \mathrm{BV}\,\langle a,b\rangle$, $i = 1,2,\dots,m$. *This solution is given by the formula* $\varphi_i = \lim_{n\to\infty} \varphi_{i,n}$ *(uniformly in* $\langle a,b\rangle$*), where* $\varphi_{i,n+1}(x) =$
$h_i(x,\varphi_{1,n}[f_{i,1}(x)],\dots,\varphi_{m,n}[f_{i,m}(x)])$, $x \in \langle a,b\rangle$, $i = 1,2,\dots,m$,
$n = 0,1,\dots$ *and* $\varphi_{i,0} \in \mathrm{BV}\,\langle a,b\rangle$ *is arbitrarily choosen. Moreover*

$$\operatorname*{Var}_{\langle a,b\rangle} \varphi_i \le \frac{\sum\limits_{\substack{k=1\\k\ne i}}^{m} \ell_{i,k} \operatorname*{Var}\limits_{\langle a,b\rangle} \varphi_k + \operatorname*{Var}\limits_{\langle a,b\rangle} H_i}{1-\ell_{i,i}}, \qquad i = 1,2,\dots,m. \tag{8}$$

Proof. In view of Lemma 1.1 from [4], there exist the numbers $r_i > 0$, $i = 1,2,\dots,m$ and t, $0 < t < 1$, such that

$$\sum_{\substack{k=1\\k\ne i}}^{m} \ell_{i,k} r_k \le t\, r_i, \quad i = 1,2,\dots,m. \tag{9}$$

Note that for every $\lambda > 0$ the numbers λr_i, $i = 1,2,\dots,m$, also satisfy (9). It follows that without any loss of generality we can assume that

$$\operatorname*{Var}_{\langle a,b\rangle} H_i \le (1-t) r_i, \quad i = 1,2,\dots,m. \tag{10}$$

Now let us define the system of functions spaces

$$X_i = \left\{ \varphi_i \in \mathrm{BV}\,\langle a,b\rangle : \operatorname*{Var}_{\langle a,b\rangle} \varphi_i \le r_i, \quad i = 1,2,\dots,m \right\} \tag{11}$$

with the metrics

$$\varrho_i\,(\bar\varphi_i,\bar{\bar\varphi}_i) = \sup_{x\in\langle a,b\rangle} d\,(\bar\varphi_i(x),\bar{\bar\varphi}_i(x))\,, \quad \bar\varphi_i,\bar{\bar\varphi}_i \in X_i, \tag{12}$$
$$\text{for every } i = 1,2,\dots,m.$$

Evidently, the space (X_i,ϱ_i) is a complete metric space,
$i = 1,2,\dots,m$.

Let us consider now the system of transformations

$$\Psi_i = T_i[\varphi_1,\dots,\varphi_m], \quad i = 1,2,\dots,m,$$

where $\Psi_i(x)$ is defined by the formula

$$\Psi_i(x) = h_i\left(x, \varphi_1[f_{i,1}(x)], \ldots, \varphi_m[f_{i,m}(x)]\right), \\ x \in \langle a, b\rangle, \quad i = 1, 2, \ldots, m.$$

We shall show that $T_i : X_1 \times X_2 \times \cdots \times X_m \to X_i,\ i = 1, 2, \ldots, m.$

Let $\varphi_i \in X_i$. We shall show that $\operatorname*{Var}_{\langle a,b\rangle} \Psi_i \le r_i,\ i = 1, 2, \ldots, m$. For this purpose, let us take into account the set of all partitions $P\langle a, b\rangle$ of interval $\langle a, b\rangle$ and estimate $\operatorname*{Var}_{\langle a,b\rangle} \Psi_i$.

