On Taylor series absolutely convergent on the circumference of the circle of convergence II.

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Let

$$f(z)=\sum_{k=0}^{\infty} a_k z^k$$

be regular for $|z|<1$ and substitute z by a conformal mapping of $|w|<1$ onto $|z|<1$ different from a rotation:

$$z=T(w)=e^{i\vartheta}\frac{w-\zeta}{1-w\bar{\zeta}} \qquad (\vartheta \text{ real}, 0<|\zeta|<1),$$

$$f[T(w)]=\sum_{n=0}^{\infty} b_n w^n.$$

L. ALPÁR [1] proved that the absolute convergence of $\sum_{k=0}^{\infty} a_k z^k$ on the periphery is not necessarily inherited by $\sum_{n=0}^{\infty} b_n w^n$. Generalizing his result, in part I. *) we constructed examples with $\sum_{k=0}^{\infty}|a_k|<+\infty$ where $\sum_{n=0}^{\infty}|b_n|$ not simply diverged but did so in a certain sense rapidly. In this paper we are concerned with extra conditions imposed on the original series $\sum_{k=0}^{\infty} a_k z^k$.

Theorem 1. *If* $\sum_{k=0}^{\infty}|a_k|$ *has a convergent majorant series of decreasing terms:*

$$|a_k|\leqq A_k, \quad A_{k+1}\leqq A_k, \quad \sum_{k=0}^{\infty} A_k<+\infty$$

or what is the same thing

$$\sum_{k=0}^{\infty}\max_{l\geqq k}|a_l|<+\infty,$$

then always

$$\sum_{n=0}^{\infty}|b_n|<+\infty.$$

*) *Publ. Math. (Debrecen)* **14** (1967), 63—68.

If the majorant is only a little greater we have, on the contrary,

Theorem 2. *If a series* $\sum_{k=0}^{\infty} A_k$ *is given with*

$$A_{k+1} \leqq A_k, \quad \sum_{k=0}^{\infty} A_k = +\infty$$

then a_k *can be chosen to satisfy the conditions* [1])

$$|a_k| \leqq A_k, \quad \sum_{k=0}^{\infty} |a_k| < +\infty, \quad \sum_{n=0}^{\infty} |b_n| = +\infty.$$

In that follows $c, c_1, c_2, \ldots$ denote positive constants which may possibly depend on quantities we think to be fixed (such as ζ in the expression of $T(w)$ e.t.c.) The notation $O(\ldots)$ will be understood similarly.

PROOF OF THEOREM 1. Putting the Taylor series of $T(w)$ into that of $f(z)$, we find

$$b_n = \sum_{k=0}^{\infty} t_{kn} a_k,$$

t_{kn} being the nth coefficient of the kth power of $T(w)$.

Let $\zeta = re^{i\varphi}$ and fix somehow p and q subject only to the conditions $0 < p < \frac{1-r}{1+r} < \frac{1+r}{1-r} < q$, e.g. by putting $p = \frac{1}{2}\frac{1-r}{1+r}$, $q = 2\frac{1+r}{1-r}$. As a first step we prove

$$\sum_{k \leqq pn} + \sum_{k \geqq qn} |t_{kn}| \leqq c_1 e^{-c_2 n}. \tag{1}$$

We may confine ourselves to the first sum, the treatment of the second being similar.

According to the definition of t_{kn}, by Cauchy's formula

$$t_{kn} = \frac{1}{2\pi i} \int_{|w|=1} \frac{T^k(w)}{w^{n+1}}\, dw.$$

Let $\delta > 0$, to be determined later and $\eta > 0$ sufficiently small depending on δ and let us deform the path of integration into the circle of radius $1 + \eta$ [2]). This is possible if the pole of $T(w)$ lies outside, that is $1 + \eta < \frac{1}{|\zeta|} = \frac{1}{r}$. A simple analysis of the

[1]) Theorem 2 disproves the following conjecture of Mr. SHIELDS: „$\Sigma|a_k| < +\infty$ and $\Sigma k|a_k|^2 < +\infty$ imply $\Sigma|b_n| < +\infty$" since we can put $A_k = \frac{1}{k \log k}$ $(k \geqq 2)$ and then $\sum_2^\infty k|a_k|^2 \leqq \sum_2^\infty k \frac{1}{k^2 \log^2 k} < +\infty$. As we were informed a better example in this direction had been constructed by Messrs. KAHANE and KATZNELSON.

