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On the structure of crossed products of groups and simple rings

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Abstract. Let $K * G$ be a crossed product of the group G over the ring K with a factor set $\rho : G \times G \to U(K)$ and a map $\sigma : G \to \operatorname{Aut} K$, and let $G_{\text{ker}} = \{g \in G \mid g\sigma \text{ is inner}\}$ be the kernel of σ. If for some central subgroup H of G the ring K has no H-invariant ideals, then there exists an one to one correspondence between the H-invariant left K-ideals of $K * G$ and $K * G_{\text{ker}}$. Thus, $K * G$ is a simple ring if and only if $K * G_{\text{ker}}$ is G-simple. If $[G : G_{\text{ker}}] < \infty$, then the ideals of $K * G$ satisfy ACC (resp. DCC) if and only if the ideals of $K * G_{\text{ker}}$ satisfy ACC (resp. DCC). In consequence, necessary and sufficient conditions are given for some classes of crossed products to be simple rings. Simple rings of skew Laurent polynomials in n variables are also studied.

Let $R = (K, G, \rho, \sigma)$ be a crossed product of the multiplicative group G and the associative ring K with a factor set ρ and a mapping σ and let H be the kernel of σ [5]. A well known result of A. A. Bovdi [6] asserts that if K is a simple ring, then the intersection $A_H = A \cap (K, H, \rho, \sigma)$ of every nonzero ideal A of R and the subring $R_H = (K, H, \rho, \sigma)$ is a nonzero G-invariant ideal of R_H. Furthermore, every G-invariant ideal A_H of R_H generates an ideal A of R such that $A \cap R_H = A_H$. If K is a field, then this correspondence is one to one [5, Theorem 3]. These results have some interesting applications when we discuss the structure of R. In this paper we investigate the correspondence between the G-invariant ideals of R_H and the ideals of R in the case when K is not a field.

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§1. Preliminary notions and definitions

The crossed products of arbitrary finite groups over fields were introduced by E. Noether in 1929 in her lectures in Göthingen (see [14]). N. Jacobson [8] extended the notion of crossed products allowing coefficient rings other that fields. Crossed products (K, G, ρ, σ) of arbitrary semigroups G over general rings K with factor sets ρ and mappings σ were introduced by A. A. Bovdi [4]. Here we recall this construction.

Let K be an associative ring with unity and let G be an arbitrary semigroup. Suppose that we are given a single-valued mapping $\sigma : G \to \operatorname{Aut} K$ of G into the group of automorphisms $\operatorname{Aut} K$ and a family $\rho = \{\rho(g,h) \mid g,h \in G\}$ of elements of K such that

$$\rho(f, gh)\rho(g,h) = \rho(fg,h)\rho(f,g)^{h\sigma}, \tag{1.1}$$

$$\alpha^{(gh)\sigma}\rho(g,h) = \rho(g,h)\alpha^{g\sigma.h\sigma} \tag{1.2}$$

for all $f, g, h \in G$ and $\alpha \in K$. Here $\alpha^{g\sigma}$ is the image of $\alpha \in K$ under the action of the automorphism $g\sigma \in \operatorname{Aut} K$. The family ρ is called a factor set of the semigroup G in the ring K with respect to the mapping σ.

We associate to each element $g \in G$ a symbol $\overline{g}$ and consider the free right K-module R, generated by the elements $\overline{g}$ $(g \in G)$. If the factor set ρ is invertible, i.e. $\rho \subset K^* = U(K)$, and the K-basis $\overline{G} = \{\overline{g} \mid g \in G\}$ satisfies the conditions

$$\overline{g}\overline{h} = \overline{gh}\rho(g,h), \quad \alpha\overline{g} = \overline{g}\alpha^{g\sigma} \quad (g,h \in G, \quad \alpha \in K), \tag{1.3}$$

then R is an associative ring, where the product of arbitrary elements of R is defined by using the distributive law and the conditions (1.3). This ring R is called a *crossed product* of G and K with respect to the factor set ρ and the mapping σ, and A. A. Bovdi denotes it by (K, G, ρ, σ) [4, 5, 6]. A number of properties of this ring can be found in [5, 15].

This definition shows how we can construct the ring (K, G, ρ, σ), if K and G are given. But, at times it is necessary to verify that some right free K-module R is a crossed product. It appears that there is no necessity at all to verify that the factor set ρ is invertible and satisfies the conditions (1.1) and (1.2).

Suppose now that R is both left and right free K-module with a basis $\overline{G} = \{\overline{g} \mid g \in G\}$, where G is the set of indices of the elements $\overline{g} \in \overline{G}$. The basis $\overline{G}$ of the free K-module R is said to be a *diagonal basis* if the elements of $\overline{G}$ satisfy the conditions (1.3) for all $g, h \in G$ and $\alpha \in K$, where $\rho(g,h)$

are nonzero elements of K, $\alpha^{g\sigma} \in K$ and $gh \in G$. The diagonal basis $\overline{G}$ of the free K-module R is called *projective* if

$$\overline{f}(\overline{g}\overline{h}) = (\overline{f}\overline{g})\overline{h}, \ \alpha(\overline{g}\overline{h}) = (\alpha\overline{g})\overline{h} \tag{1.4}$$

for all $f, g, h \in G$ and $\alpha \in K$. It is clear that if R is an associative ring containing K, then every diagonal K-basis of R is projective. Furthermore, if the K-module R has at least one projective K-basis, then R is an associative ring.

Obviously, if $\overline{G}$ is a projective basis of the K-module R, then the mapping $(g, h) \mapsto gh$ defines an associative binary operation of G and therefore G is a semigroup. Moreover, the mapping $\alpha \mapsto \alpha^{g\sigma}$ is an automorphism of K and σ maps G into the group of authomorphisms $\operatorname{Aut} K$ of the ring K. Furthermore, from (1.3) and (1.4) we obtain that the factor set $\rho = \{\rho(g, h) \mid g, h \in G\}$ of the basis $\overline{G}$ satisfies the conditions (1.1) and (1.2). Hence it follows that ρ is a factor set of the semigroup G in the ring K with respect to the mapping σ. And so, the mapping $g \mapsto \overline{g}$ is a projective representation of G into the multiplicative semigroup of R (see [8], p. 154), and this explains the name of the projective bases.

