On the regularity of the locally integrable solutions of the functional equations

$$\sum_i a_i(x, t)f(x+\varphi_i(t)) = 0$$

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The equations

$$(1) \qquad \sum_{i=1}^{k} a_i(x, t)f(x+\varphi_i(t)) = 0,$$

where $x=(x_1, \ldots, x_n)$, $\varphi_i(t)=(\varphi_{i1}(t), \ldots, \varphi_{in}(t))$, $n>1$ are a generalization of some known functional equations (cf. e.g.[2]).

General theorems on the regularity of the integrable solutions of similar equations with $n=1$ are known (cf. ACZÉL [1]).

We shall prove theorems on the regularity of the locally integrable solutions of equations (1) under some general assumptions on the functions $a_i(x, t)$ and $\varphi_i(t)$ $(i=1, \ldots, k)$. The idea of proof is the following: to show that every locally integrable solution of equation (1) satisfies some differential equation in the distributional sense and that this differential equation is hypoelliptic i. e. every distributional solution of it is a function of class C^∞.

The theory of distributions was already used in connection with functional equations by FENYŐ. In [5] and [7] he solved distributional equations adequate to some known functional equations in which the unknown function is dependent on one variable. In [6] he solved the distributional equation adequate to the functional equation

$$f(x_1+x_2, x_3)+f(x_1, x_2) = f(x_1, x_2+x_3)+f(x_2, x_3)$$

which was solved by HOSSZÚ ([3]) whithout any regularity assumptions. The most general solution of this functional equation has a form different from the most general distributional solution of the adequate distributional equation which was considered by Fenyő. This is not strange since not all the functions can be identified with distributions. Distributional methods can be used only to find locally integrable solutions of the functional equations.

We are not looking for the locally integrable solutions of special functional equations but for the classes of functional equations which can easily be solved in the class of locally integrable functions in view of the regularity of such solutions.

Basic definitions and theorems which we shall use can be found e. g. in [9], the notation will be the same as in [8].

Notation and definitions

$x=(x_1, \ldots, x_n)\in R^n$, $y=(y_1, \ldots, y_n)\in R^n$, $0=(0, \ldots, 0)\in R^n$, $p=(p_1, \ldots, p_n)$, where p_i are integers $\geqq 0$,

$$|p| = p_1 + \ldots + p_n,$$

$$D^p = D^p_x = \frac{\partial^{|p|}}{\partial x_1^{p_1} \ldots \partial x_n^{p_n}},$$

$\mathscr{D}=\{\psi: \psi \in C^\infty \text{ in } R^n, \text{ supp } \psi \text{ is compact}\}$.

Definition 1. We say that $\psi_\nu \to 0$ in $\mathscr{D}$ if there exists a compact set $E\subset R^n$ such that $\operatorname{supp}\psi_\nu \subset E$ and $D^p\psi_\nu \to 0$ uniformly for every p.

Definition 2. We say that the operator

$$T: \psi \to (T, \psi), \quad \text{where} \quad \psi\in\mathscr{D}, (T, \psi)\in R$$

is a distribution if it is

1° linear,

2° continuous in the following sense:

If $\psi_\nu \to 0$ in $\mathscr{D}$, $(T, \psi_\nu)\to 0$ in R.

The set of all the distributions in R^n will be denoted by $\mathscr{D}'$. (T, ψ) will be denoted also by $(T(x), \psi(x))_x$. By $\Psi_t(x)$ we shall denote the function $\psi(x, t)$ with a flxed $t\in\Delta\subset R$.

$$\gamma(t)=\big(T(x), \psi(x, t)\big)_x=(T, \Psi_t),$$

$$\Phi(t)=\big(\Phi_1(t), \ldots, \Phi_n(t)\big),$$

$$\varphi_i(t)=\big(\varphi_{i1}(t), \ldots, \varphi_{in}(t)\big) \qquad (i=1, \ldots, n).$$

Definition 3. We say that $\Phi(t)\in C^m$ (or $\varphi_i(t)\in C^m$) in R if $\Phi_j(t)\in C^m$ (or $\varphi_{ij}(t)\in C^m$) for $j=1, \ldots, n$.