[2]) For the range $k \geqq qn$ we would choose $1 - \eta$.

behaviour of the mapping $T(w)$ tells us that on this circle it takes the largest absolute value at $(1+\eta)e^{i\varphi}$ ($\varphi = \arg\zeta$) namely

$$|T([1+\eta]e^{i\varphi})| = \left|\frac{(1+\eta)e^{i\varphi} - re^{i\varphi}}{1-(1+\eta)e^{i\varphi}re^{-i\varphi}}\right| = \frac{1-r+\eta}{1-r-\eta r} = 1+\eta\frac{1+r}{1-r-\eta r} \leqq e^{\eta\left(\frac{1+r}{1-r}+\delta\right)}$$

while for the constant $|w|$

$$|w| = 1+\eta \geqq e^{\eta(1-\delta)}$$

so that we have

$$|t_{kn}| \leqq \frac{1}{2\pi}\int\limits_{|w|=1+\eta} \frac{|T(w)|^k}{|w|^{n+1}}\,|dw| \leqq e^{k\eta\left(\frac{1+r}{1-r}+\delta\right)} e^{-n\eta(1-\delta)},$$

$$\sum_{k\leqq pn} |t_{kn}| \leqq e^{-n\eta(1-\delta)} \sum_{k\leqq pn} e^{k\eta\left(\frac{1+r}{1-r}+\delta\right)} \leqq e^{-n\eta(1-\delta)} \frac{e^{pn\eta\left(\frac{1+r}{1-r}+\delta\right)}}{1-e^{-\eta\left(\frac{1+r}{1-r}+\delta\right)}} =$$

$$= \frac{1}{1-e^{-\eta\left(\frac{1+r}{1-r}+\delta\right)}}\, e^{-n\eta\left(1-p\frac{1+r}{1-r}-[1+p]\delta\right)} = c_1 e^{-c_2 n}$$

if we first fix $\delta > 0$ with $1-p\dfrac{1+r}{1-r}-[1+p]\delta > 0$ $\left(\text{take notice of } p < \dfrac{1-r}{1+r}\right)$ and then $\eta > 0$ in accordance with δ and $1+\eta < \dfrac{1}{r}$.

Now we deal with b_n on the range $N < n \leqq 2N$. Let us put

$$a_k^* = a_k \quad \text{if} \quad pN < k < 2qN$$

$$a_k^* = 0 \quad \text{otherwise,}$$

$$f^*(z) = \sum_{k=0}^{\infty} a_k^* z^k = \sum_{pN<k<2qN} a_k z^k,$$

$$f^*(T[w]) = \sum_{n=0}^{\infty} b_n^* w^n.$$

We have for $N < n \leqq 2N$ (since $|a_k| \leqq c_3$)

$$|b_n - b_n^*| = \left|\sum_{k=0}^{\infty} t_{kn}(a_k - a_k^*)\right| = \Big|\sum_{k\leqq pN} + \sum_{k\geqq 2qN} t_{kn} a_k\Big| \leqq c_3\Big\{\sum_{k\leqq pN} + \sum_{k\geqq 2qN} |t_{kn}|\Big\} \leqq$$

$$\leqq c_3\Big\{\sum_{k\leqq pn} + \sum_{k\geqq qn} |t_{kn}|\Big\} \leqq c_4 e^{-c_2 n} \leqq c_4 e^{-c_2 N}$$

by what we have just proved. This estimation suggests that b_n can be replaced by b_n^*.

For this latter by Parseval's formula

$$\sum_{n=0}^{\infty}|b_n^*|^2 = \frac{1}{2\pi}\int\limits_{|w|=1}|f^*(T[w])|^2\,|dw| = \frac{1}{2\pi}\int\limits_{|z|=1}|f^*(z)|^2\left|\frac{dw}{dz}\right||dz| \leqq$$

$$\leqq \max_{|z|=1}\left|\frac{dw}{dz}\right|\cdot\frac{1}{2\pi}\int\limits_{|z|=1}|f^*(z)|^2\,|dz| = c_5\sum_{k=0}^{\infty}|a_k^*|^2 = c_5\sum_{pN<k<2qN}|a_k|^2.$$

