On polylogarithms*

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1. The polylogarithm function of arbitrary complex order $\nu=\nu'+i\nu''$ is defined for $|z|<1$ by the special Dirichlet series

$$\mathrm{Li}_\nu(z) = \sum_{n=1}^{\infty} z^n/n^\nu. \tag{1.1}$$

Since $|Li_\nu(z)| \leqq \sum_{n=1}^{\infty} |z|^n/n^{\nu'}$, the following bounds are readily derived by using elementary techniques and the Cauchy—Maclaurin integral method [3]:

$$|\mathrm{Li}_\nu(z)| \leqq |z| \left\{1+\Psi\left(1, 2-\nu'; \ln\frac{1}{|z|}\right)\right\} \quad \text{for} \quad |z|<1,\ \nu'>0; \tag{1.2a}$$

$$|\mathrm{Li}_\nu(z)| \leqq |z|/(1-|z|) \quad \text{for} \quad |z|<1,\ \nu'=0; \tag{1.2b}$$

$$|\mathrm{Li}_\nu(z)| \leqq \frac{\Gamma(1-\nu')}{(-\ln|z|)^{1-\nu'}}+(-\nu')^{-\nu'}(-e\ln|z|)^{\nu'} \quad \text{for} \quad |z|<1,\ \nu'<0, \tag{1.2c}$$

where Ψ denotes the second solution of the confluent hypergeometric equation [7].

It is easily shown by differentiating (1.1) that

$$\frac{d\,\mathrm{Li}_\nu(z)}{dz} = \frac{1}{z}\,\mathrm{Li}_{\nu-1}(z). \tag{1.3}$$

By analytic continuation (1.3) will hold over all regions of the complex z-plane where the polylogarithm is defined. Higher derivatives are readily computed from (1.3) as

$$\frac{d^p\,\mathrm{Li}_\nu(z)}{dz^p} = \frac{1}{z^p}\sum_{m=1}^{p} S_p^{(m)}\,\mathrm{Li}_{\nu-m}(z) \tag{1.4}$$

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where the $S_p^{(m)}$ are defined by

$$(1.5a)\qquad S_{p+1}^{(m)} = S_p^{(m-1)} - pS_p^{(m)} \quad \text{for} \quad 0 < m < p,$$

$$(1.5b)\qquad S_p^{(m)} = 1 \quad \text{for} \quad m = p > 0$$

$$(1.5c)\qquad S_p^{(m)} = 0 \quad \text{for} \quad m = 0.$$

The equations (1. 5) serve to generate the Stirling numbers of the first kind as defined and tabulated by MIKSA [12]; it should be noted that there is still no uniformity of definition (or symbolism) for these numbers, and usage varies from tabulation to tabulation [9], [13], [14], [15], [17].

There are two integral representations which should be cited. The first, deriving from APPELL [20], is

$$(1.6)\qquad \mathrm{Li}_\nu(z) = \frac{z}{\Gamma(\nu)} \int_0^\infty \frac{t^{\nu-1}}{e^t - z}\, dt.$$

where z is any complex number not on the cut $(1, \infty)$ and $\nu' > 0$; it is clear from (1. 6) that the polylogarithm function is closely related to both the Fermi—Dirac and Bose—Einstein integrals of mathematical physics, and additional information can be obtained by consulting standard references an these integrals [4], [5]. The second, deriving from BARNES [1], is

$$(1.7)\qquad \mathrm{Li}_\nu(z) = \frac{iz}{2} \int_{\mathscr{C}} \frac{(-z)^s}{(1+s)^\nu \sin \pi s}\, ds$$

where it is assumed that $-\pi < \arg(-z) < \pi$ unless $\nu' > 1$ [1]). The contour $\mathscr{C}$ is shown in Figure 1.; by closing the contour in the right half-plane one obtains (1. 1) and by closing it in the left half-plane one obtains (3. 9).