These arguments show that we have the following proposition:

Proposition 1.1. *Let K be an associative ring with unity. The K-module R is a crossed product of K and some semigroup G if and only if R is both left and right free K-module with a projective basis $\overline{G}$ and an invertible factor set ρ.*

Thus we obtain

Corollary 1.2. *Let R be an associative ring containing the subring K with unity. Then R is a crossed product of K and some semigroup G if and only if R is both left and right free K-module with a diagonal basis $\overline{G}$ and an invertible factor set ρ.*

Indeed, each diagonal K-basis of R is projective.

For example, let KG be an ordinary group ring and let H be a normal subgroup of G. Then KG is both a left and a right free KH-module with a diagonal basis $T(G/H)$, a transversal of H in G, and an invertible factor set $\rho \subset H$. Thus, as it is well known, KG is a crossed product of G/H and KH. By analogy, (K, G, ρ, σ) is a crossed product of G/H over (K, H, ρ, σ) with a basis $\overline{G/H} = \{\overline{g} \mid g \in T(G/H)\}$.

Throughout in this paper we shall assume that $1 \in K$. Recall that the element $\alpha \in K$ is said to be *regular* if α is neither a left nor a right divisor of zero. The following proposition shows that it is not often necessary to verify that the factor set ρ is invertible.

Proposition 1.3. *Let K be a simple ring (or a direct sum of simple rings). The K-module R is a crossed product of K and some semigroup G if and only if R is both left and right free K-module with projective basis $\overline{G}$ such that each element of the factor set ρ is nonzero (or respectively, regular) element of K.*

Indeed, by Proposition 1.1, it is enough to prove that each factor $\rho(g,h)$ is an invertible element of K. Since $g\sigma, h\sigma \in \operatorname{Aut} K$, the condition (1.2) shows that $\rho(g,h)K = K\rho(g,h)$ for all $\rho(g,h) \in \rho$. Thus, if K is a simple ring, then $\rho(g,h)K = K$ and $\rho(g,h)$ is invertible. Moreover, if K is a direct sum of simple rings, then $\rho(g,h)K$ is also an ideal of K and $\rho(g,h)K = K$, because $\rho(g,h)$ is a regular element. Hence, $\rho(g,h) \in K^*$.

Observe that if in the preceding proposition R is an associative ring containing K, then as in Corollary 1.2 the condition (1.4) for projectivity of G can be replaced by condition (1.3) for diagonality. If $\overline{G}$ is not projective, then R is nonassociative crossed product. Such constructions are studied by Tihomirov [17] and Albert [1].

We shall assume throughout below that G is a group. The crossed product of G and K with a factor set ρ and a mapping σ we shall denote also by $K^\sigma_\rho G$ or simply $K{*}G$. One knows that then $K{*}G$ has an identity element $e = \overline{1}\rho(1,1)^{-1}$ and that each $\overline{g}$ is invertible in $K{*}G$ [5].

After replacing the basis element $\overline{1}$ $(1 \in G)$ by the identity element $e \in K{*}G$ we can assume that the factor set ρ is normalized [15], i.e.

$$\rho(g,1) = \rho(1,g) = \rho(1,1) = 1, \quad \alpha^{1\sigma} = \alpha \quad (g \in G, \alpha \in K).$$

Moreover,

$$\overline{g}^{-1} = \rho\left(g^{-1}, g\right)^{-1} \overline{g^{-1}} = \overline{g^{-1}} \rho\left(g, g^{-1}\right)^{-1} \quad (g \in G).$$

Each element $a \in K{*}G$ is uniquely expressible in the form $a = \sum \overline{g}\alpha_g$ $(g \in G,\ \alpha_g \in K)$, where $\operatorname{Supp} a = \{g \in G \mid \alpha_g \neq 0\}$ is a finite set, called the *support* of a. The subgroup $\langle \operatorname{Supp} a \rangle$, generated by the elements of $\operatorname{Supp} a$ is said to be the *supporting subgroup* of a.

It is worth to mention some special cases of this construction. If we assume that $\rho = 1$, i.e. $\rho(g,h) = 1$ for all $g, h \in G$, then we obtain a skew group ring, denoted by $K^\sigma G$. In this case the mapping $g \mapsto \overline{g}$ is an affine representation (see [8], p. 155) of G into the multiplicative group of units in $K^\sigma G$. If we assume that $\sigma = 1$, i.e. $\alpha^{g\sigma} = \alpha$ for all $g \in G$ and $\alpha \in K$, then we obtain a twisted group ring $K_\rho G$. Here it is clear by (1.2) that each $\rho(g,h)$ must belong to the center of K. Finally, if $\rho = 1$ and $\sigma = 1$, then the mapping $g \mapsto \overline{g}$ is a linear representation of G. In this case we obtain the ordinary group ring which we denote by KG.

If H is a subgroup of G, then the subset S of $K*G$ is said to be H-*invariant* (respectively H-*fixed*) if $\overline{h}^{-1}s\overline{h} \in S$ (respectively $\overline{h}^{-1}s\overline{h} = s$) for all $h \in H$ and $s \in S$. The set of all H-fixed elements of S is denoted by S^H. We say that K is H-*simple* if K has no H-invariant ideals.

Let $\mathrm{Inn}(K)$ be the group of the inner authomorphisms of K. Then $G_{ker} = \{g \in G \mid g\sigma \in \mathrm{Inn}(K)\}$ is a normal subgroup of G and G_{ker} is called a *kernel* of the mapping σ [5]. It is clear that $G_{\mathrm{ker}} \subseteq G_{\mathrm{inn}}$ [15] and if K is a simple ring or a commutative domain, then $G_{\mathrm{ker}} = G_{\mathrm{inn}}$ [15].

If H is any subgroup of G, then $K*H = \{a \in K*G \mid \mathrm{Supp}\, a \subseteq H\}$ is also a crossed product of H over K with $H_{\mathrm{ker}} = H \cap G_{\mathrm{ker}}$.

If L is a ring or a group, then the center of L will be denoted by $C(L)$ and we set $C_W(L) = \{x \in W \mid xr = rx \text{ for each } r \in L\}$. By $\Delta(G)$ we denote the maximal FC-subgroup of G [10]. We recall that a group G is said to be *hypercentral* or ZA-*group* [10] if every nontrivial factor group of G has nontrivial center. One example of a hypercentral group is of course a nilpotent group.