Basic theorems

I If $T\in\mathscr{D}'$, then the equality

$$(D^pT, \psi)=(-1)^{|p|}\,(T, D^p\psi) \quad \text{for each} \quad \psi\in\mathscr{D}$$

defines the distribution D^pT.

II If $T\in\mathscr{D}', a\in C^\infty$ in R^n, then the equality

$$(aT, \psi)=(T, a\psi) \quad \text{for each} \quad \psi\in\mathscr{D}$$

defines the distribution aT.

III If $T\in\mathscr{D}'$ $a\in C^\infty$ in R^n, then

$$\frac{\partial}{\partial x_j}(aT) = a\frac{\partial}{\partial x_j}T+\frac{\partial}{\partial x_j}a T.$$

IV If $T\in\mathscr{D}'$ and the mapping $y\to x(y)$ is a diffeomorphism, then

$$\big(T(y(x)), \psi(x)\big)_x = \left(T(y), \psi\big(x(y)\big)\left|\det\frac{\partial x_\nu}{\partial y_\mu}\right|\right)_y.$$

V If $T \in \mathscr{D}'$ $\psi(x,t) \in C^m$ in $R^n \times \Delta$, where Δ is an open interval in R, $\Psi_t(x) \in C^\infty$ in R^n for each $t \in \Delta$, and there exists a compact set $E \subset R^n$ such that

$$\operatorname{supp} \Psi_t \subset E \quad \text{for each} \quad t \in \Delta,$$

then the function

$$\gamma(t) = \big(T(x), \psi(x, t)\big)_x$$

is of the class C^m in Δ and

$$\gamma^{(l)}(t) = \left(T(x), \frac{\partial^l}{\partial t^l} \psi(x, t)\right)_x$$

for $l = 1, \ldots, m$.

Now, we can prove the following

Lemma. *If $T \in \mathscr{D}'$, $a(x, t) \in C^\infty$ in R^n for every fixed t from an open interval $\Delta \subset R$, $a(x, t) \in C^m$ in $R^n \times \Delta$, $\Phi \in C^m$ in Δ, there exists $\alpha \in \Delta$ such that $\Phi(\alpha) = 0$, and $\psi \in \mathscr{D}$, then the function*

$$\gamma(t) = \big(a(x, t)\, T(x + \Phi(t)), \psi(x)\big)_x$$

is of the class C^m in Δ and

$$\gamma^{(l)}(\alpha) = \left(a(x, \alpha) \sum_{j_1, \ldots, j_l = 1}^{n} \Phi'_{j_1}(\alpha) \ldots \Phi'_{j_l}(\alpha) \frac{\partial^l}{\partial x_{j_1} \ldots \partial x_{j_l}} T(x), \psi(x)\right)_x + \\ + \Big(\sum_{|p| < l} c_p(x, \alpha) D^p T(x), \psi(x)\Big)_x,$$

where $c_p(x, \alpha)$ depend on $\Phi_j^{(\lambda)}(\alpha)$, $D_x^p a(x, \alpha)$, $\frac{\partial^\lambda}{\partial t^\lambda} a(x, \alpha)$ $(\lambda = 1, \ldots, l,\ j = 1, \ldots, n,$ $|p| \leq l,\ l = 1, \ldots, m)$.

PROOF. In view of II, we have

$$\gamma(t) = \big(a(x, t)\, T(x + \Phi(t))\big)_x = \big(T(x + \Phi(t)), a(x, t) \psi(x)\big)_x.$$

Making use of IV, we obtain

$$\gamma(t) = \big(T(y), a(y - \Phi(t), t) \psi(y - \Phi(t))\big)_y.$$

Now, let us take into account an arbitrary closed and bounded interval $[a, b] \subset \Delta$ such that $\alpha \in (a, b)$. Since the function $\Phi\colon \Delta \to R^n$ is continuous, the image of the interval $[a, b]$ is bounded and closed in R^n.