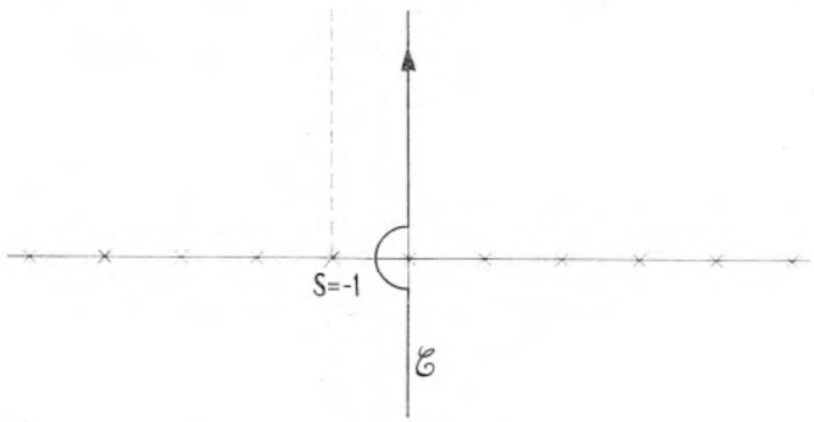

Figure 1

[1]) The convention to be followed in this paper is $-1 = e^{-i\pi}$. When $\nu' > 1$, the restriction on z is $-\pi \leq \arg(-z) \leq \pi$.

2. The factorization theorem

$$N^{1-\nu}\,\mathrm{Li}_\nu(z^N)=\sum_{m=1}^{N}\mathrm{Li}_\nu(\omega_m^{(N)}z). \tag{2.1}$$

where the $\omega_m^{(N)}$ are the N^{th} roots of unity, is well known for real z and positive integer ν [10]. That it also holds for arbitrary complex z and ν will now be shown.

The proof of (2.1) hangs upon the lemma

$$\sum_{m=1}^{N}[\omega_m^{(N)}]^p=0 \quad \text{for} \quad p\not\equiv 0 \pmod N, \tag{2.2a}$$

$$\sum_{m=1}^{N}[\omega_m^{(N)}]^p=N \quad \text{for} \quad p\equiv 0 \pmod N. \tag{2.2b}$$

(2.2b) is obvious. To demonstrate (2.2a) rewrite the sum as $\sum_{m=1}^{N}e^{2\pi i\frac{m}{N}p}$ and let k be the greatest common divisor of $p(=qk)$ and $N(=Mk)$. Then

$$\sum_{m=1}^{Mk}e^{2\pi i\frac{m}{M}q}=k\sum_{m=1}^{M}e^{2\pi i\frac{m}{M}q}.$$

The sum on the right is obviously zero for $q=1$, since in this case it is simply the coefficient of z^{M-1} in $\prod_{m=1}^{M}(z-\omega_m^{(M)})=z^M-1=0$. To show that the sum vanishes for $q\neq 1$ it is sufficient to prove that $\{mq\}\equiv\{m\} \pmod M$. If this last relation is not true, then there must be two integers n_1 and n_2 belonging to $\{m\}$ and such that $n_1q\equiv n_2q \pmod M$. Since q is relatively prime to M, it follows from the theory of congruences [2] that $n_1\equiv n_2 \pmod M$, or, since both n_1 and n_2 are less than M, $n_1=n_2$. Hence, $\{mq\}\equiv\{m\} \pmod M$, and (2.2a) holds. Given (2.2), it is easy to complete the proof of (2.1) by expanding the polylogarithms on the right-hand side of (2.1) using (1.1), interchanging summation, applying (2.2) and once, again using (1.1).

A useful corollary to (2.1) is

$$2^{1-\nu}\,\mathrm{Li}_\nu(z^2)=\mathrm{Li}_\nu(z)+\mathrm{Li}_\nu(-z). \tag{2.3}$$

A more general factorization theorem can be shown by noting that, by (1.3), $dLi_\nu(z^\lambda)/d\lambda=\ln z\, Li_{\nu-1}(z^\lambda)$ or, more generally, $d^mLi_\nu(z^\lambda)/d\lambda^m=\ln^m z\, Li_{\nu-m}(z^\lambda)$. This leads to the Taylor expansion

$$\mathrm{Li}_\nu(z^\lambda)=\sum_{m=0}^{\infty}\frac{(\lambda-\lambda_0)^m}{m!}\ln^m z\,\mathrm{Li}_{\nu-m}(z^{\lambda_0}) \tag{2.4}$$

which will be valid when $Li_\nu(z^\lambda)$ is analytic in and on a circle of radius $(\lambda-\lambda_0)$ about λ_0. As an example of a convergent expansion of the type (2.4), it can easily be shown by real variable techniques that for z, ν, λ, and λ_0 real and for $0<z<1$, $0<\nu<\infty$, $0<\lambda<\infty$, and $0<\lambda_0<\infty$ a sufficient condition for the validity of (2.4) is $|(\lambda-\lambda_0)/\xi|<1$, where ξ lies between λ and λ_0.