Using the majorant of a_k and Schwartz's inequality, we get further

$$\sum_{N<n\leqq 2N}|b_n^*| \leqq \sqrt{N\sum_{n=0}^{\infty}|b_n^*|^2} \leqq \sqrt{Nc_5\sum_{pN<k<2qN}|a_k|^2} \leqq \sqrt{Nc_5 2Nq A_{[pN]}^2} = c_6 NA_{[pN]} \leqq$$

$$\leqq c_7\sum_{\frac{1}{2}pN<k\leqq pN}A_k \qquad \left(N \geqq \frac{1}{p}\right).$$

Finally, using this and the bound for $|b_n - b_n^*|$

$$\sum_{N<n\leqq 2N}|b_n| \leqq c_8\{Ne^{-c_2N} + \sum_{\frac{1}{2}pN<k\leqq pN}A_k\}$$

and summing for $N=1, 2, 4, 8, \ldots$

$$\sum_{n=2}^{\infty}|b_n| \leqq c_8\left\{\sum_{l=0}^{\infty}2^l e^{-c_2 2^l} + \left|\sum_{k=1}^{\infty}A_k\right\}\right. < +\infty,$$

q.e.d.

Remark. From the proof we see, choosing $A_k = \max_{l\geqq k}|a_l|$ that

$$\sum_{n=0}^{\infty}|b_n| \leqq c\sum_{k=0}^{\infty}\max_{l\geqq k}|a_l|$$

where c is depending only on the transformation $T(w)$.

Proof of Theorem 2. Let us regard as a test function

$$g_m(z) = \frac{1}{1-(\varrho z)^m} = \sum_{l=0}^{\infty}(\varrho z)^{lm} \stackrel{\text{def}}{=} \sum_{k=0}^{\infty}u_k^{(m)}z^k, \qquad (|\varrho| = 1\,\text{const}, m \geqq 1)$$

and put

$$g_m(T[w]) = \frac{1}{1-[\varrho T(w)]^m} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty}v_n^{(m)}w^n. \tag{2}$$

We have

$$\sum_{k=0}^{N}|u_k^{(m)}| = \frac{N}{m} + O(1)$$

and if we can show that with ϱ suitably chosen and for large m

$$\sum_{n=0}^{N}|v_n^{(m)}|$$

will be much greater than this, then we have good enough functions $g_m(z)$ for which $g_m(T[w])$ behave much worse. Once we are in possession of such functions the actual construction of an example required by Theorem 2 will be a straighforward one.

Actually we prove the following. There exists a ϱ ($|\varrho|=1$) depending only on $T(w)$ such that for every $m \geqq 1$

$$\sum_{\alpha N < n \leqq N} |v_n^{(m)}| \geqq c_9 \frac{N}{\sqrt{m}} \quad \text{whenever} \quad N \geqq N_0(m). \tag{3}$$

$0 \leqq \alpha < 1$ will be fixed in the application and so c_9 is also fixed.

To begin with, let us derive this inequality from

$$\sum_{\alpha N < n \leqq N} |v_n^{(m)}|^2 \geqq c_{10} \frac{N}{m}, \tag{4}$$

$$\sum_{\alpha N < n \leqq N} |v_n^{(m)}|^4 \leqq c_{11} \frac{N}{m^2} \tag{5}$$

$(N \geqq N_0(m))$. In fact, by Hölder's inequality

$$\sum_{\alpha N < n \leqq N} |v_n^{(m)}|^{2/3} |v_n^{(m)}|^{4/3} \leqq \Big(\sum_{\alpha N < n \leqq N} |v_n^{(m)}|\Big)^{2/3} \Big(\sum_{\alpha N < n \leqq N} |v_n^{(m)}|^4\Big)^{1/3},$$

$$\sum_{\alpha N < n \leqq N} |v_n^{(m)}| \geqq \frac{\Big(\sum_{\alpha N < n \leqq N} |v_n^{(m)}|^2\Big)^{3/2}}{\Big(\sum_{\alpha N < n \leqq N} |v_n^{(m)}|^4\Big)^{1/2}} \geqq \frac{\left(c_{10}\frac{N}{m}\right)^{3/2}}{\left(c_{11}\frac{N}{m^2}\right)^{1/2}} = c_9 \frac{N}{\sqrt{m}}.$$

