Toeplitz matrices in quasi Hilbert algebras I.

By BÉLA GYIRES (Debrecen)

The present paper is connected with the following theorem of GÁBOR SZEGŐ ([7], 206):

Let $f(\lambda)$ be a real valued function, with period 2π, bounded and measurable, $m \leqq f(\lambda) \leqq M$, and let $f(\lambda)$ have the Fourier expansion

$$f(\lambda) \sim \sum_{-\infty}^{\infty} c_k e^{ikx} \qquad (c_{-k} = \bar{c}_k).$$

Let

$$T_n(f) = (c_{r-s})_{r,s=0}^{n}$$

the so called Toeplitz matrix of order $n+1$ constructed with the aid of the Fourier coefficients c_k have the eigenvalues $\lambda_1^{(n)}, \ldots, \lambda_{n+1}^{(n)}$. Since the matrix $T_n(f)$ is Hermitian, its eigenvalues are real numbers. We write each of them a number of times corresponding to its multiplicity. Szegő has shown that all of the eigenvalues $\lambda_k^{(n)}$ fall into the interval $[m, M]$, and that for any function $F(\lambda)$, continuous in this interval, the relation

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=1}^{n+1} F(\lambda_k^{(n)}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(f(\lambda))\, d\lambda \tag{1}$$

holds.

In his paper [5] the author has generalized this theorem for the case, when the values of the function $f(\lambda)$ are Hermitian matrices of order p $(p \geqq 1)$. In this case the condition $m \leqq f(\lambda) \leqq M$ is to be understood so that the range of the Hermitian form belonging to $f(\lambda)$, taken on the unit shpere of the p-dimensional (complex) space falls into the interval $[m, M]$. The Fourier coefficients c_k will also be matrices of order p, $c_{-k} = c_k^*$, where * denotes transpose-conjugate. The Toeplitz matrix $T_n(f)$ will then be a hypermatrix of order $n+1$, consisting of blocks of order p, i. e. a matrix of order $(n+1)p$, and clearly again hermitian; the eigenvalues of $T_n(f)$ will be denoted by $\lambda_1^{(n)}, \ldots, \lambda_{(n+1)p}^{(n)}$.

In this paper above mentioned the author has proved that if the matrix-valued function $f(\lambda)$ of order p is a measurable bounded function of λ, then as a generalization of (1) the following limit relation holds:

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=1}^{(n+1)p} F(\lambda_k^{(n)}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} tr F(f(\lambda))\, d\lambda. \tag{2}$$

Here tr denotes the sum of the elements standing on the main diagonal of the matrix, and $F(f(\lambda))$ is defined by the formula

$$F(f(\lambda))=U(\lambda)(F(\lambda_1(\lambda)), \dots, F(\lambda_p(\lambda)))U^*(\lambda),$$

when $f(\lambda)$ is reduced to its diagonal form $(\lambda_1(\lambda), \dots, \lambda_p(\lambda))$ by the unitary matrix $U(\lambda)$ (i. e. $f(\lambda)=U(\lambda)(\lambda_1(\lambda), \dots, \lambda_p(\lambda))U^*(\lambda)$).

In his paper already mentioned, the author gives more general definition of Toeplitz matrices; the limit theorem expressed by (2) is then obtained as a special case of theorems on these more general Toeplitz matrices. The main feature of the generalization is the following: starting with a matrix-valued function of order p, defined on a arbitrary finite or infinite interval of the real line, square integrable and bounded, Toeplitz matrices are defined with the aid of a full orthonormal system of matrix-valued functions defined on this interval. The discussion of the whole problem makes use of a generalization of Hilbert space, where the value of inner product is a matrix of order p with complex elements. In formulating the generalized theorems, there occurs a condition, which for the case $p=1$ was considered by U. GRENANDER, and called by him „trace complete" property ([1], 130). Finally, it is to be mentioned that the limit (2) was given by the author also in the more general case, when the trace of the Toeplitz matrix was defined as the sum of the diagonal blocks of this hypermatrix, i.e. when the trace itself is a matrix of order p.

The present paper grew out from the authors observation that the main part of the results set forth in his paper [5] retains validity if one starts with more general normed algebras. The natural generalization is based on the so called quasi Hilbert algebras, introduced first here. Our aim is to work out a theory of generalized Toeplitz matrizes, from this point of view. In § 1 we give the axioms of the quasi Hilbert algebra and we expose those consepts and facts about normed algebras, which are used in the sequel. In § 2 we start from the axioms of quasi Hilbert space, and formulate and prove those theorems which are needed in defining the generalized Toeplitz matrices. In § 3 we deal with the generalized Toeplitz matrices generated by the elements of a quasi Hilbert algebra; several properties are established and then the proof of the two theorems which form the main result of the paper are presented. Finally § 4 treats the limit theorem expressed by (2) (which is a generalization of Szegő's theorem) as a special case of these theorems.