3. Since $Li_\nu(z)$ is a function of two complex variables, it is possible to consider expansions of it in either. In particular, one can consider expansions as the variable becomes large or small or expansions about some arbitrary value.

As $z \to 0$, $Li_\nu(z)$ is given by (cf. (1. 1))

$$\text{Li}_\nu(z) = \sum_{n=1}^{N-1} z^n/n^\nu + R_N \qquad (|z|<1), \tag{3. 1}$$

where $R_N = \sum_{n=N}^{\infty} z^n/n^\nu$. Two bounds will be given for $|R_N|$. Proceeding in a manner analogous to that used in deriving (1. 2a), one has

$$|R_N| \leqq |z|^N \sum_{n=0}^{\infty} \frac{|z|^n}{(N+n)^{\nu'}} \leqq |z|^N \int_0^\infty \frac{e^{-\lambda \ln \frac{1}{|z|}}}{(N+\lambda)^{\nu'}}\, d\lambda + \frac{|z|^{P+N}}{(P+N)^{\nu'}},$$

where P is that value of n for which $\frac{|z|^n}{(N+m)^{\nu'}}$ is maximal and $N \geqq 1$; and utilizing an integral representation of the confluent hypergeometric function of the second kind [7], one has

$$|R_N| \leqq \frac{|z|^N}{N^{\nu'}} \left\{ N\Psi\left(1, 2-\nu';\ N \ln \frac{1}{|z|}\right) + \frac{|z|^P}{(1+P/N)^{\nu'}} \right\}. \tag{3. 2}$$

Limits can also be derived in terms of polylogarithms.

For $\nu' > 0$

$$|R_N| \leqq \frac{|z|^N}{N^{\nu'}} \sum_{n=0}^{\infty} \frac{|z|^n}{(1+n/N)^{\nu'}} \leqq \frac{|z|^N}{N^{\nu'}} \left\{ \sum_{n=0}^{N-1} \frac{|z|^n}{1^{\nu'}} + \sum_{n=N}^{2N-1} \frac{|z|^n}{2^{\nu'}} + \dots \right\}$$

or

$$|R_N| \leqq \frac{1}{N^{\nu'}} \frac{1-|z|^N}{1-|z|} \text{Li}_{\nu'}(|z|^N). \qquad (N \geqq 1,\ |z|<1). \tag{3. 3a}$$

For $\nu' = 0$ it is obvious that

$$|R_N| \leqq |z|^N \frac{1}{1-|z|} = |z|^{N-1} \text{Li}_0(|z|). \qquad (N \geqq 1,\ |z|<1) \tag{3. 3b}$$

For $\nu' < 0$ an approach similar to that used for (3. 3a) yields

$$|R_N| \leqq \frac{|z|^{-N}}{N^{\nu'}} \frac{1-|z|^N}{1-|z|} \left[\text{Li}_{\nu'}(|z|^N) - |z|^N\right] \qquad (N \geqq 1,\ |z|<1). \tag{3. 3c}$$

For expansions about some point z_0 one can utilize a Taylor series

$$Li_\nu(z) = Li_\nu(z_0) + (z-z_0)Li_\nu'(z_0) + \dots + R_N,$$

where $R_N = \sum_{n=N}^{\infty} \frac{(z-z_0)^n}{n!} Li_\nu^{(n)}(z_0)$, and the derivatives $Li_\nu^{(n)}(z_0)$ $(n<N)$ can be evaluated by using the series (1. 4). To bound R_N assume that on a circle of radius r about z_0, within and on which $Li_\nu(z)$ is analytic, the maximum modulus of $Li_\nu(z)$ is $M(r)$. It then follows at once from Cauchy's inequality that

$$|R_N| \leqq M(r) \frac{A^N}{1-A}, \tag{3. 4}$$

where $A = \left|\frac{z-z_0}{r}\right|$.

The question of asymptotic expansions is more difficult. Two will be derived here, valid, respectively, for $\nu'>0$ and $\nu'<0$.