In virtue of hypotheses (i)–(iv) and inequalities (9) and (10), we have

$$\begin{aligned}
\operatorname*{Var}_{\langle a,b\rangle} \Psi_i &= \sup_{P\langle a,b\rangle} \sum_{j=1}^{s} d(\Psi_i(x_j), \Psi_i(x_{j-1})) \\
&= \sup_{P\langle a,b\rangle} \sum_{j=1}^{s} d\Big(h_i(x_j, \varphi_1[f_{i,1}(x_j)], \ldots, \varphi_m[f_{i,m}(x_j)]), \\
&\qquad\qquad h_i(x_{j-1}, \varphi_1[f_{i,1}(x_{j-1})], \ldots, \varphi_m[f_{i,m}(x_{j-1})])\Big) \\
&\le \sup_{P\langle a,b\rangle} \sum_{j=1}^{s} \Bigg\{ \sum_{k=1}^{m} \ell_{i,k} d(\varphi_k[f_{i,k}(x_j)], \varphi_k[f_{i,k}(x_{j-1})]) \\
&\qquad\qquad + d(H_i(x_j), H_i(x_{j-1})) \Bigg\} \\
&\le \sup_{P\langle a,b\rangle} \sum_{j=1}^{s} \sum_{k=1}^{m} \ell_{i,k} d(\varphi_k[f_{i,k}(x_j)], \varphi_k[f_{i,k}(x_{j-1})]) \\
&\qquad\qquad + \sup_{P\langle a,b\rangle} \sum_{j=1}^{s} d(H_i(x_j), H_i(x_{j-1})) \\
&\le \sum_{k=1}^{m} \left\{ \ell_{i,k} \sup_{P\langle a,b\rangle} \sum_{j=1}^{s} d\left(\varphi_k[f_{i,k}(x_j)], \varphi_k[f_{i,k}(x_{j-1})]\right) \right\} \\
&\qquad\qquad + \sup_{P\langle a,b\rangle} \sum_{j=1}^{s} d(H_i(x_j), H_i(x_{j-1})) \\
&= \sum_{k=1}^{m} \ell_{i,k} \operatorname*{Var}_{\langle f_{i,k}(a), f_{i,k}(b)\rangle} \varphi_k + \operatorname*{Var}_{\langle a,b\rangle} H_i
\end{aligned}$$

$$\le \sum_{k=1}^{m} \ell_{i,k} \operatorname*{Var}_{\langle a,b\rangle} \varphi_k + \operatorname*{Var}_{\langle a,b\rangle} H_i \le \sum_{k=1}^{m} \ell_{i,k} r_k + (1-t) r_i$$
$$\le t r_i + (1-t) r_i = r_i.$$

Next, we shall prove that the transformation Ψ_i is a contraction map, $i = 1, 2, \ldots, m$.

Let us take an arbitrary $\bar{\varphi}_i, \bar{\bar{\varphi}}_i \in X_i$ and, taking into account (ii), (5), (9) and (12), estimate

$$\begin{aligned}
&\varrho_i(T_i[\bar{\varphi}_1, \ldots, \bar{\varphi}_m], T_i[\bar{\bar{\varphi}}_1, \ldots, \bar{\bar{\varphi}}_m]) = \varrho_i(\bar{\Psi}_i, \bar{\bar{\Psi}}_i)\\
&\quad = \sup_{x \in \langle a,b\rangle} d(\bar{\Psi}_i(x), \bar{\bar{\Psi}}_i(x))\\
&\quad = \sup_{x \in \langle a,b\rangle} d\Big(h_i(x, \bar{\varphi}_1[f_{i,1}(x)], \ldots, \bar{\varphi}_m[f_{i,m}(x)]),\\
&\qquad\qquad h_i(x, \bar{\bar{\varphi}}_i[f_{i,1}(x)], \ldots, \bar{\bar{\varphi}}_m[f_{i,m}(x)])\Big)\\
&\quad \le \sup_{x \in \langle a,b\rangle} \sum_{k=1}^{m} \ell_{i,k}\, d(\bar{\varphi}_k[f_{i,k}(x)], \bar{\bar{\varphi}}_k[f_{i,k}(x)])\\
&\quad \le \sup_{x \in \langle a,b\rangle} \sum_{k=1}^{m} \ell_{i,k}\, d(\bar{\varphi}_k(x), \bar{\bar{\varphi}}_k(x))\\
&\quad \le \sum_{k=1}^{m} \ell_{i,k} \sup_{x \in \langle a,b\rangle} d(\bar{\varphi}_k(x), \bar{\bar{\varphi}}_k(x)) = \sum_{k=1}^{m} \ell_{i,k}\, \varrho_k(\bar{\varphi}_k, \bar{\bar{\varphi}}_k).
\end{aligned}$$

Thus, the first statement of the theorem results from Theorem 1.4 in [4].