§ 1. Definitions

1. Let R be a normed algebra with unit element.[1]) We shall call the set X a normed space over R, if

X is an R-module,
X is a normed space,
$$|\alpha x| \leqq |\alpha|\,|x| \qquad (\alpha \in R,\ x \in X).$$

2. Let R be a symmetric Banach algebra [2]) where the norm is determined

[1]) For the concepts used see [9] and [11].

[2]) A Banach algebra R with involution is said to be symmetric (in [11] completely symmetric) if it has a unit element 1 and if $(1+\alpha\alpha^*)^{-1}$ exists for all $\alpha \in R$.

by a positive functional.[3]) The normed space H over R will be called a quasi Hilber space, if there is defined on it an inner product (x, y) taking values in R, and having the following properties:

$$(x+y, z)=(x, z)+(y, z) \qquad (x, y, z \in H),$$

$$(\alpha x, y)=\alpha(x, y) \qquad (\alpha \in R),$$

$$(x, y)=(y, x)^*,$$

$$(x, x)>0, \text{ if } x \neq 0,$$

$$|x|=|(x, x)^{\frac{1}{2}}|.$$

3. Let R be a symmetric Banach algebra in which the norm is determined by a positive functional. The quasi Hilbert space H over R will be called a quasi Hilbert algebra H^*, if there is defined a multiplication and an involution on H, in such a way that the following requirements are fulfilled:

H^* is a normed algebra with involution with respect to these operations and to the addition, scalar multiplication and norm defined originally on H, and moreover $(xy, z)=(x, zy^*)$ holds for all elements x, y, z of H^* and $xx^* \neq 0$ if $x \neq 0$.

Now we remind of a few concepts concerning normed algebras. If R is an algebra with unit element ε then by the spectrum (with respect to R) of the element x of this algebra we mean the set of those complex numbers λ, for wich $x-\lambda\varepsilon$ has no inverse in R. If R is an algebra with involution then the element x is said to be Hermitian element, if $x=x^*$ holds. The element $x \in R$ is said to be a bounded element, if x is a Hermitian element and its spectrum is contained in a finite (real) interval. The elements of the form xx^* of R are called nonnegative elements. If R has a unit element, then the nonnegative invertable elements of R are called positive elements. It is known that if R is a normed algebra with involution, then for any positive functional f defined on R the inequality $|f(x, y)|^2 \leqq f(xx^*)f(yy^*)$ holds for any elements $x, y \in R$. If R is a symmetrical Banach algebra with unit element, then any positive functional defined on R is bounded. If R is a normed algebra with involution and the norm of the elements of R is determined by the positive functional f, then by the trace of the element $x \in R$ we mean the number $tr\, x=f(x)$. It can be shown that if R is a symmetric Banach algebra in which the norm is determined by a positive functional, then any nonnegative element x of R has a nonnegative square root, and if x is positive then its square roots is also positive.

Let finally H^* be a quasi Hilbert algebra with unit element over the symmetric Banach algebra R, and let the norm of the elements of R be determined by the positive functional f. Then it is clear that H^* is a (not necessarily complete) Hilbert space with respect to the inner product $\langle x, y\rangle=f((x, y))$. Moreover, for any element $c \in H^*$ the mapping $x \to cx$ is a bounded operator of H^*. So it has a uniquely determined extension to a bounded operator of H^* the smalest (complete) Hilbert space $\overline{H}^*$ containing H^*. If we make correspond to the element $c \in H^*$ this operator of $\overline{H}^*$, then we get a *-isomorphism of the algebra H^* into the algebra of bounded

[3]) If R is a normed algebra with involution then we say that the norm of the elements of R is determined by the positive linear functional f defined on R, if $|\alpha| = \sqrt{f(\alpha\alpha^*)}$ holds for all elements $\alpha \in R$.

operators of the Hilbert space $\overline{H}^*$. This mapping is continuous in both directions. Therefore it is clear that the spectrum of the operator corresponding (in the above sense) the element $c \in H^*$ is contained in the spectrum (with respect to H^*) of the element c. So, if the element $c \in H^*$ is bounded, then the operator belonging to c can also be written in the form $\int_a^b \lambda dE(\lambda)$, where $[a, b]$ is a finite interval containing the spectrum of c and $E(\lambda)$ is the spectral function belonging to the operator. So in this case for any real continuous function $F(\lambda)$ $(a \leqq \lambda \leqq b)$ we can define $F(c)$ by the formula

$$F(c) = \int_a^b F(\lambda)\, dE(\lambda).$$

If there is an element $c' \in H^*$ for which the corresponding operator coincides with $F(c)$, then we say that $F(c) \in H^*$ and $F(c) = c'$.