For $\nu'>0$ one proceeds from the integral representation (1. 6). Let

$$I_1=\int_0^{\gamma}\frac{t^{\nu-1}}{1-1/ze^{-t}}\,dt,\quad I_2=\int_{\gamma}^{\infty}t^{\nu-1}\frac{ze^{-t}}{1-ze^{-t}}\,dt,$$

where $\gamma=\ln|z|>0$, and z does not lie on the line $(1,\infty)$; obviously, $Li_\nu(z)=$ $=\frac{1}{\Gamma(\nu)}[-I_1+I_2)$. Consider first I_1. Since the geometric series is of bounded convergence in and on the unit circle except at unity itself, it follows that, the denominator of the integral having been expressed as a geometric series, integration and summation can be interchanged to yield

$$I_1=\sum_{n=0}^{\infty}z^{-n}\int_0^{\gamma}t^{\nu-1}e^{nt}\,dt.$$

Integrating by parts,

$$\int t^{\nu-1}e^{nt}\,dt=e^{nt}t^{\nu}\frac{1}{\nu}-\frac{n}{\nu}\int t^{\nu}e^{nt}\,dt=e^{nt}\frac{t^{\nu}}{\nu}\sum_{p=0}^{\infty}(-nt)^p\frac{1}{(\nu+1)_p}=$$

$$=e^{nt}\frac{t^{\nu}}{\nu}\Phi(1,\nu+1;-nt),$$

where $(\omega)_m$ is the Pochhammer symbol $\Gamma(\omega+m)/\Gamma(\omega)$. Hence

$$I_1=\frac{\gamma^{\nu}}{\nu}+\frac{\gamma^{\nu}}{\nu}\sum_{n=1}^{\infty}e^{-in\theta}\,\Phi(1,\nu+1;-n\gamma),$$

where $\theta=\arg z$. The asymptotic expansion of the confluent hypergeometric function is known to be [18]

$$\Phi(1,\nu+1;-n\gamma)=\frac{1}{n\gamma}\frac{\Gamma(\nu+1)}{\Gamma(\nu)}\left\{\sum_{m=0}^{M}(1-\nu)_m(n\gamma)^m+\mathcal{O}\big((n\gamma)^{-M-1}\big)\right\},$$

Hence

$$I_1\sim\frac{\gamma^{\nu}}{\nu}+\gamma^{\nu-1}\sum_{n=1}^{\infty}e^{-in\theta}\frac{1}{n}\sum_{m=0}^{M}(1-\nu)_m(n\gamma)^{-m},$$

where the summation in m can be continued until the series bottoms and the symbol $\sim$ is used to denote asymptotic equality.

Consider next I_2. Here a series expansion of the integral yields

$$I_2=\sum_{n=1}^{\infty}z^n\int_{\gamma}^{\infty}t^{\nu-1}e^{-nt}\,dt.$$

Direct integration by parts then yields the desired asymptotic form

$$I_2\sim\gamma^{\nu-1}\sum_{n=1}^{\infty}e^{in\theta}\frac{1}{n}\sum_{m=0}^{M}(1-\nu)_m(-n\gamma)^{-m}.$$

Hence

(3. 5)
$$\mathrm{Li}_\nu(z) \sim -\frac{\gamma^\nu}{\Gamma(\nu+1)} - \frac{\gamma^{\nu-1}}{\Gamma(\nu)} \sum_{m=0}^{M} (1-\nu)_m \gamma^{-m} \cdot \left(\mathrm{Li}_{m+1}(e^{-i\theta}) + (-1)^{m+1}\,\mathrm{Li}_{m+1}(e^{i\theta})\right),$$

where $0 < \theta < 2\pi$, $\gamma > 1$, and the summation in m can be continued with profit until the series bottoms.

For $\nu' < 0$ one proceeds from the integral representation (1. 7). The contour of integration in the s-plane is $\mathscr{C}$ (cf. Figure 1.) plus an arc from $\infty e^{i\pi/2}$ to $\infty e^{3i\pi/2}$ in the left half-plane, the arc being suitably indented about the branch line emanating