To prove that the estimation (8) holds let us take the set $P\langle a,b\rangle$ of all partitions of the interval $\langle a,b\rangle$. Using succesively (1), (iv) and (ii) we obtain

$$\begin{aligned}
\operatorname*{Var}_{\langle a,b\rangle} \varphi_i &= \sup_{P\langle a,b\rangle} \sum_{j=1}^{s} d(\varphi_i(x_j), \varphi_i(x_{j-1}))\\
&= \sup_{P\langle a,b\rangle} \sum_{j=1}^{s} d\Big(h_i(x_j, \varphi_1[f_{i,1}(x_j)], \ldots, \varphi_m[f_{i,m}(x_j)]),\\
&\qquad\qquad h_i(x_{j-1}, \varphi_1[f_{i,1}(x_{j-1})], \ldots, \varphi_m[f_{i,m}(x_{j-1})])\Big)
\end{aligned}$$

$$
\begin{aligned}
&\le \sup_{P\langle a,b\rangle} \sum_{j=1}^{s} \Bigg\{ \sum_{k=1}^{m} \ell_{i,k}\, d(\varphi_k[f_{i,k}(x_j)], \varphi_k[f_{i,k}(x_{j-1})]) \\
&\qquad\qquad + d(H_i(x_j), H_i(x_{j-1})) \Bigg\} \\
&\le \sup_{P\langle a,b\rangle} \sum_{j=1}^{s} \sum_{k=1}^{m} \ell_{i,k}\, d(\varphi_k[f_{i,k}(x_j)], \varphi_k[f_{i,k}(x_{j-1})]) \\
&\qquad + \sup_{P\langle a,b\rangle} \sum_{j=1}^{s} d(H_i(x_j), H_i(x_{j-1})) \\
&\le \sum_{k=1}^{m} \Bigg\{ \ell_{i,k} \sup_{P\langle a,b\rangle} \sum_{j=1}^{s} d(\varphi_k[f_{i,k}(x_j)], \varphi_k[f_{i,k}(x_{j-1})]) \Bigg\} \\
&\qquad + \sup_{P\langle a,b\rangle} \sum_{j=1}^{s} d(H_i(x_j), H_i(x_{j-1})) \\
&= \sum_{k=1}^{m} \ell_{i,k} \operatorname*{Var}_{\langle f_{i,k}(a), f_{i,k}(b)\rangle} \varphi_k + \operatorname*{Var}_{\langle a,b\rangle} H_i \\
&\le \sum_{k=1}^{m} \ell_{i,k} \operatorname*{Var}_{\langle a,b\rangle} \varphi_k + \operatorname*{Var}_{\langle a,b\rangle} H_i \ .
\end{aligned}
$$

This completes the proof of the Theorem 1.

Remark. If in the hypothesis (iv), instead conditions (7), we assume that the characteristic roots of the matrix $[\ell_{i,k}]$ have absolute values less than 1 then Theorem 1 remain true. It follows from the Lemma 1.2 and Theorem 1.5 in [4].

2. Now let us consider the system (2) and assume the following hypotheses:

(i) (X, d) is a complete metric space,

(ii) $f : \langle a, b\rangle \to \langle a, b\rangle$ is continuous, strictly increasing in $\langle a, b\rangle$ and $a < f(x) < x$ for $x \in (a, b\rangle$,

(iii) $g_i : \langle a, b\rangle \times X^m \to X$, $i = 1, 2, \ldots, m$,

(iv) There are functions $G_i \in \mathrm{BV}(a, b\rangle$ and positive constants $\ell_{i,k}$, $i, k = 1, 2, \ldots, m$, such that

$$\begin{aligned} d(g_i(x, y_1, \ldots, y_m), g_i(\bar{x}, \bar{y}_1, \ldots, \bar{y}_m)) \\ \le \sum_{k=1}^{m} \ell_{i,k}\, d(y_k, \bar{y}_k) + d(G_i(x), G_i(\bar{x})), \end{aligned}$$
$$i = 1, 2, \ldots, m,\ (x, y_1, \ldots, y_m), (\bar{x}, \bar{y}_1, \ldots, \bar{y}_m) \in (a, b\rangle \times X^m,$$

(v) All the characteristic roots of the matrix $[\ell_{i,k}]$ have absolute values less than 1.