To prove (4) and (5), we represent $v_n^{(m)}$ as a finite sum. Its definition (2) and Cauchy's formula give us

$$v_n^{(m)} = \frac{1}{2\pi i}\int \frac{g_m(T[w])}{w^{n+1}}\,dw = \frac{1}{2\pi i}\int \frac{1}{w^{n+1}\big(1-[\varrho T(w)]^m\big)}\,dw,$$

integrated on a closed curve round the origin in $|w|<1$. Substituting $w=S(z)$ where $S(z)$ is the inverse mapping of $T(w)$:

$$S(z) = e^{-i\vartheta}\frac{z+\zeta e^{i\vartheta}}{1+z\bar{\zeta}e^{-i\vartheta}}, \tag{6}$$

$$v_n^{(m)} = \frac{1}{2\pi i}\int \frac{1}{S^{n+1}(z)[1-(\varrho z)^m]}\,S'(z)\,dz,$$

the integration is over a closed curve round $-\zeta e^{i\vartheta}$ in $|z|<1$. Replacing this path by $|z|=R$, $R\to\infty$ we find that the integral on $|z|=R$ tends to zero so that we have only to collect residues outside the original path. $\dfrac{S'(z)}{S^{n+1}(z)}$ is (for $n \geqq 1$) regular there, while $\frac{1}{1-(\varrho z)^m}$ has poles of the first order at $\varrho^{-1}\eta$ (η denoting mth unit roots) with residues $-\frac{1}{\varrho m \eta^{m-1}} = -\frac{\eta}{m\varrho}$ hence

$$v_n^{(m)} = \frac{1}{m\varrho} \sum_{\eta} \frac{S'(\varrho^{-1}\eta)}{S^{n+1}(\varrho^{-1}\eta)} \eta \qquad (n \geqq 1).$$

Here $S(\varrho^{-1}\eta)$ are numbers of unit absolute value.

Turning to (4) we have

$$\sum_{\alpha N < n \leqq N} |v_n^{(m)}|^2 = \sum_{\alpha N < n \leqq N} \frac{1}{m^2} \sum_{\eta_1, \eta_2} \frac{S'(\varrho^{-1}\eta_1)\bar{S}'(\varrho^{-1}\eta_2)}{S^{n+1}(\varrho^{-1}\eta_1)\bar{S}^{n+1}(\varrho^{-1}\eta_2)} \eta_1 \bar{\eta}_2 = \frac{1}{m^2} \sum_{\eta_1, \eta_2} \sum_{n} .$$

The inner sum is a geometric series with quotient $\frac{1}{S(\varrho^{-1}\eta_1)\bar{S}(\varrho^{-1}\eta_2)} = \frac{S(\varrho^{-1}\eta_2)}{S(\varrho^{-1}\eta_1)}$. This is 1 if $\eta_1 = \eta_2$ and all such terms give

$$\frac{1}{m^2} \sum_{\eta} |S'(\varrho^{-1}\eta)|^2 (N - [\alpha N]) \geqq \frac{N(1-\alpha)}{m^2} \cdot m \cdot \min_{|z|=1} |S'(z)|^2 = c_{13} \frac{N}{m}.$$

Here by conformity $S'(z) \neq 0$, $c_{13} > 0$.

Again by conformity, the numbers $S(\varrho^{-1}\eta)$ are all different hence, on the other hand, the quotient is different from 1 if $\eta_1 \neq \eta_2$, though of unit absolute value and the geometric series is bounded by a quantity independent of N. Summation for such (η_1, η_2) is also independent of N, therefore the remaining terms give a quantity bounded by a constant depending only on m (once ϱ is determined). If N is so large that $c_{13} \frac{N}{m}$ is greater than the double of this bound we get (4) with $c_{10} = \frac{c_{13}}{2}$.

For (5) we have

$$\sum_{\alpha N < n \leqq N} |v_n^{(m)}|^4 =$$

$$= \sum_{\alpha N < n \leqq N} \frac{1}{m^4} \sum_{\eta_1, \eta_2, \eta_3, \eta_4} \frac{S'(\varrho^{-1}\eta_1)S'(\varrho^{-1}\eta_2)\bar{S}'(\varrho^{-1}\eta_3)\bar{S}'(\varrho^{-1}\eta_4)}{[S(\varrho^{-1}\eta_1)S(\varrho^{-1}\eta_2)\bar{S}(\varrho^{-1}\eta_3)\bar{S}(\varrho^{-1}\eta_4)]^{n+1}} \eta_1\eta_2\bar{\eta}_3\bar{\eta}_4 =$$

$$= \frac{1}{m^4} \sum_{\eta_1, \eta_2, \eta_3, \eta_4} \sum_{n} .$$

The inner sum is again a geometric series. Its quotient is certainly 1 if $\eta_1 = \eta_3$, $\eta_2 = \eta_4$ or $\eta_1 = \eta_4$, $\eta_2 = \eta_3$. The contribution of these terms is less than

$$2 \frac{1}{m^4} \sum_{\eta_1, \eta_2} |S'(\varrho^{-1}\eta_1)|^2 |S'(\varrho^{-1}\eta_2)|^2 (N - [\alpha N]) \leqq \frac{2N}{m^4} m^2 \max_{|z|=1} |S'(z)|^4 = c_{14} \frac{N}{m^2}.$$