§2. Relations between the ideals of $K*G$ and $K*G_{\mathrm{ker}}$

In this section we discuss the bonds between the ideals of $K*G$ and $K*G_{\mathrm{ker}}$.

Lemma 2.1. *Let $K*G$ be a crossed product of G over K and let K be a H-simple ring for some central subgroup H of G. If the H-invariant K-subbimodule L of $K*G$ contains an element a with $g_0 \in \mathrm{Supp}\, a$, then L contains an element $b = \overline{g}_0 + a_1$ such that $g_0 \notin \mathrm{Supp}\, a_1$ and $\mathrm{Supp}\, b \subseteq \mathrm{Supp}\, a$.*

Proof. Let $a = \overline{g}_0\alpha_0 + \overline{g}_1\alpha_1 + \cdots + \overline{g}_n\alpha_n \in L$ ($\alpha_i \in K$) and $\alpha_0 \neq 0$. We define

$$L_a = \{x \in L \mid \mathrm{Supp}\, x \subseteq \mathrm{Supp}\, a\}\,.$$

Obviously, $a \in L_a$ and L_a is an H-invariant K-subbimodule of L. Now let

$$\theta(L_a) = \left\{ \beta \in K \;\middle|\; \text{there exists } \sum_0^n \overline{g}_i\beta_i \in L \text{ with } \beta_0 = \beta \right\}.$$

It is easy to see that $\theta(L_a)$ is an ideal of K. Moreover, if $\beta_0 \in \theta(L_a)$ and

$$b = \overline{g}_0\beta_0 + \overline{g}_1\beta_1 + \cdots + \overline{g}_n\beta_n \in L_a \quad (\beta_i \in K),$$

then for each element $g \in H$ we have

$$b^{\overline{g}} = \overline{g}^{-1} b \overline{g} = \sum_0^n \overline{g}^{-1} \overline{g}_i \beta_i \overline{g} = \sum_0^n \overline{g}^{-1} \overline{g_i g} \rho(g_i, g) \beta_i^{g\sigma}$$
$$= \sum_0^n \overline{g}_i \rho\left(g, g_i\right)^{-1} \rho(g_i, g) \beta_i^{g\sigma}.$$

Since $b^{\overline{g}} \in L$ and $\operatorname{Supp}(b^{\overline{g}}) \subseteq \operatorname{Supp} a$, we conclude that $b^{\overline{g}} \in L_a$ and $\rho(g, g_i^{-1})\rho(g_i, g)\beta_0^{g\sigma} \in \theta(L_a)$. Thus we obtain that $\beta_0^{g\sigma} \in \theta(L_a)$ and therefore $\theta(L_a)$ is an H-invariant ideal of K. Hence it follows that $\theta(L_a) = K$ and $1 \in \theta(L_a)$. This implies that $L_a \subseteq L$ contains an element $b = \overline{g}_0 + \overline{g}_1\beta_1 + \cdots + \overline{g}_n\beta_n \quad (\beta_i \in K)$, and the lemma is proved.

Lemma 2.2. *Let $K*G$ be a crossed product and let H be a central subgroup of G. If the ring K is H-simple, then every nonzero H-invariant K-subbimodule L of $K*G$ contains an element $a = \overline{g}_1\alpha_1 + \overline{g}_2\alpha_2 + \cdots + \overline{g}_n\alpha_n \quad (\alpha_i \in K)$ such that $g_1, g_2, \dots, g_n \in g_1 G_{\text{ker}}$ and $\alpha_1, \alpha_2, \dots \alpha_n \in K^*$.*

Proof. Suppose that among all nonzero elements of L the element $a = \sum_1^n \overline{g}_i\alpha_i \quad (\alpha \in K)$ has a minimal support size. In view of Lemma 2.1 we may assume that $\alpha_1 = 1$. Then for all $g \in H$ and $\varepsilon \in K^*$ the element

$$x = \varepsilon a - \overline{g}^{-1} a \overline{g} = \sum_1^n \overline{g}_i \left[\varepsilon^{g_i\sigma}\alpha_i - \rho(g, g_i)^{-1}\rho(g_i, g)\alpha_i^{g\sigma}\right]$$

belongs to L. If we take $\varepsilon = \rho(g, g_1)^{-1}\rho(g_1, g)^{(g_1\sigma)^{-1}}$, then we obtain that the coefficient of g_1 of the element x is zero. Hence $|\operatorname{Supp} x| < |\operatorname{Supp} a|$ and the minimality of $\operatorname{Supp} a$ implies $x = 0$, i.e. all coefficients of x are equal to zero. Therefore, there exist elements $\varepsilon_i(g) \in K^*$ such that

$$\alpha_i^{g\sigma} = \varepsilon_i(g)\alpha_i \quad (g \in H; \quad i = 1, 2, \dots, n). \tag{2.1}$$

Moreover, the element

$$y = \gamma a - a\gamma^{g_1\sigma} = \sum_1^n \overline{g}_i \left(\gamma^{g_i\sigma}\alpha_i - \alpha_i\gamma^{g_1\sigma}\right)$$

belongs to L for all $\gamma \in K$ and $|\operatorname{Supp} y| < |\operatorname{Supp} a|$ because $\gamma^{g_1\sigma}\alpha_1 - \alpha_1\gamma^{g_1\sigma} = 0$. Thus $y = 0$ and

$$\gamma^{g_i\sigma}\alpha_i = \alpha_i\gamma^{g_1\sigma} \quad (i = 1, 2, \dots, n). \tag{2.2}$$

Since $g_1\sigma$ and $g_i\sigma$ are automorphisms of K, by (2.2) we conclude that $K\alpha_i$ is a two-sided ideal of K and the condition (2.1) shows that this ideal is

H-invariant. And so, $K\alpha_i = \alpha_i K = K \quad (i = 1, 2, \dots, n)$ and therefore $\alpha_1, \alpha_2, \dots, \alpha_n$ are invertible elements of K. Then by (2.2) and (1.2) we obtain that

$$\gamma^{(g_1^{-1} g_i)\sigma} = \beta_i \gamma \beta_i^{-1} \quad (\beta_i \in K^*)$$

and therefore $g_1^{-1} g_i \in G_{\mathrm{ker}} \;\; (i = 2, 3, \dots, n)$. Hence $g_1, g_2, \dots, g_n \in g_1 G_{\mathrm{ker}}$ and the lemma is proved.