We shall prove the following:

Theorem 2. *Let hypotheses* (i)–(v) *be fulfilled. Then the system of equations* (2) *has in the interval* $(a, b\rangle$ *a solution* $\varphi = (\varphi_1, \ldots, \varphi_m)$, $\varphi_i \in \mathrm{BV}(a, b\rangle$, $i = 1, 2, \ldots, m$, *depending on an arbitrary function. More precisely: for any system of functions* $\{\varphi_{i,0}\} : \langle f(b), b\rangle \to \mathbb{R}$, *such that* $\varphi_{i,0} \in \mathrm{BV}\langle f(b), b\rangle$, $i = 1, 2, \ldots, m$, *there exists the unique system of functions* $\varphi = (\varphi_1, \ldots, \varphi_m)$, $\varphi_i \in \mathrm{BV}(a, b\rangle$ *satisfying the system of equations* (2) *in* $(a, b\rangle$ *and such that* $\varphi_i = \varphi_{i,0}$ *in the interval* $\langle f(b), b\rangle$, $i = 1, 2, \ldots, m$.

Proof. Let $I_n = \langle f^{n+1}(b), f^n(b)\rangle$, $n = 0, 1, 2, \ldots$, where $f^n(x)$ denotes the n-th iterate of the function $f(x)$. In virtue of hypothesis (ii) we have $\bigcup_{n=0}^{\infty} I_n = (a, b\rangle$. In the interval $I_0 = \langle f(b), b\rangle$ we define an arbitrary system of functions $\{\varphi_{i,0}\}$, $i = 1, 2, \ldots, m$, fulfilling the conditions of the theorem. Notice, that if $x \in I_1$ then $f^{-1}(x) \in I_0$. So let us define the function $\varphi_{i,1} : I_1 \to \mathbb{R}$ in the following way:

$$\begin{aligned} \varphi_{i,1}(x) = g_i(f^{-1}(x),\ \varphi_{1,0}[f^{-1}(x)], \ldots,\ \varphi_{m,0}[f^{-1}(x)]), \\ x \in I_1, \quad i = 1, 2, \ldots, m. \end{aligned}$$

We shall show that $\varphi_{i,1} \in \mathrm{BV}(I_1)$. Let $P(I_1)$ be the set of all finite partitions of the interval I_1. From the hypotheses (iv) and (ii) we get succesively

$$\begin{aligned} \operatorname*{Var}_{I_1} \varphi_{i,1} &= \sup_{P(I_1)} \sum_{j=1}^{s} d(\varphi_{i,1}(x_j), \varphi_{i,1}, (x_{j-1})) \\ &= \sup_{P(I_1)} \sum_{j=1}^{s} d\Big(g_i(f^{-1}(x_j), \varphi_{1,0}\left[f^{-1}(x_j)\right], \ldots, \varphi_{m,0}\left[f^{-1}(x_j)\right]), \\ &\qquad g_i(f^{-1}(x_{j-1}), \varphi_{1,0}\left[f^{-1}(x_{j-1})\right], \ldots, \varphi_{m,0}\left[f^{-1}(x_{j-1})\right]) \Big) \end{aligned}$$

$$
\begin{aligned}
&\le \sup_{P(I_1)} \sum_{j=1}^{s} \Bigg\{ \sum_{k=1}^{m} \ell_{i,k}\, d(\varphi_{k,0}[f^{-1}(x_j)], \varphi_{k,0}[f^{-1}(x_{j-1})] \\
&\qquad + d\left(G_i\left[f^{-1}(x_j)\right], G_i\left[f^{-1}(x_{j-1})\right]\right)\Bigg\} \\
&\le \sup_{P(I_1)} \sum_{j=1}^{s} \sum_{k=1}^{m} \ell_{i,k}\, d\left(\varphi_{k,0}\left[f^{-1}(x_j)\right], \varphi_{k,0}\left[f^{-1}(x_{j-1})\right]\right) \\
&\qquad + \sup_{P(I_1)} \sum_{j=1}^{s} d\left(G_i\left[f^{-1}(x_j)\right], G_i\left[f^{-1}(x_{j-1})\right]\right) \\
&\le \sum_{k=1}^{m} \left\{ \ell_{i,k} \sup_{P(I_1)} \sum_{j=1}^{s} d\left(\varphi_{k,0}\left[f^{-1}(x_j)\right], \varphi_{k,0}\left[f^{-1}(x_{j-1})\right]\right) \right\} \\
&\qquad + \sup_{P(I_1)} \sum_{j=1}^{s} d\left(G_i\left[f^{-1}(x_j)\right], G_i\left[f^{-1}(x_{j-1})\right]\right) \\
&= \sum_{k=1}^{m} \ell_{i,k} \operatorname*{Var}_{\langle f(b), b\rangle} \varphi_{k,0} + \operatorname*{Var}_{\langle f(b), b\rangle} G_i = \sum_{k=1}^{m} \ell_{i,k} \operatorname*{Var}_{I_0} \varphi_{k,0} + \operatorname*{Var}_{I_0} G_i .
\end{aligned}
$$