If we can choose $|\varrho| = 1$ so that the quotient differ from 1 for all the other quadruples $(\eta_1, \eta_2, \eta_3, \eta_4)$ then just as in the case of (4) we get that the contribution of all the other terms does not exceed a bound depending only on m which can be included in the estimation $c_{11} \frac{N}{m^2}$ for large enough N. So we have but to choose ϱ and it is here that we use the fact that $T(w)$ and consequently its inverse $S(z)$ is not a rotation. Suppose that for fixed $\eta_1, \eta_2, \eta_3, \eta_4$ the quotient in question is 1 that is

$$S(\varrho^{-1}\eta_1)S(\varrho^{-1}\eta_2)=S(\varrho^{-1}\eta_3)S(\varrho^{-1}\eta_4).$$

Varying ϱ on the circle $|\varrho|=1$ this can happen only for a finite number of ϱ since otherwise, by the analyticity of $S(z)$ it would follow

$$S(z\eta_1)S(z\eta_2)=S(z\eta_3)S(z\eta_4)$$

everywhere. The left hand side has zeros at $-\zeta e^{i\vartheta}\eta_1^{-1}$ and $-\zeta e^{i\vartheta}\eta_2^{-1}$ and only there (see (6)), the right hand side at $-\zeta e^{i\vartheta}\eta_3^{-1}$, $-\zeta e^{i\vartheta}\eta_4^{-1}$. Since the zeros of the left and right hand side have to coincide and $\zeta\neq 0$ this implies $\eta_1=\eta_3$, $\eta_2=\eta_4$ or $\eta_1=\eta_4$, $\eta_2=\eta_3$ which cases are now excluded. So there are only a finite number of „wrong" ϱ for each quadruplex $(\eta_1, \eta_2, \eta_3, \eta_4)$. But to each m there are only a finite number of quadruples while there are only denumerably many m. This means at most denumerably many „wrong" ϱ and we can choose a ϱ from the remaining part of $|\varrho|=1$ which makes all the quotients in question different from 1.

Now follows the actual construction. We use some notations and the first result from the proof of Theorem 1. Let s be a fixed integer definitely greater than $\frac{q}{p}$.
Successively we can select numbers N_m with the following properties:

N_m is a power of s, $N_{m+1}>N_m$,

$N_m\geqq qN_0(m)$, $N_0(m)$ as defined in (3).

Further, since

$$\sum_{l=0}^{\infty} s^l A_{s^l} \geqq \sum_{l=0}^{\infty}\frac{1}{s-1}\sum_{k=s^l}^{s^{l+1}-1} A_k = \frac{1}{s-1}\sum_{k=1}^{\infty} A_k = +\infty,$$

choosing N_{m+1} in the $(m+1)$th step sufficiently large we can achieve that in addition

$$\sum_{N_m<s^l\leqq N_{m+1}} s^l A_{s^l} \geqq \frac{1}{\sqrt{m}}$$

so that if

$$d_m \sum_{N_m<s^l\leqq N_{m+1}} s^l A_{s^l} = \frac{1}{\sqrt{m}}, \tag{7}$$

then $0<d_m\leqq 1$.

After these preparations we define a_k as follows. Let $k\in(N_m, N_{m+1}]$. The endpoints of this interval are powers of s and possibly the interval is subdivided by other powers of s. Suppose that k belongs to the subinterval $(s^{l-1}, s^l]$. To define a_k for such k we use our test function corresponding to m and the prescribed majorant A_n corresponding to s^l:

$$a_k \overset{\text{def}}{=} d_m A_{s^l} u_k^{(m)}. \tag{8}$$

If for $k\leqq N_1$ we put $a_k=0$, a_k is defined completely and we have to verify the three requirements of our theorem.