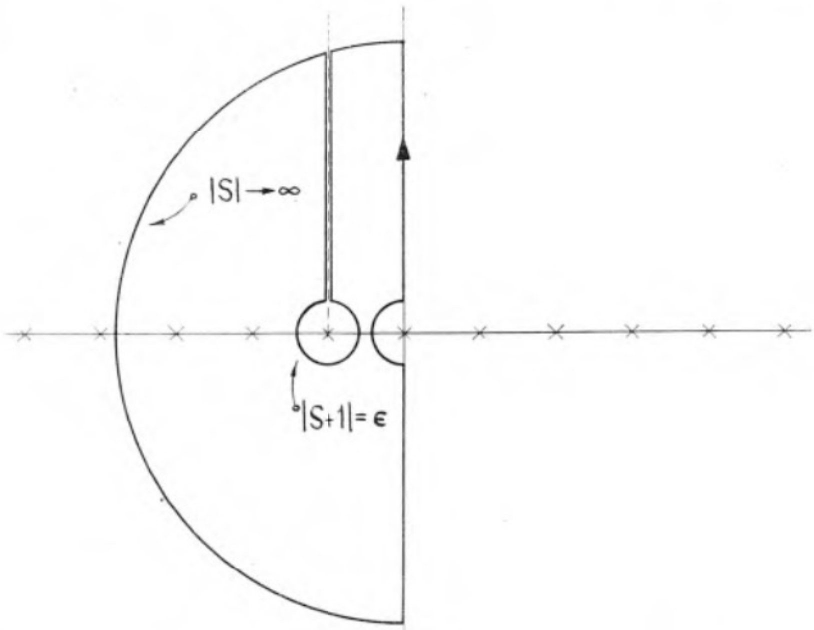

Figure 2

from $s = -1$; this is shown in Figure 2. Since the residue at $s = m$ $(m \neq -1)$ is $-\frac{1}{2\pi i} z^{m+1}(m+1)^{-\nu}$, it follows at once that

$$Li_\nu(z) + e^{i\pi\nu} Li\left(\frac{1}{z}\right) = -(\text{integral around the branch line})$$

$$(|z|) > 1 \text{ or } |z| \geqq 1,\ \nu' > 1).$$

The branch line integral is simply expressible as

$$\frac{i}{2} \frac{e^{i2\pi\nu} - 1}{e^{i\frac{\pi}{2}\nu}} \int_\varepsilon^\infty \frac{(-z)^{i\zeta}}{\zeta^\nu} \frac{d\zeta}{\sinh \pi\zeta} + \int_{\pi/2}^{-3\pi/2} \frac{i}{2} \frac{(-z)^{\varepsilon e^{i\psi}}}{(\varepsilon e^{i\psi})^\nu} \frac{i\varepsilon e^{i\psi}}{\sin(\pi\varepsilon e^{i\psi})}\, d\psi.$$

It is desirable to let $\varepsilon \to 0$, and when this is done it is seen that simple forms obtain only if ν is a positive integer or if $\nu' < 0$. The case of positive integral ν has been discussed elsewhere [1] [7], [10], [19]: the resultant formula for the polylogarithms is

$$\text{(3. 6)} \quad \mathrm{Li}_p(z)+(-1)^p\,\mathrm{Li}_p\left(\frac{1}{z}\right) = -\frac{1}{p!}\ln^p(-z)+2\sum_{r=1}^{[\frac{1}{2}p]}\mathrm{Li}_{2r}(-1)\frac{\ln^{p-2r}(-z)}{(p-2r)!},$$

where $p=2, 3, \ldots$ and $[\frac{1}{2}p]$ denotes the largest integer contained in $\frac{1}{2}p$. The well-known formulae for $\nu=0$ and $\nu=1$, given here for completeness, are

$$\text{(3. 7)} \qquad Li_0(z)+Li_0(1/z) = -1,$$

$$\text{(3. 8)} \qquad Li_1(z)-Li_1(1/z) = -\ln(-z).$$

When $\nu' < 0$ the branch-line integral can be reduced to

$$-e^{i\frac{\pi}{2}\nu}\sin\pi\nu\int_0^\infty(-z)^{i\zeta}\zeta^{-\nu}\frac{d\zeta}{\sinh\pi\zeta}.$$

After the substitution $\zeta=\dfrac{t}{2\pi}$ and a rearrangement of terms, the integral can be seen to represent [7] a generalized zeta function. Hence

$$\text{(3. 9)} \quad \mathrm{Li}_\nu(z)+e^{i\pi\nu}\,\mathrm{Li}_\nu(1/z) = \frac{e^{i\frac{\pi}{2}\nu}(2\pi)^\nu}{\Gamma(\nu)}\,\zeta\left(1-\nu, \frac{1}{2\pi i}\ln z\right) \qquad (\nu'<0,\ 0<\arg z<2\pi);$$