Recall that the left ideal I of $K*G$ is said to be a *left K-ideal* [5] if $a\alpha \in I$ for each $a \in I$ and $\alpha \in K$, i.e. the elements of K act on I as rigth operators.

The next result is a useful consequence of the above lemma.

Proposition 2.3. *Let $K*G$ be a crossed product and let H be a central subgroup of G. If the ring K is H-simple, then for every nonzero H-invariant left K-ideal A of $K*G$ the intersection $A \cap K*G_{\mathrm{ker}}$ is a nonzero H-invariant left K-ideal of $K*G_{\mathrm{ker}}$.*

Proof. If A is a nonzero H-invariant left K-ideal of $K*G$, then by preceding lemma there exists a nonzero element $a \in A$ such that $\operatorname{Supp} a \subseteq gG_{\mathrm{ker}}$ where $g \in \operatorname{Supp} a$. Then $\overline{g}^{-1} a \in A \cap K*G_{\mathrm{ker}}$. Moreover, $A \cap K*G_{\mathrm{ker}}$ is an H-invariant left K-ideal of $K*G_{\mathrm{ker}}$.

Lemma 2.4. *Let $K*G$ be a crossed product of G over K and let K be a H-simple ring for some central subgroup H of G. If A_1 and A_2 are H-invariant left K-ideals of $K*G$ such that $A_1 \subset A_2$, then $A_1 \cap K*G_{\mathrm{ker}} \subset A_2 \cap K*G_{\mathrm{ker}}$.*

Proof. Suppose that $A_1 \cap K*G_{\mathrm{ker}} = A_2 \cap K*G_{\mathrm{ker}}$. Let $a = \sum_1^n \overline{g}_i \alpha_i \quad (\alpha_i \in K)$ be an element with a minimal support size n such that $a \in A_2 \backslash A_1$. Since A_2 is a left ideal of $K*G$ we may assume that $g_1 = 1$. In view of Lemma 2.1, A_2 contains an element

$$b = \overline{g}_1 + \overline{g}_2 \beta_2 + \cdots + \overline{g}_n \beta_n \quad (\beta_i \in K).$$

Suppose that $b \in A_1$. Then $b\alpha_1 \in A_1 \subset A_2$ and $c = a - b\alpha_1 \in A_2$, where the element c has a support size smaller than n. Thus we conclude that $c \in A_1$ and hence $a = c + b\alpha_1 \in A_1$. But this contradicts the condition $a \in A_2 \setminus A_1$. Therefore $b \in A_2 \setminus A_1$ and for the element a we may assume that $g_1 = 1$ and $\alpha_1 = 1$. Now we define

$$\overline{A}_1 = \{x \in A_1 \mid \operatorname{Supp} x \subseteq \operatorname{Supp} a \setminus \{g_1\}\}.$$

Certainly, we think that $0 \in \overline{A}_1$.

Suppose that $\overline{A}_1 = 0$. Since A_2 is a K-bimodule and $a \in A_2$, the element

$$x = \gamma a - a\gamma = \textstyle\sum_1^n \overline{g}_i(\gamma^{g_i\sigma}\alpha_i - \alpha_i\gamma)$$

belongs to A_2 for all $\gamma \in K$. Furthermore x has a support size smaller than n, since the coefficient of $\overline{g}_1$ is zero. Hence $x \in A_1$ and $x \in \overline{A}_1$ for all $\gamma \in K$. But $\overline{A}_1 = 0$ and thus $x = 0$, i.e.

$$\gamma^{g_i\sigma}\alpha_i = \alpha_i\gamma \quad (\gamma \in K, \quad i = 1, 2, \ldots, n). \tag{2.3}$$

Since A_2 is an H-invariant left ideal of $K{*}G$, we have $\overline{g}^{-1}a\overline{g} \in A_2$ for all $g \in H$. Then the element

$$y = a - \overline{g}^{-1}a\overline{g} = \sum \overline{g}_i \left[\alpha_i - \rho(g, g_i)^{-1}\rho(g_i, g)\alpha_i^{g\sigma}\right]$$

belongs to A_2 for all $g \in H$ and $|\operatorname{Supp} y| < n$. Thus we conclude that $y \in \overline{A}_1$ and $y = 0$, i.e.

$$\alpha_i^{g\sigma} = \rho(g_i, g)^{-1}\rho(g, g_i)\alpha_i \quad (g \in H, \quad i = 1, 2, \ldots, n). \tag{2.4}$$

From (2.3) and (2.4) we obtain that $K\alpha_i = \alpha_i K$ is an H-invariant ideal of K for all $i = 2, 3, \ldots, n$. So $a_2, a_3, \ldots, a_n$ are invertible elements of K. Then (2.3) implies that $g_1, g_2, \ldots, g_n \in G_{\mathrm{ker}}$ and

$$a \in A_2 \cap K{*}G_{\mathrm{ker}} = A_1 \cap K{*}G_{\mathrm{ker}},$$

i.e. $a \in A_1$. But this is impossible and therefore $\overline{A}_1 \neq 0$. Let $d = \sum_2^n \overline{g}_i\gamma_i \quad (\gamma_i \in K)$ be a nonzero element of $\overline{A}_1 \subset A_1$ and suppose that $\gamma_2 \neq 0$. In view of Lemma 2.1 we may assume that $\gamma_2 = 1$. Then the conditions $a \in A_2$ and $d \in A_1 \subset A_2$ imply that $z = a - d\alpha_2 \in A_2$ and $|\operatorname{Supp} z| < n$. Therefore $z \in A_1$ and $a = z + d\alpha_2 \in A_1$. But this is also impossible and the lemma is proved.

Theorem 2.5. *Let $K{*}G$ be a crossed product of G over K and let K be an H-simple ring for some central subgroup H of G. Then the mappings*

$$A \xrightarrow{\varphi} A \cap (K * G_{\mathrm{ker}}) \text{ and } B \xrightarrow{\psi} (K{*}G)B$$

set up a one to one correspondence between the H-invariant left K-ideals of $K{*}G$ and the H-invariant left K-ideals of $K{*}G_{\mathrm{ker}}$. Furthermore, under this correspondence, ideals of $K{*}G$ correspond to G-invariant ideals of $K{*}G_{\mathrm{ker}}$.