First of all $|a_k|\leqq A_k$ is trivial by $0<d_m\leqq 1$, $|u_k^{(m)}|\leqq 1$ (see definition (2) of $u_k^{(m)}$) and $A_{s^l}\leqq A_k$ for $k\leqq s^l$.

Let us turn to $\sum_{k=0}^{\infty}|a_k|<+\infty$.

$$\sum_{s^{l-1}<k\leqq s^l}|a_k| = d_m A_{s^l}\sum_{s^{l-1}<k\leqq s^l}|u_k^{(m)}| \leqq d_m A_{s^l}\frac{s^l}{m}.$$

Here m is defined by $(s^{l-1}, s^l]\subset(N_m, N_{m+1}]$. Keeping m fixed and summing for all such intervals $(s^{l-1}, s^l]$

$$\sum_{N_m<k\leqq N_{m+1}}|a_k| \leqq \frac{d_m}{m}\sum_{N_m<s^l\leqq N_{m+1}} s^l A_{s^l} = \frac{1}{m}\frac{1}{\sqrt{m}} = \frac{1}{m^{3/2}}$$

owing to (7) and summation for m gives finally

$$\sum_{k=0}^{\infty}|a_k| \leqq \sum_{m=1}^{\infty}\frac{1}{m^{3/2}}<+\infty.$$

Let us regard now the transformed series with coefficients

$$b_n = \sum_{k=0}^{\infty} t_{kn}a_k.$$

Recalling (1) we see that similarly as it was used for Theorem 1, in the expression of b_n we can change a_k for $k\leqq pn$ and $k\geqq qn$ anyhow, being careful only that the changed values should not exceed a fixed bound. The error made by this change is namely $O(e^{-c_2 n})$ and denoting the modified b_n by b_n^* it is sufficient to prove that $\sum_{n=0}^{\infty}|b_n^*| = +\infty$.

Let $\frac{s^{l-1}}{p}<n\leqq\frac{s^l}{q}$. This is possible by the choice $s>\frac{q}{p}$ of s. For $s^{l-1}<k\leqq s^l$ a_k was defined by (8) and since for our n $s^{l-1}<pn$, $s^l\geqq qn$ we change a_k in the way said before if we put in place of a_k (8) with the same l and m even for $k\leqq s^{l-1}$ and $k>s^l$. (A bound of the changed values is provided by $|d_m A_{s^l}u_k^{(m)}|\leqq A_0$). This means that we put

$$b_n^* = d_m A_{s^l}\sum_{k=0}^{\infty} t_{kn}u_k^{(m)} = d_m A_{s^l}v_n^{(m)}$$

recalling that the relation between $u_k^{(m)}$ and $v_n^{(m)}$ is the same as between a_k and b_n. Using (3) with $N=\frac{s^l}{q}\left(\geqq\frac{N_m}{q}\geqq N_0(m)\right)$, $\alpha=\frac{q}{ps}$

$$\sum_{\frac{s^{l-1}}{p}<n\leqq\frac{s^l}{q}}|b_n^*| = d_m A_{s^l}\sum_{\frac{s^{l-1}}{p}<n\leqq\frac{s^l}{q}}|v_n^{(m)}| \geqq c_9\, d_m A_{s^l}\frac{s^l}{q\sqrt{m}} = \frac{c_{15}}{\sqrt{m}}\, d_m A_{s^l}s^l.$$

Summing for all l with $(s^{l-1}, s^l]\subset(N_m, N_{m+1}]$ and taking into account that the intervals $\left(\frac{s^{l-1}}{p}, \frac{s^l}{q}\right]$ are disjoint $(p<q)$

$$\sum_{\frac{N_m}{p}<n\leqq\frac{N_{m+1}}{q}}|b_n^*| \geqq c_{15}\frac{1}{\sqrt{m}}\, d_m\sum_{N_m<s^l\leqq N_{m+1}} s^l A_{s^l} = c_{15}\frac{1}{\sqrt{m}}\frac{1}{\sqrt{m}} = \frac{c_{15}}{m}$$

owing to (7). Summing for m

$$\sum_{n=0}^{\infty} |b_n^*| \geqq c_{15} \sum_{m=1}^{\infty} \frac{1}{m} = +\infty,$$

q.e.d.

Reference

[1] ALPÁR, L., Sur certaines transformées des séries de puissance absolument convergentes sur la frontière de leur cercle de convergence, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **7** (1962), 287—316.

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