Hence and from the fact that the support of the function $\psi(y)$ is compact it follows that there exists a compact set $E \subset R^n$ which contains the supports of all the functions $\psi_t(y) = \psi(y - \Phi(t))$ $(t \in [a, b])$.

Since, moreover, the function $\psi(y - \Phi(t))$ is of the class C^m in $R^n \times \Delta$, we may apply V for the function $\gamma(t)$ considered in (a, b), and we obtain:

$$\gamma^{(l)}(t) = \\ = \left(T(y), (-1)^l a(y - \Phi(t), t) \sum_{j_1, \ldots, j_l = 1}^{n} \Phi'_{j_1}(t) \ldots \Phi'_{j_l}(t) \frac{\partial^l}{\partial y_{j_1} \ldots \partial y_{j_l}} \psi(y - \Phi(t))\right)_y + \\ + \Big(T(y), \sum_{|p| < l} c_p^*(y - \Phi(t), t) D^p \psi(y - \Phi(t))\Big)_y,$$

where $c_p^*(y-\Phi(t), t)$ depend on $\Phi_j^{(\lambda)}(t)$, $D_y^p a(y-\Phi(t), t)$, $\frac{\partial\lambda}{\partial t^\lambda} a(y-\Phi(t), t)$ $(\lambda = 1, \dots, l,\ j = 1, \dots, n,\ |p| \leq l,\ l = 1, \dots, m)$. Hence

$$\gamma^{(l)}(\alpha) = \left(T(y), (-1)^l a(y, \alpha) \sum_{j_1, \dots, j_l = 1}^{n} \Phi'_{j_1}(\alpha) \dots \Phi'_{j_l}(\alpha) \frac{\partial^l}{\partial y_{j_1} \dots \partial y_{j_l}} \psi(y)\right)_y + \\ + \big(T(y), \sum_{|p|<l} c_p^*(y, \alpha) D^p \psi(y)\big)_y,$$

where $c_p^*(y, \alpha)$ depend on $\Phi_j^{(\lambda)}(\alpha)$, $D_y^p a(y, \alpha)$, $\frac{\partial^\lambda}{\partial t^\lambda} a(y, \alpha)$ $(\lambda = 1, \dots, l,\ j = 1, \dots, n,$ $|p| \leq l,\ l = 1, \dots, m)$.

In view of II and I

$$\gamma^{(l)}(\alpha) = \left(\sum_{j_1, \dots, j_l = 1}^{n} \Phi'_{j_1}(\alpha) \dots \Phi'_{j_l}(\alpha) \frac{\partial^l}{\partial y_{j_1} \dots \partial y_{j_l}} (a(y, \alpha) T(y)), \psi(y)\right)_y + \\ + \big(\sum_{|p|<l} (-1)^{|p|} D_y^p (c_p^*(y, \alpha) T(y)), \psi(y)\big)_y$$

and by III

$$\gamma^{(l)}(\alpha) = \left(a(y, a) \sum_{j_1, \dots, j_l = 1}^{n} \Phi'_{j_1}(\alpha) \dots \Phi'_{j_l}(\alpha) \frac{\partial^l}{\partial y_{j_1} \dots \partial y_{j_l}} T(y), \psi(y)\right)_y + \\ + \big(\sum_{|p|<l} c_p(y, \alpha) D^p T(y), \psi(y)\big)_y,$$

where $c_p(y, \alpha)$ depend on $\Phi_j^{(\lambda)}(\alpha)$, $D_y^p a(y, \alpha)$, $\frac{\partial^\lambda}{\partial t^\lambda} a(y, \alpha)$ $(\lambda = 1, \dots, l,\ j = 1, \dots, n,$ $|p| \leq l,\ l = 1, \dots, m)$.