Proof. If A is a nonzero H-invariant left K-ideal of $K{*}G$, then, by Proposition 2.3, $\varphi(A) = A \cap K{*}G_{\mathrm{ker}}$ is a nonzero H-invariant left K-ideal of $K*G_{\mathrm{ker}}$. On the other hand, if B is a nonzero H-invariant left K-ideal of $K*G_{\mathrm{ker}}$, then

$$\psi(B) = (K*G)B = \sum_{g\in T(G/G_{\mathrm{ker}})} \overline{g}B$$

is a nonzero H-invariant left K-ideal of $K*G$ and

$$(K*G)B \cap K*G_{\mathrm{ker}} = B. \tag{2.5}$$

Hence

$$B \xrightarrow{\psi} (K*G)B \xrightarrow{\varphi} (K*G)B \cap K*G_{\mathrm{ker}} = B,$$

i.e. $\varphi\psi = id$. To see that $\psi\varphi = id$, it is enough to show that

$$A \xrightarrow{\varphi} (A \cap (K*G_{\mathrm{ker}})) \xrightarrow{\psi} (K*G)(A \cap (K*G_{\mathrm{ker}})) = A$$

for each nonzero H-invariant left K-ideal A of $K*G$.

It is clear that $A_1 = (K*G)(A \cap K*G_{\mathrm{ker}}) \subseteq A$. Suppose that $A_1 \subset A$. Then in view of Lemma 2.4 we have $A_1 \cap (K*G_{\mathrm{ker}}) \subset A \cap K*G_{\mathrm{ker}}$. Now applying the equality (2.5) for the left K-ideal $B = A \cap K*G_{\mathrm{ker}}$ we obtain

$$B \supset A_1 \cap K*G_{\mathrm{ker}} = (K*G)B \cap K*G_{\mathrm{ker}} = B.$$

But this is impossible. Therefore, $A_1 = A$ and the equality $\psi\varphi = id$ is proved. Finally we observe that if A is an ideal of $K*G$, then A is simultaneously an H-invariant and G-invariant K-ideal of $K*G$. Thus $A \cap K*G_{\mathrm{ker}}$ is a G-invariant ideal of $K*G_{\mathrm{ker}}$, since $K*G_{\mathrm{ker}}$ is a G-invariant subring of $K*G$. This completes the proof.

The above theorem yields the necessary reduction of some problems from $K*G$ to $K*G_{\mathrm{ker}}$. As will be apparent soon, some results facilitate the further reduction to $C(K)*G_{\mathrm{ker}}$.

First let us recall a well known standard definition. A family of subsets $\{M_i \mid i \in I\}$ in a set M is said to satisfy the *Ascending Chain Condition* (ACC) if in the family does not exist an infinite strictly ascending chain $M_{i_1} \subset M_{i_2} \subset \ldots \subset M_{i_n} \subset \ldots$. The *Descending Chain Condition* (DCC) for a family of subsets of M is defined similarly.

Corollary 2.6. *Let $K{*}G$ be a crossed product of G over K and let K be an H-simple ring for some central subgroup H of G.*

(i) The ideals of $K*G$ satisfy ACC (resp. DCC) if and only if the G-invariant ideals of $K*G_{\text{ker}}$ satisfy ACC (resp. DCC);

(ii) If $[G : G_{\text{ker}}] < \infty$, then the ideals of $K*G$ satisfy ACC (resp. DCC) if and only if the ideals of $K*G_{\text{ker}}$ satisfy ACC (resp. DCC);

(iii) If K is a simple ring, then the left K-ideals of $K{*}G$ satisfy ACC (resp. DCC) if and only if the left K-ideals of $K*G_{\text{ker}}$ satisfy ACC (resp. DCC).

Proof. (i) and (iii) follow immediately from the preceding theorem, as $H = 1$ in (iii). Therefore it is enough to prove (ii). Let A be any ideal of $K{*}G_{\text{ker}}$ and $g \in G$. Then $g = hg_1$, where $h \in G_{\text{ker}}$ and $g_1 \in T(G/G_{\text{ker}})$, a transversal of G_{ker} in G. Hence we obtain $A^{\overline{g}} = \overline{g}^{-1}A\overline{g} = \overline{g}_1{}^{-1}A\overline{g}_1$. It is clear that $A^{\overline{g}}$ is an ideal of $K{*}G_{\text{ker}}$. Thus $\overline{G}$ acts on the set of all ideals of $K*G_{\text{ker}}$ by conjugation as finite group of authomorphisms. Then (ii) follows by the preceding theorem and [7, Corollary 2.1].

The following propositions enable us to enlarge some well known results on the semiprimitiveness of twisted group rings (see [16]) to the case of crossed products. Recall that the ring R is said to be *semiprimitive* if the Jacobson radical $J(R)$ of R is the zero ideal of R [2].

Proposition 2.7. *Let $K{*}G$ be a crossed product of G over K. Then*

(i) $K*G_{\text{ker}}$ is a twisted group ring with a central factor set ρ and $R = C(K)_\rho G_{\text{ker}}$ is a subring of $K_\rho G_{\text{ker}}$;

(ii) If K is a $C(G)$-simple ring, then $A \cap R$ is a nonzero ideal of R for each nonzero ideal A of $K*G$;

(iii) If K is a central simple F-algebra, then $J(K*G) \cap F_\rho G_{\text{ker}} \subseteq J(F_\rho G_{\text{ker}})$. In particular, if $F_\rho G_{\text{ker}}$ is semiprimitive, then $K{*}G$ is semiprimitive too.

Proof. (i). If $g \in G_{\text{ker}}$ then by definition there exists an element $\varepsilon_g \in K^*$ such that $\alpha^{g\sigma} = \varepsilon_g \alpha \varepsilon_g^{-1}$ for all $\alpha \in K$. Now we define $\widetilde{G} = \{\widetilde{g} = \overline{g}\varepsilon_g \mid g \in G_{\text{ker}}\}$. It is clear that $\alpha\widetilde{g} = \widetilde{g}\alpha$ for all $\alpha \in K$ and $\widetilde{g} \in \widetilde{G}$. Therefore, $K{*}G_{\text{ker}}$ is a twisted group ring of G_{ker} over K with K-basis $\widetilde{G}$, i.e. $K*G_{\text{ker}} = K_\rho G_{\text{ker}}$ where ρ is a central system of factors. Hence we conclude that $R = C(K)_\rho G_{\text{ker}}$ is a subring of $K*G_{\text{ker}}$.