(3. 9) is the well-known JONQUIÈRE relation [7], [8]. The sought after asymptotic expansion for $|z|\to\infty$ and $\nu'<0$ can be obtained from (3. 9) by using the known asymptotic forms [7] for the generalized zeta function and the elementary relations [7] between the BERNOULLI numbers and the polylogarithms of unity argument

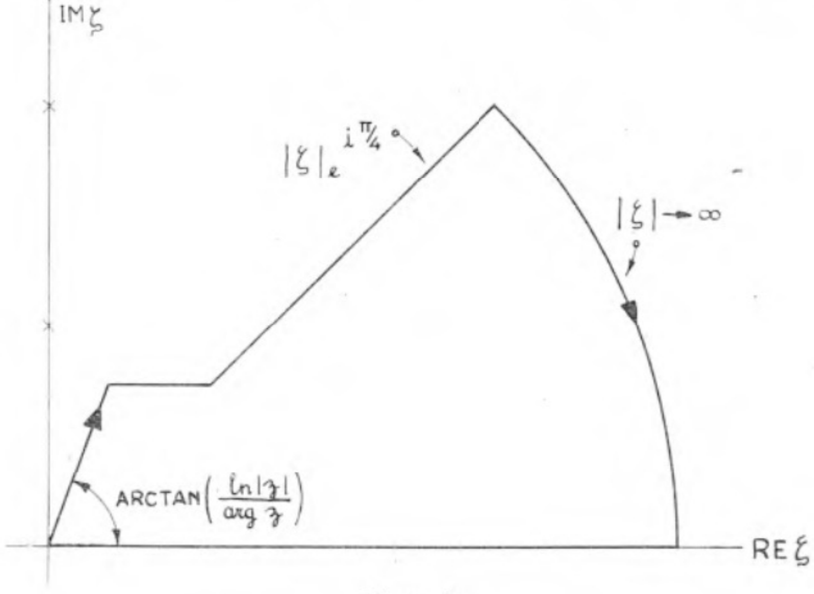

Figure 3

(zeta functions). It can also be obtained directly from the branch-line integral by integrating along a constant phase trajectory in Quadrant I until the integrand is small and then switching onto a ray toward $\infty e^{1/4i\pi}$ in order to avoid the singularity at $\zeta = +i$, this contour is shown in Figure 3. Either method yields

$$\text{(3. 10)} \qquad \mathrm{Li}_\nu(z) + e^{i\pi\nu}\,\mathrm{Li}_\nu(1/z) \sim$$
$$\frac{\sin \pi\nu}{\pi}\, e^{i\pi\nu}(\ln z)^\nu \left\{ \Gamma(-\nu) + \frac{\pi i}{\ln z}\,\Gamma(1-\nu) - 2\sum_{n=1}^{N} \mathrm{Li}_{2n}(1)\,\Gamma(2n-\nu)(\ln z)^{-2n} \right\}$$
$$(\nu' < 0,\ \ 0 < \arg z < 2\pi).$$

$Li_\nu(1/z)$ can often be neglected, or it can be expanded by using (1.1); N can be increased profitably until the series bottoms. An analogous expansion has been derived by Barnes [1] who did not, however, note that the asymptotic series must be terminated because it eventually diverges as $N \to \infty$.

4. An expansion valid as $|\nu| \to 0$ can be obtained from the Maclaurin expansion

$$\mathrm{Li}_\nu(z) = \mathrm{Li}_0(z) + \left.\frac{d\,\mathrm{Li}_\nu(z)}{d\nu}\right|_{\nu=0} \nu + R_2,$$

where R_2 has the integral representation [11]

$$R_2 = \nu^2 \int_0^1 (1-t)\left(d^2 Li_{\nu t}(z)/d(\nu t)^2\right) dt.$$

Since, for $|\nu'| \leqq 1/2$,

$$|d^2\,\mathrm{Li}_\lambda(z)/d\lambda^2| \leqq \sum_{n=1}^{\infty} |z^n \ln^2 n/n^\lambda| \leqq \frac{9}{4}\sum_{n=1}^{\infty} |z|^n n \leqq \frac{9}{4}\,\mathrm{Li}_{-1}(|z|) = \frac{9}{4}\,|z|/(1-|z|)^2,$$