If $F(c) \in H^*$, then for any $y \in H^*$

$$f((F(c)y, y)) = \langle F(c)y, y \rangle = \int_a^b F(\lambda) d \langle E(\lambda)y, y \rangle =$$
$$= \int_a^b F(\lambda)\, df((E(\lambda)y, y)).$$

§ 2. The quasi Hilbert space

In this section we give the foundation of the linear algebra of quasi Hilbert spaces; moreover we investigate some metric properties of these spaces.

1. Let R be an algebra with unit element and with an involution. Let V_n be the space of all sequences $(\alpha_1, \ldots, \alpha_n)$ formed of elements from R. (The addition of sequences and their multiplication by elements of R are defined componentwise.) By inner product $(\mathfrak{a}, \mathfrak{b})$ of two such elements $\mathfrak{a} = (\alpha_1, \ldots, \alpha_n)$, $\mathfrak{b} = (\beta_1, \ldots, \beta_n)$ we mean the element $\alpha_1 \beta_1^* + \ldots + \alpha_n \beta_n^* \in R$.

The matrices of order n, formed by elements from R constitute a symmetric algebra with unit element with respect to the following operations:

By the sum of two such matrices $A = (\alpha_{kl})$, $B = (\beta_{kl})$ of order n we mean the matrix

$$A + B = (\alpha_{kl} + \beta_{kl}).$$

By the product of $\alpha \in R$ and of A we mean the matrix

$$\alpha A = (\alpha \alpha_{kl}).$$

By the product of A and of B we mean the matrix

$$AB = \left(\sum_{\nu=1}^{n} \alpha_{k\nu} \beta_{\nu l} \right).$$

By the conjugate of A we mean the matrix

$$A^* = (\gamma_{kl}), \ \gamma_{kl} = \alpha_{lk}^*.$$

Now, among the finite matrices which can be built from elements of R we define positive semidefinite and positive definite matrices.

First of all by the product of the vector $\mathfrak{x}=(\xi_1, \ldots, \xi_n)\in V_n$ and of the matrix A consisting of elements from R we mean the vector

$$\mathfrak{x}A = \left(\sum_{\nu=1}^{n} \xi_\nu \alpha_{\nu 1}, \ldots, \sum_{\nu=1}^{n} \xi_\nu \alpha_{\nu n}\right) \in V_n.$$

We say the matrix A to be positive semidefinite if for any vector $\mathfrak{x}\in V_n$ the inner product $(\mathfrak{x}A, \mathfrak{x})$ is a nonnegative element of the algebra R. If, on the other hand, $(\mathfrak{x}A, \mathfrak{x})$ is always a positive element of the algebra R (the trivial case of $\mathfrak{x}=0$ exepted), then we call A a positive definite matrix.

2. Let H be a quasi Hilbert space over a symmetric Banach algebra in which the norm is determined by a positive functional. We say that the elements $x_k\in H$ $(k=1, \ldots, n)$ are linearly independent (with respect to R), if

$$\alpha_1 x_1 + \ldots + \alpha_n x_n = 0 \qquad (\alpha_k \in R,\ k=1, \ldots, n)$$

can hold only for $\alpha_1=\ldots=\alpha_n=0$. We say that the elements $e_k\in H$ $(k=1,2,\ldots)$ form an orthonormal system (with respect to R) if (e_k, e_l) is equal to the unit element or to the zero element of R respectively, according as $k=l$ or $k\neq l$. In investigating independent respectively orthonormal systems of elements of H, the so called Gram-matrices play a certain role.

By the Gram-matrix of the system of elements $x_k\in H$ $(k=1, \ldots, n)$ we understand the matrix

$$G(x_1, \ldots, x_n) = \begin{pmatrix} (x_1, x_1)\ldots(x_1, x_n) \\ \cdot \quad \cdots \quad \cdot \\ (x_n, x_1)\ldots(x_n, x_n) \end{pmatrix}$$

formed by elements of R.