(ii). Now suppose that K is a $C(G)$-simple ring and let A be any nonzero ideal of $K{*}G$. In view of the above theorem we obtain that $A_1 =$

$A \cap K_\rho G_{\mathrm{ker}}$ is a nonzero ideal of $K_\rho G_{\mathrm{ker}}$. Let $x = \sum_1^n \widetilde{g}_i \alpha_i \neq 0$ $(\alpha_i \in K)$ be an element of minimal nonzero support size in A_1. Since we can multiply x by any $\widetilde{g} \in \widetilde{G}_{\mathrm{ker}}$ without changing the support size, we may assume that $g_1 = 1$. From Lemma 2.1 we may assume also that $\alpha_1 = 1$. Then A_1 contains the element

$$y = \alpha x - x\alpha = \sum_1^n \widetilde{g}_i(\alpha\alpha_i - \alpha_i\alpha)$$

for all $\alpha \in K$ and $|\operatorname{Supp} y| < |\operatorname{Supp} x|$. This shows that $y = 0$ and $\alpha\alpha_i = \alpha_i\alpha$ for $\alpha \in K$ and $1 = 1, 2, \dots, n$. Hence $\alpha_i \in C(K)$ and $x \in R$, i.e. $A \cap R \neq 0$ and (ii) is proved.

(iii). Next assume that $J(K*G) \neq 0$. In view of (ii), it is enough to show that $I = J(K*G) \cap F_\rho G_{\mathrm{ker}}$ is a quasi-regular ideal of $F_\rho G_{\mathrm{ker}}$. Since K is a linear space over F, as a right F-module F is a direct summand of K. Write $K = F \oplus L$, where L is a suitable right F-submodule of the F-module K. Hence

$$K*G_{\mathrm{ker}} = K_\rho G_{\mathrm{ker}} = F_\rho G_{\mathrm{ker}} \oplus L_\rho G_{\mathrm{ker}}$$

is a direct sum of F-modules, where

$$L_\rho G_{\mathrm{ker}} = \left\{ a = \sum \overline{g}\alpha_g \mid g \in G_{\mathrm{ker}}, \quad \alpha_g \in L \right\}.$$

The proof will be completed if we can show that $r \in I$ implies that $1 + r$ is left-invertible in $F_\rho G_{\mathrm{ker}}$. The element $1 + r$ is invertible in $K*G$ because $r \in J(K*G)$. Moreover, $1 + r \in F_\rho G_{\mathrm{ker}}$ so that $t = (1+r)^{-1} \in K_\rho G_{\mathrm{ker}}$. Let $t = t_0 + t_1$, where $t_0 \in F_\rho G_{\mathrm{ker}}$ and $t_1 \in L_\rho G_{\mathrm{ker}}$. Then

$$1 = t(1+r) = t_0(1+r) + t_1(1+r).$$

Since $1, \quad t_0(1+r) \in F_\rho G_{\mathrm{ker}}$ and $t_1(1+r) \in L_\rho G_{\mathrm{ker}}$, this implies that $1 = t_0(1+r)$, as desired.

If $K*G$ is a skew group ring, then we have the following proposition.

Proposition 2.8. *Let $K^\sigma G$ be a skew group ring of G over the commutative $C(G)$-simple ring K. Then*

(i) $F = K^{C(G)}$ is a field and $A \cap FG_{\mathrm{ker}}$ is a nonzero ideal of the group ring FG_{ker} for each nonzero ideal A of $K^\sigma G$;

(ii) $J(K^\sigma G) \cap FG_{\mathrm{ker}} \subseteq J(FG_{\mathrm{ker}})$. In particular, if FG_{ker} is semiprimitive, then $K^\sigma G$ is semiprimitive too.

Proof. (i). Obviously, F is a subring of K. If $\alpha \in F$ is a nonzero element, then αK is a nonzero $C(G)$-invariant ideal of K and therefore

$\alpha K = K$. This yields $\alpha \in K^*$ and $\alpha^{-1} \in F$. Thus F is a subfield of K. Let $x = \sum_1^n \overline{g}_i \alpha_i \neq 0$ $(\alpha_i \in K)$ be an element of minimal nonzero support size in the nonzero ideal A of $K^\sigma G$. Obviously, we may assume that $g_1 = 1$ and $\alpha_1 = 1$. Then A contains the elements $y = hx - xh$ and $z = \alpha x - x\alpha$ for all $\alpha \in K$ and $h \in C(G)$. Moreover, $|\operatorname{Supp} y| < |\operatorname{Supp} x|$ and $|\operatorname{Supp} z| < |\operatorname{Supp} x|$. Thus we obtain that $y = z = 0$ and therefore $\alpha_i^{h\sigma} = \alpha_i$ and $\alpha^{g_i\sigma}\alpha_i = \alpha_i\alpha$ $(\alpha \in K;\ \ h \in C(G);\ \ i = 1, 2, \ldots, n)$. Hence we conclude that $\alpha_i \in F$, $\alpha^{g_i\sigma} = \alpha_i\alpha\alpha_i^{-1}$ and $g_i \in G_{\mathrm{ker}}$ $(i = 1, 2, \ldots, n)$. This shows that $x \in F^\sigma G \cap F^\sigma G_{\mathrm{ker}}$ and (i) is proved.

The part (ii) may be proved as (iii) in the preceding proposition.

§3. Simple crossed products of groups and rings

In this final section we study crossed products $K*G$ which are simple rings, i.e. $K*G$ contains no proper ideals. From Theorem 2.5 we obtain immediately the following theorem.

Theorem 3.1. *Let $K*G$ be a crossed product of the group G over the $C(G)$-simple ring K. Then $K*G$ is simple if and only if $K*G_{\mathrm{ker}}$ is G-simple.*

Observe that this theorem is proved in [5] and [6] when K is a skew field or a simple ring respectively.

It is clear that if A is a G-invariant ideal of K, then $(K*G)A$ is an ideal of $K*G$. Therefore, if $K*G$ is simple, then K is G-simple. In a different way, in [3, Proposition 5.7] it is proved a result, which is analogical to the following proposition. In [3] G is an Abelian or a torsion free ZA-group [10] and K is a commutative G-simple ring. Here K is G-simple, but G is arbitrary.