one has $|R_2| \leqq \frac{9}{8}\,|\nu^2|\,\frac{|z|}{(1-|z|)^2}$. To evaluate the first derivative for $|z| < 1$ one proceeds as follows:

$$\left.\frac{d\,\mathrm{Li}_\nu(z)}{d\nu}\right|_{\nu=0} = -\sum_{n=1}^{\infty} z^n \ln n = -\frac{z}{1-z}\sum_{n=1}^{\infty} z^n \ln(1+1/n) =$$
$$= \frac{z}{1-z}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} z^n \left(\frac{-1}{n}\right)^m \frac{1}{m} = \frac{z}{1-z}\sum_{m=1}^{\infty} \frac{(-1)^m}{m}\,\mathrm{Li}_m(z),$$

the interchange of summation having been justified by a simple application of Pringsheim's theorem [3]. Hence

$$\text{(4. 1)} \quad \mathrm{Li}_\nu(z) = \mathrm{Li}_0(z) + \nu\frac{z}{1-z}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\,\mathrm{Li}_m(z) + R_2 \qquad \left(|\nu'| \leqq \frac{1}{2},\ |z| < 1\right).$$

An expansion valid near some arbitrary ν_0 can be derived in a manner exactly analogous to that used for (4. 1):

$$\text{(4. 2)} \qquad \mathrm{Li}_\nu(z) = \mathrm{Li}_{\nu_0}(z) + (\nu-\nu_0)\frac{z}{1-z}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\,\mathrm{Li}_{\nu_0+m}(z) + R_2 \qquad (|z| < 1),$$

where R_2 can be bounded by using Cauchy's inequality in a manner analogous to that employed in deriving (3. 4).

An expansion valid as $\nu' \to \infty$ can be derived from (1. 6), for, expanding $(e^t - z)^{-1}$ in powers of e^{-t}, one obtains

$$\text{(4. 3)} \qquad \mathrm{Li}_\nu(z) = \sum_{n=1}^{N} \frac{z^n}{n^\nu} + \frac{z^{N+1}}{N^\nu \Gamma(\nu)} \int_0^\infty \lambda^{\nu-1} e^{-\lambda} \frac{d\lambda}{e^{\lambda/N} - z} \qquad (\nu' \to \infty).$$

The remainder R_{N+1} is easily bounded:

$$\text{(4. 4)} \qquad |R_{N+1}| \leqq \frac{|z|^{N+1}}{N^{\nu'}} \frac{\Gamma(\nu')}{|\Gamma(\nu)|} \frac{1}{B},$$

where B is the minimum value assumed by $|e^{\lambda/N} - z|$ over the λ-range $(0, \infty)$. It is easy to show by using Euler's integral for the gamma function that $\Gamma(\nu')/|\Gamma(\nu)| \geqq 1$ and by using Stirling's series for the gamma function that $\Gamma(\nu')/|\Gamma(\nu)| \doteq \exp\left[(\nu' - 1/2)(\ln \nu' - \ln|\nu|) + \nu'' \arctan \frac{\nu''}{\nu'}\right] = O\,\{\exp\,[|\nu| f(\varphi)]\}$, where $\varphi = \arg \nu$ and $f(\varphi) = \cos\varphi \ln\cos\varphi + \varphi \sin\varphi$. However, $\pi/2 \geqq f(\varphi) \geqq 0$ when $|\varphi| \leqq \pi/2$, and hence the remainder as given by (4. 4) may be objectionably large for some nonreal ν.

5. It is of course obvious that certain functions are expressible in terms of sums of polylogarithms, and indeed examples of this have been given by Eastham [6], or may be seen in (1. 4), (2. 1), and (2. 4). The problem of interest is to determine the conditions under which such expansions can be made. A theorem and an additional example will be given here.