We may replace y by x in the last equality and this finishes the proof.

Theorem I. *Suppose that*

1° $a_i(x, t) \in C^\infty$ *in* R^n *for every fixed* t *from an open interval* $\Delta \subset R$ $(i = 1, \dots, k)$,
2° $a_i(x, t) \in C^m$ *in* $R^n \times \Delta$ $(i = 1, \dots, k)$,
3° $\varphi_i(t) = (\varphi_{i1}(t), \dots, \varphi_{in}(t)) \in C^m$ *in* Δ $(i = 1, \dots, k)$,
4° *there exists* $\alpha \in \Delta$ *such that* $\varphi_i(\alpha) = 0$ *for* $i = 1, \dots, k$,

5° $$\sum_{j_1, \dots, j_m = 1}^{n} \sum_{i=1}^{k} a_i(x, \alpha) \varphi'_{ij_1}(\alpha) \dots \varphi'_{ij_m}(\alpha) \xi_{j_1} \dots \xi_{j_m} \neq 0$$

for every $\xi = (\xi_1, \dots, \xi_n) \neq 0$ *and for every* $x \in R^n$.

Then every locally integrable solution f *of the equation*

$$\sum_{i=1}^{k} a_i(x, t) f(x + \varphi_i(t)) = 0 \tag{1}$$

is a function of class C^∞ *almost everywhere.*

PROOF. The locally integrable function $f(x)$ can be identified with the distribution f defined by the equality

$$(f, \psi) = \int_{R^n} f(x) \psi(x)\, dx \quad \text{for each} \quad \psi \in \mathcal{D}.$$

If the function f is locally integrable, the functions $a_i(x, t)f(x+\varphi_i(t))$ $(i=1, \dots, k)$ are locally integrable with respect to x, too. They can be identified with the distributions defined by the equalities

$$(a_i(x, t)f(x+\varphi_i(t)), \psi(x))_x = \int_{R^n} a_i(x, t)f(x+\varphi_i(t))\psi(x)\, dx \quad \text{for each} \quad \psi \in \mathscr{D}.$$

The functions

$$\gamma_i(t, \psi) = (a_i(x, t)f(x+\varphi_i(t)), \psi(x))$$

satisfy the assumptions of the Lemma and since it follows from (1) that

$$\sum_{i=1}^{k} \gamma_i(t, \psi) = 0$$

we have

$$\sum_{i=1}^{k} \left(a_i(x, \alpha) \sum_{j_1, \dots, j_m = 1}^{n} \varphi'_{ij_1}(\alpha) \dots \varphi'_{ij_m}(\alpha) \frac{\partial^m}{\partial x_{j_1} \dots \partial x_{j_m}} f(x), \psi(x) \right)_x +$$
$$+ \sum_{i=1}^{k} \Big(\sum_{|p|<m} c_{ip}(x, \alpha) D^p f(x), \psi(x) \Big)_x = 0 \quad \text{for each} \quad \psi \in \mathscr{D}.$$

i. e.

$$\left(\sum_{i=1}^{k} a_i(x, \alpha) \sum_{j_1, \dots, j_m = 1}^{n} \varphi'_{ij_1}(\alpha) \dots \varphi'_{ij_m}(\alpha) \frac{\partial^m}{\partial x_{j_1} \dots \partial x_{j_m}} f(x), \psi(x) \right)_x +$$
$$+ \Big(\sum_{|p|<m} c_p(x, \alpha) D^p f(x), \psi(x) \Big)_x = 0 \quad \text{for each} \quad \psi \in \mathscr{D}.$$

The last equality means that f is a distributional solution of the differential equation