Proposition 3.2. *Let $K^\sigma G$ be a skew group ring of G over a commutative $C(G)$-simple ring K. Then $K^\sigma G$ is a simple ring if and only if $G_{\mathrm{ker}} = 1$.*

Proof. If $G_{\mathrm{ker}} = 1$, then $K^\sigma G_{\mathrm{ker}} = K$ contains no G-invariant ideals, because K is $C(G)$-simple. In view of the preceding theorem we conclude that $K^\sigma G$ is a simple ring. Conversely, let $K^\sigma G$ be a simple skew group ring and assume that $G_{\mathrm{ker}} \neq 1$. Then $K^\sigma G_{\mathrm{ker}} = KG_{\mathrm{ker}}$ is a group ring which contains the proper G-invariant ideal $\omega(G_{\mathrm{ker}})$ generated by the elements $\{h - 1 \mid h \in G_{\mathrm{ker}}\}$. Indeed, if $a \in \omega(G_{\mathrm{ker}})$ then

$$a = \sum_{h \in G_{\mathrm{ker}}} \alpha_h(h - 1) \quad (\alpha_h \in K).$$

Since G_{ker} is a normal subgroup of G,

$$g^{-1}ag = \sum_{h \in G_{\mathrm{ker}}} \alpha_h^{g\sigma}(g^{-1}hg - 1)$$

is an element of $\omega(G_{\mathrm{ker}})$ for each $g \in G$. But this contradicts the preceding theorem and the result follows.

Corollary 3.3. *Let $K^\sigma G$ be a simple skew group ring of an Abelian group G over a commutative ring K. Then $K^\sigma G$ is simple if and only if K is G-simple and $G_{\mathrm{ker}} = 1$.*

Proof. The necessity follows from Theorem 3.1 and the sufficiency follows from Proposition 3.2.

Proposition 3.2 is incorrect for arbitrary crossed products. For example, let $K*G$ be a field (see [16]). Then $G_{\mathrm{ker}} = G$, but $K*G$ is simple.

The next assertion is announced in [12, Theorem 7] for torsion free Abelian groups. For torsion free ZA-groups it is proved in [3, Theorem 5.6].

Lemma 3.4. *Let $K*G$ be a crossed product of a torsion free ZA-group G over a ring K. Then $K*G$ is simple if and only if K is G-simple and there is not a nonidentity central element $h \in H$ such that*

$$\alpha^{h\sigma} = \varepsilon_h \alpha \varepsilon_h^{-1}, \quad \varepsilon_h^{g\sigma} = \rho(h,g)^{-1}\rho(g,h)\varepsilon_h$$

for all $\alpha \in K$ and $g \in G$, where ε_h is an invertible element of K.

The proof of this lemma is based on the fact that each ideal of $K*G$ contains a nonzero central element of $K*G$ (see [13, Corollary 2.2]). Moreover, if an element $h \in G$ satisfies the conditions of the lemma, then the element $a = 1 + h\varepsilon_h$ is central, but it is not invertible in $K*G$.

Recall that the factor set ρ of the crossed product $K*G$ is *symmetrical* [5], if $gh = hg$ yields $\rho(g,h) = \rho(h,g)$ for all $g,h \in G$. Then from Lemma 3.4 we obtain immediately the following corollary.

Corollary 3.5. *Let $K*G$ be a crossed product of the torsion free ZA-group G over the commutative ring K with a symmetric factor set ρ. Then $K*G$ is a simple ring if and only if K is G-simple and $G_{\mathrm{ker}} = 1$.*

Observe that the preceding corollary takes place also in the case when K is not commutative, but the factor group $K^*/C(K)^*$ is torsion and $C(K)$ contains the factor set ρ. Indeed, if $1 \neq h \in G_{\mathrm{ker}}$ and $\alpha^{h\sigma} = \varepsilon_h \alpha \varepsilon_h^{-1}$ $(\alpha \in K, \ \varepsilon_h \in K^*)$, then there exists an integer n such that $\varepsilon_h^n \in C(K)$. Thus $\alpha^{h^n\sigma} = \alpha$ $(\alpha \in K)$ and the elements $h^n \in G$ and $\varepsilon_{h^n} = 1$ satisfy the condition of Lemma 3.4. The rest of the proof is clear.

If G is a torsion group, then the field of complex numbers shows that Corollary 3.5 is incorrect.

Now we will prove the following theorem.

Theorem 3.6. *Let $K*G$ be a crossed product of a finitely generated torsion free Abelian group G over an arbitrary ring K with a symmetric factor set ρ. Then $K*G$ is not simple if and only if either*

(i) K is not G-simple, or

(ii) G has a basis $g_1, g_2, \dots, g_n$ such that $\alpha^{g_1^r\sigma} = \varepsilon\alpha\varepsilon^{-1}$ $(\varepsilon \in K^*)$ for some natural number r and every $\alpha \in K$, and $\varepsilon^{g_i\sigma} = \varepsilon$ for $i = 2, 3, \dots, n$.

Proof. First assume that $K*G$ is not simple, but K is G-simple. In view of the above lemma there exists an element $h \in G$ such that

$$\alpha^{h\sigma} = \varepsilon\alpha\varepsilon^{-1}, \quad \varepsilon^{g\sigma} = \varepsilon \quad (\varepsilon \in K^*)$$

for all $\alpha \in K$ and $g \in G$, because the factor set ρ is symmetrical. Let $H = \langle h \rangle$ be the cyclic subgroup of G, generated by h. Then there exists a basis $g_1, g_2, \dots, g_n$ of G such that $H = \langle g_1^r \rangle$ for some natural number r [10, p. 120]. Thus $h = g_1^r$ or $h = (g_1^{-1})^r$ and the condition (ii) follows.