Theorem 1. *Let H be a quasi Hilbert space over a symmetric Banach algebra R, in which the norm is determined by a positive functional. The Gram-matrix of any finite system of elements of H is positive semidefinit and it is positive definite if and only if the system of elements is independent.*

Proof. Let $x_k\in H$ $(k=1, \ldots, n)$ be an arbitrary system of elements. Then for Gram-matrix G of this system and for any vector $\mathfrak{x}=(\xi_1, \ldots, \xi_n)\in V_n$

$$(\mathfrak{x}G, \mathfrak{x}) = \sum_{k,l=1}^{n} \xi_k (x_k, x_l)\xi_l^* = \left(\sum_{k=1}^{n} \xi_k x_k, \sum_{k=1}^{n} \xi_k x_k\right)$$

is a nonnegative element of R, and so G is a positive semidefinite matrix.

In order to prove the second assertion of the theorem, we first consider a linearly dependent system of elements $x_1, \ldots, x_n$ of H; this means that for suitable elements $\alpha_k\in R$ $(k=1, \ldots, n)$ not all of which are equal to zero, $\alpha_1 x_1+\ldots+\alpha_n x_n=0$ holds. If we form the inner product with $\alpha_k x_k$ of both sides of the equality and then adding these relations we obtain with the vector $\mathfrak{a}=(\alpha_1, \ldots, \alpha_n)\neq 0$ the relation

$$(\mathfrak{a}G, \mathfrak{a}) = \sum_{k,l=1}^{n} \alpha_k (x_k, x_l)\alpha_l^* = 0,$$

i. e. G is not a positive definite matrix. Conversely, if the Gram-matrix G of the elements $x_1, \ldots, x_n$ of H is not positive definite, then for a suitable vector $\mathfrak{a}=$

$=(\alpha_1, \ldots, \alpha_n) \neq 0$ of V_n the relation

$$(\mathfrak{a}G, \mathfrak{a}) = \left(\sum_{k=1}^{n} \alpha_k x_k, \sum_{k=1}^{n} \alpha_k x_k \right) = 0$$

holds, so $\alpha_1 x_1 + \ldots + \alpha_n x_n = 0$ and consequently the elements $x_1, \ldots, x_n$ are linearly dependent.

Theorem 2. *Let H be a quasi Hilbert space over a symmetric Banach algebra R, inwich the norm is determined by a positive functional. In the case of any linearly independent system of elements $x_k \in H$ $(k=1, \ldots, n)$ there exists an orthonormal system $e_k \in H$ $(k=1, \ldots, n)$ such that e_m can be linearly combined from $x_1, \ldots, x_m$ and x_m can be linearly combined from $e_1, \ldots, e_m$.*

PROOF. If $(x, x) > 0$ then we denote by $\|x\|$ the positive square root of the element (x, x). Clearly, for the element $e = \|x\|^{-1} x$ the equality $\|e\| = \varepsilon$ holds. Indeed, we have

$$(e, e) = (\|x\|^{-1} x, \|x\|^{-1} x) = \|x\|^{-1} (x, x)(\|x\|^{-1})^* = \|x\|^{-1} (\|x\|^{-1})^* (x, x) = \varepsilon,$$

since $\|x\|^{-1}$ is also a positive element of R.

Let $e_1 = \|x_1\|^{-1} x_1$. In the expression $y_2 = x_2 - \alpha_1 e_1$ we determine the element $\alpha_1 \in R$ so that $(y_2, e_1) = 0$ holds. Hence we get $\alpha_1 = (x_2, e_1)$. Clearly, (y_2, y_2) is a positive element in R. Indeed, in the contrary case there would exist by Theorem 1 an element $\beta \neq 0$ of R, such that $\beta y_2 = \beta x_2 - \beta \alpha_1 \|x_1\|^{-1} x_1 = 0$ thus contradicting the linear independence of the elements x_k. Let $e_2 = \|y_2\|^{-1} y_2$. Suppose our assertion to be true for $m-1 < n$. Let us determine the elements $\alpha_k \in R$ $(k=1, \ldots, m-1)$ so that the element

$$y_m = x_m - \alpha_{m-1} e_{m-1} - \ldots - \alpha_1 e_1 \tag{3}$$

should be orthogonal to the elements e_k $(k=1, \ldots, m-1)$, where by the induction hypothesis $e_1, \ldots, e_{m-1}$ is an orthonormal system. The realtion $(y_m, e_k) = 0$ implies $\alpha_k = (x_m, e_k)$. Similarly as in the case of (y_2, y_2), it can be shown also here that (y_m, y_m) is a positive element of R. Let $e_m = \|y_m\|^{-1} y_m$. Then $(e_m, e_k) = 0$ $(k=1, \ldots, m-1)$ and by (3) x_m can be expressed linearly by the vectors e_j $(j=1, \ldots, m)$. If now, on the basis of our induction hypothesis, we express the vector e_k $(k=1, \ldots, m-1)$ with the vectors x_j $(j=1, \ldots, k)$, and put these expressions into the equality (3) multiplied by $\|y\|^{-1}$, then we obtain that e_m can be expressed by a linear combination of the elements x_k $(k=1, \ldots, n)$.