Hence $\varphi_{i,1} \in \mathrm{BV}(I_1)$.

More generally, let us take the interval I_{n+1}. If $x \in I_{n+1}$ then $f^{-1}(x) \in I_n$. Let us define a function $\varphi_{i,n+1} : I_{n+1} \to \mathbb{R}$ in the following way:

$$
\varphi_{i,n+1}(x) = g_i\left(f^{-1}(x), \varphi_{1,n}\left[f^{-1}(x)\right], \ldots, \varphi_{m,n}\left[f^{-1}(x)\right]\right), \quad x \in I_{n+1}, \quad i = 1, 2, \ldots, m. \tag{13}
$$

By similar calculation as above one can prove that $\varphi_{i,n+1} \in \mathrm{BV}(I_{n+1})$ and that

$$
\operatorname*{Var}_{I_{n+1}} \varphi_{i,n+1} \le \sum_{k=1}^{m} \ell_{i,k} \operatorname*{Var}_{I_n} \varphi_{k,n} + \operatorname*{Var}_{I_n} G_i, \quad i = 1, 2, \ldots, m. \tag{14}
$$

Now we introduce the following notations:

$$
a_{i,k} = \operatorname*{Var}_{I_k} \varphi_{i,k} \qquad b_{i,k} = \operatorname*{Var}_{I_k} G_i \tag{15}
$$

Using these notations we can write the inequality (14) as follows:

$$
a_{i,n+1} \le \sum_{k=1}^{m} \ell_{i,k} a_{k,n} + b_{i,n}, \quad i = 1, 2, \ldots, m. \tag{16}
$$

In the interval $(a,b\rangle$ we define the function $\varphi = (\varphi_1, \dots, \varphi_m)$ in the following way:

$$\varphi_i(x) = \begin{cases} \varphi_{i,0}(x) & \text{for } x \in I_0, \quad i = 1,2,\dots,m \\ \varphi_{i,n+1}(x) & \text{for } x \in I_{n+1}, \quad i = 1,2,\dots,m, \\ & \qquad n = 0,1,\dots. \end{cases} \tag{17}$$

From (13) and (17) it follows that φ satisfies the system of equations (2). The uniqueness of the solution is obvious.

We have to prove that $\varphi_i \in \mathrm{BV}(a,b\rangle$, $i = 1,2,\dots,m$. From (15) and (17) it follows that

$$\begin{aligned}\operatorname*{Var}_{(a,b\rangle} \varphi_i &= \sum_{n=0}^{\infty} \operatorname*{Var}_{I_n} \varphi_i = \operatorname*{Var}_{I_0} \varphi_i + \sum_{n=1}^{\infty} \operatorname*{Var}_{I_n} \varphi_i \\ &= \operatorname*{Var}_{I_0} \varphi_{i,0} + \sum_{n=1}^{\infty} \operatorname*{Var}_{I_n} \varphi_{i,n} = \operatorname*{Var}_{I_0} \varphi_{i,0} + \sum_{n=i}^{\infty} a_{i,n}.\end{aligned}$$

Since, by hypotheses of the theorem, $\operatorname*{Var}_{I_0} \varphi_{i,0}$ is finite, and every component of the series $\sum_{n=1}^{\infty} a_{i,n}$ fulfils inequality (16), in view of Lemma 4.1 from the paper [4], this series converges. This completes the proof of Theorem 2.

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(Received May 2, 1994; revised October 16, 1995)