Theorem. *If $F(z)$ is expressible in the form*

$$F(z) = \sum_{m=1}^{\infty} z^m f(m) \qquad (|z| < R),$$

where $f(m)$ can be expanded in a Laurent series of the form

$$f(m) = \sum_{-\infty}^{\infty} a_p \frac{1}{m^p} \sum_{k=1}^{K} \frac{b_k}{m^{\sigma_k}} \qquad (K \text{ finite}),$$

and if the order of summation can be interchanged, then

$$F(z) = \sum_{k=1}^{K} b_k \sum_{-\infty}^{\infty} a_p \,\mathrm{Li}_{\sigma_k + p}(z).$$

The proof is trivial, the real questions being when the summation interchange can be accomplished and when the $f(m)$ can be suitably expanded. The problem of summation interchange has not been fully resolved, but many cases of interest can be treated by using rules given by Bromwich [3]. $f(m)$ can obviously be expanded as desired if $f(\lambda)$ is analytic except (perhaps) at the origin and at infinity. A simple example of practical importance in hydraulics is provided by the backwater function [16]

$$\mathscr{B}_{\gamma}^{\binom{1}{3}}(z) = \sum_{m=0}^{\infty} (\pm 1)^m \frac{z^{m+\gamma}}{m+\gamma} = z^{\gamma} \left\{ \frac{1}{\gamma} + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{(\pm z)^m}{m} \left(\frac{-\gamma}{m} \right)^n \right\} =$$

$$= z^{\gamma} \left\{ \frac{1}{\gamma} + \sum_{n=0}^{\infty} (-\gamma)^n \operatorname{Li}_{n+1}(\pm z) \right\} \qquad (|\gamma| < 1, \ |z| < 1).$$

where the $(+)$ sign is to the used for $\mathscr{B}_{\gamma}^{(1)}(z)$ and the $(-)$ sign for $\mathscr{B}_{\gamma}^{(3)}(z)$.

It has been shown by EASTHAM [6] that there is no recurrence relation of the form $0 = \sum_{p=0}^{P} A_p(z)\, Li_{M-p}(z)$, where the $A_p(z)$ are algebraic, $A_0(z)$ is not identically zero, M is an integer and $M \geqq 1$, and $P \geqq M$ is allowed. However, when $\nu' < 0$ it is possible to prove the following relation

$$\text{(5. 1)} \qquad \operatorname{Li}_{\nu}(z) = \frac{z}{1-z} \left\{ 1 + \sum_{p=1}^{\infty} a_p(\nu) \operatorname{Li}_{\nu+p}(z) \right\}, \qquad (|z| < 1, \ \nu' < 0),$$

where

$$\text{(5. 2)} \qquad a_p(\nu) = \frac{\Gamma(1-\nu)}{\Gamma(1-\nu-p)} \frac{1}{\Gamma(1+p)}.$$

This follows from (1.1), since

$$\operatorname{Li}_{\nu}(z) = z + z \sum_{n=1}^{\infty} \frac{z^n}{n^{\nu}} \left(1 + \frac{1}{n} \right)^{-\nu} = z + z \sum_{n=1}^{\infty} \frac{z^n}{n^{\nu}} \sum_{p=0}^{\infty} a_p(\nu) \frac{1}{n^p} = z + z \sum_{p=0}^{\infty} a_p(\nu) \operatorname{Li}_{\nu+p}(z),$$

where the absolute convergence of the binomial expansion and of the double series has been ensured by requiring $\nu' < 0$. It is thus possible to express any polylogarithms, function not of positive integer order as a series of other polylogarithms, the coefficients of the series being algebraic. When $\nu = -m$, the series is finite and contains $(m+1)$ terms.

6. In concluding it is appropriate to cite a number of areas in which knowledge of the polylogarithm is especially incomplete.

First, little is known about behavior in the ν-plane except along and near the line $(0, \infty)$. Of interest would be a Maclaurin expansion of the form $\sum_{p=0}^{\infty} \nu^p b_p(z)$, an asymptotic expansion for $\nu' \to -\infty$, and information regarding the functions of purely imaginary order.

Second, integral transforms involving these functions seem not to have been studied.

Finally, there is but little information on the function's zeros. Considered as a function of z, only the zero at the origin is well known; it is, however, easy to demonstrate from (1.1) that as $z \to 0$ the zeros are given by $\nu = \frac{1}{\ln 2}(\ln |z| + i \arg z + i\pi(2p-1))$. Considered as a function of ν, it has, since $Li_{\nu}(1) = \zeta(\nu)$, infinitely many zeros along the line $\nu' = 1/2$. One is tempted to conjecture that all zeros in the complex (ν, z)-plane lie on trajectories that lead to the line $\nu' = \ln |z| / \ln 2$ as $z \to 0$ and the line $\nu' = 1/2$ as $z \to 1$.

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