3. Let H be a quasi Hilbert space over a symmetric Banach algebra R, in which the norm is determined by a positive functional. By the Fourier-series of the element $x \in H$ with respect to the orthonormal system e_k $(k=1, 2, \ldots)$ we understand the series

$$x \sim \sum_{\nu=1}^{\infty} \alpha_\nu e_\nu$$

where $\alpha_\nu = (x, e_\nu)$.

The following Bessel-inequality holds:

$$0 \leqq \sum_{\nu=1}^{n} \alpha_\nu \alpha_\nu^* \leqq (x, x). \tag{4}$$

In fact the equalities

$$\left(\sum_{\nu=1}^{n} \alpha_\nu e_\nu, \sum_{\nu=1}^{n} \alpha_\nu e_\nu\right) = \left(x, \sum_{\nu=1}^{n} \alpha_\nu e_\nu\right) = \left(\sum_{\nu=1}^{n} \alpha_\nu e_\nu, x\right) = \sum_{\nu=1}^{n} \alpha_\nu \alpha_\nu^*$$

imply

$$0 \leqq \left(x - \sum_{\nu=1}^{n} \alpha_\nu e_\nu, x - \sum_{\nu=1}^{n} \alpha_\nu e_\nu\right) = (x, x) - \sum_{\alpha=1}^{n} \alpha_\nu \alpha_\nu^*,$$

and so (4) holds.

If $x, y \in H$ and (y, y) is a positive element of R, then the following inequality of Cauchy holds:

$$(5) \qquad (x, y)(y, y)^{-1}(y, x) \leqq (x, x).$$

Indeed, iquality (5) results if we apply the Bessel-inequality (4) for $n=1$ to the element $e_1 = (y, y)^{-\frac{1}{2}} y$.

We say the system $e_k \in H$ $(k=1, 2, \dots)$ to be a complete orthonormal system, if for any $x \in H$ the Parseval equality

$$(6) \qquad \sum_{\nu=1}^{\infty} \alpha_\nu \alpha_\nu^* = (x, x)$$

holds.

If $e_k \in H$ $(k=1, 2, \dots)$ is a complete orthonormal system, then it is easy to see that the equality

$$(7) \qquad (x, y) = \sum_{\nu=1}^{\infty} \alpha_\alpha \beta_\nu^*$$

holds, where α_ν and β_ν are the Fourier-coefficients of $x \in H$ and of $y \in H$ respectively.

4. Let H be a quasi Hilbert space over a symmetric Banach algebra R in which the norm is determined by a positive functional. We say that the sequence $x_n \in H$ $(n=1, 2, \dots)$ weakly converges to the element $x \in H$, if for any element $y \in H$, for which (y, y) is a positive element of R, the relation $(x_n - x, y) \to 0$ holds.

Theorem 3. *Let R be a symmetric Banach algebra in which the norm is determined by a positive functional. If the sequence of elements x_n $(n=1, 2, \dots)$ of the quasi Hilbert space H over the Banach algebra R converges to the element $x \in H$, then the sequence x_n $(n=1, 2, \dots)$ weakly converges to x.*

Proof. Let first y be an element of H, such that (y, y) is the unit element of R. Then from the Cauchy-inequality (5) the relation

$$0 \leqq (x_n - x, y)(x_n - x, y)^* \leqq (x_n - x, x_n - x)$$

follows. So

$$0 \leqq |(x_n - x, y)|^2 \leqq |x_n - x|^2,$$

and therefore, in view of $x_n \to x$, we have $|(x_n - x, y)|^2 \to 0$, and this can hold only for $(x_n - x, y) \to 0$. If now for the element y of H, which can otherwise be arbitrary, (y, y) is a positive element of R, then the element $(y, y)^{-\frac{1}{2}} y = z$ is such that (z, z) is the unit element of R, and so by

$$|(x_n - x, y)| \leqq |(x_n - x, z)(y, y)^{-\frac{1}{2}}| \leqq |(x_n - x, z)|\, |(y, y)^{-\frac{1}{2}}|$$

we can infer from $x_n \to x$ in view of $(x_n - x, z) \to 0$, that $(x_n - x, y) \to 0$.

(Received December 28, 1966.)