Conversely, it is clear that if K is not G-simple, then $K*G$ is not simple. Assume that K is G-simple, but G satisfies the condition (ii). Via a change of the basis $\overline{G} = \left\{\overline{g_1^{r_1} g_2^{r_2} \dots g_n^{r_n}} \mid r_i \in Z\right\}$ with the basis $\widetilde{G} = \{\overline{g}_1^{r_1}\overline{g}_2^{r_2} \dots \overline{g}_n^{r_n} \mid r_i \in Z\}$, there is really no loss of generality in assuming that $K*G$ is a skew group ring $K^\sigma G$. This is possible since G is a torsion free and ρ is a summetric factor set. We will prove that $K^\sigma G$ is not simple using the method of Jordan [9]. Indeed, if K is commutative, then the assertion follows from Corollary 3.3. If K is a noncommutative ring, then we define $\beta = \varepsilon\varepsilon^{g_1\sigma}\varepsilon^{g_1^2\sigma} \dots \varepsilon^{g_1^{r-1}\sigma}$. Since $\varepsilon^{g_1^r\sigma} = \varepsilon$ we conclude that $\varepsilon^{g_1^{r+l}\sigma} = \varepsilon^{g_1^l\sigma}$ for $l = 1, 2, \dots, r-1$ and

$$\begin{aligned}\beta^{g_1\sigma} &= \varepsilon^{g_1\sigma}\varepsilon^{g_1^2\sigma} \dots \varepsilon^{g_1^r\sigma} = \left(\varepsilon^{g_1\sigma}\varepsilon^{g_1^2\sigma} \dots \varepsilon^{g_1^{r-1}\sigma}\right)^{g_1^r\sigma} \varepsilon \\ &= \varepsilon\left(\varepsilon^{g_1\sigma}\varepsilon^{g_1^2\sigma} \dots \varepsilon^{g_1^{r-1}\sigma}\right) = \beta.\end{aligned}$$

Furthermore, the conditions $g_1^s\sigma.g_i\sigma = g_i\sigma g_1^s\sigma$ and $\varepsilon^{g_i\sigma} = \varepsilon$ yield $(\varepsilon^{g_1^s})^{g_i\sigma} = \varepsilon^{g_1^s\sigma}$ for $s = 1, 2, \dots, r-1$ and $i = 2, 3, \dots, n$. Thus we obtain that β is a G-fixed invertible element of K. Moreover,

$$\begin{aligned}\beta\alpha\beta^{-1} &= \varepsilon\varepsilon^{g_1\sigma} \dots \varepsilon^{g_1^{r-1}\sigma}\alpha\left(\varepsilon\varepsilon^{g_1\sigma} \dots \varepsilon^{g_1^{r-1}\sigma}\right)^{-1} \\ &= \left[\varepsilon^{g_1\sigma} \dots \varepsilon^{g_1^{r-1}\sigma}\alpha\left(\varepsilon^{g_1\sigma} \dots \varepsilon^{g_1^{r-1}\sigma}\right)^{-1}\right]^{g_1^r\sigma}\end{aligned}$$

$$= \left[\varepsilon\varepsilon^{g_1\sigma}\ldots\varepsilon^{g_1^{r-2}\sigma}\alpha^{g_1^{-1}\sigma}(\varepsilon\varepsilon^{g_1\sigma}\ldots\varepsilon^{g_1^{r-2}\sigma})^{-1}\right]^{g_1^{r+1}\sigma}$$

$$= \left[\varepsilon^{g_1\sigma}\varepsilon^{g_1^2\sigma}\ldots\varepsilon^{g_1^{r-2}\sigma}\alpha^{g_1^{-1}\sigma}\left(\varepsilon^{g_1\sigma}\varepsilon^{g_1^2\sigma}\ldots\varepsilon^{g_1^{r-2}\sigma}\right)^{-1}\right]^{g_1^{2r+1}\sigma}.$$

Thus we conclude that

$$\beta\alpha\beta^{-1} = \left[\varepsilon^{g_1\sigma}\varepsilon^{g_1^2\sigma}\ldots\varepsilon^{g_1^{r-s}\sigma}\alpha^{g_1^{1-s}\sigma}\left(\varepsilon^{g_1\sigma}\varepsilon^{g_1^2\sigma}\ldots\varepsilon^{g_1^{r-s}\sigma}\right)^{-1}\right]^{g_1^{rs+s-1}\sigma}$$

for all $s = 1, 2, \ldots, r$. In particular, if $s = r$ then

$$\beta\alpha\beta^{-1} = \left(\alpha^{g_1^{1-r}\sigma}\right)^{g_1^{r^2+r-1}\sigma} = \alpha^{g_1^{r^2}\sigma}.$$

Therefore, the element $h = g_1^{r^2} \in G$ satisfies the conditions of Lemma 3.4 with $\varepsilon_h = \beta$ and the proof is completed.

From here we obtain the following corollary.

Corollary 3.7. *The crossed product $K{*}G$ of the infinite cyclic group G over the ring K is a simple ring if and only if K is G-simple and $G_{\ker} = 1$.*

Proof. If G is a cyclic group, then it is easy to see that the factor set ρ of $K{*}G$ is symmetrical. Then the assertion follows from Theorem 3.6 with $n = 1$.

It is clear that the ring $K[x, x^{-1}; \sigma]$ of skew Laurent polynomials is a skew group ring of the infinite cyclic group over K. Hence, the main result of [9] (see also [11, Theorem 3.18]) follows from the above corollary.

Obviously, the ring $K[x_i, x_i^{-1}; \sigma_i \mid i = 1, 2, \ldots, n]$ of skew Laurent polynomials of $x_1, x_2, \ldots, x_n$ is a skew group ring of the torsion free Abelian group $G = \langle x_1\rangle \times \langle x_2\rangle \times \ldots \times \langle x_n\rangle$ over K, where $\alpha x_i^k = x_i^k \alpha_i^{\sigma_i^k}$ $(\alpha \in K)$, i.e. $x_i^k\sigma = \sigma_i^k$ $(1 \le i \le n,\ \ k \in Z)$. Then with the help of Lemma 3.4. we obtain the following result, which is well known when K is commutative (see [18]).

Proposition 3.8. *The ring $K[x_i, x_i^{-1}; \sigma_i \mid i = 1, 2, \ldots, n]$ of skew Laurent polynomials of $x_1, x_2, \ldots, x_n$ over K is simple if and only if there exists no nonzero system of integers $m_1, m_2, \ldots, m_n$ and a nonzero ideal A of K such that $\alpha^{\sigma_1^{m_1}\sigma_2^{m_2}\ldots\sigma_n^{m_n}} = \varepsilon\alpha\varepsilon^{-1}$, $\varepsilon^{\sigma_i} = \varepsilon$ and $A^{\sigma_i} = A$ $(1 \le i \le n)$ for some $\varepsilon \in K^*$ and all $\alpha \in K$.*

If K is commutative, then it is clear that in the above proposition we have $\varepsilon = 1$.

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