(2)
$$\sum_{i=1}^{k} a_i(x, \alpha) \sum_{j_1, \dots, j_m = 1}^{n} \varphi'_{ij_1}(\alpha) \dots \varphi'_{ij_m}(\alpha) \frac{\partial^m}{\partial x_{j_1} \dots \partial x_{j_m}} f(x) +$$
$$+ \sum_{|p|<m} c_p(x, \alpha) D^p f(x) = 0.$$

In view of 5° this differential equation is elliptic. In view of other assumptions of this theorem its coefficients are functions of class C^∞. It is known (cf. e.g. Hörmander [4], pages 101, 102) that every distributional solution of such a differential equation is a function of class C^∞. *)

We have shown that an arbitrary locally integrable solution of equation (1) satisfies in a distributional sense an elliptic differential equation with coefficients of class C^∞. Therefore it must be a function of class C^∞, almost everywhere.

Remark 1. Notice that we obtained equation (2) making use of 1°, 2°, 3°, 4° only. Condition 5° means that equation (2) is elliptic.

*) The distribution f is a function of class C^∞ i.e. there exists a function $\tilde{f} \in C^\infty$ such that

$$(f, \psi) = \int_{R^n} \tilde{f}(x)\psi(x)\, dx \quad \text{for each} \quad \psi \in \mathscr{D}.$$

Remark 2. Condition 5° can be satisfied only in the case if m is an even number.

Remark 3. Condition 5° can be replaced, in the case $a_i(x, t) = a_i(t)$, by an arbitrary condition which guarantees the hypoellipticity of the resulting differential equation (cf. Hörmander [4], pages 99. 100).

Now, we shall prove (very easy in applications)

Theorem II. *Suppose that assumptions* 1°, 2°, 3°, 4° *of Theorem I are satisfied and, moreover,*

5°° *the equation*

$$\frac{\partial^m}{\partial t^m}\left(\sum_{i=1}^{k} a_i(x, t) f(x+\varphi_i(t))\right) = 0 \tag{3}$$

(where the unknown function $f \in C^m$ in R^n) is elliptic for $t = \alpha$.

Then every locally integrable solution of equation (1) *is a function of class C^∞ almost everywhere.*

Proof. In view of Remark 1, every locally integrable solution of equation (1) statisfies equation (2) in the distributional sense and to prove our theorem it is enough to show that condition 5°° implies condition 5°. To establish this fact we shall show that equation (2) is elliptic if and only if equation (3) with $t = \alpha$ is elliptic i.e. that the principal parts of these equations are the same.

The principal part of equation (2)

$$\sum_{i=1}^{k} a_i(x, \alpha) \sum_{j_1, \ldots, j_m = 1}^{n} \varphi'_{ij_1}(\alpha) \ldots \varphi'_{ij_m}(\alpha) \frac{\partial^m}{\partial x_{j_1} \ldots \partial x_{j_m}} f(x)$$

can be written as

$$\sum_{i=1}^{k} \sum_{|p|=m} a_i(x, \alpha)\big(\varphi'_{i_1}(\alpha)\big)^{p_1} \ldots \big(\varphi'_{i_n}(\alpha)\big)^{p_n} D^p f(x). \tag{4}$$

The left-hand side of equation (3) can be written as

$$\frac{\partial^m}{\partial t^m}\left(\sum_{i=1}^{k} a_i(x, t) f(x+\varphi_i(t))\right) = \sum_{i=1}^{k} \sum_{l=0}^{m} \binom{m}{l} \frac{\partial^l}{\partial t^l} a_i(x, t) \frac{\partial^{m-l}}{\partial t^{m-l}} f(x+\varphi_i(t)) =$$

$$= \sum_{i=1}^{k} a_i(x, t) \frac{\partial^m}{\partial t^m} f(x+\varphi_i(t)) + \sum_{i=1}^{k} \sum_{l=1}^{m} \binom{m}{l} \frac{\partial^l}{\partial t^l} a_i(x, t) \frac{\partial^{m-l}}{\partial t^{m-l}} f(x+\varphi_i(t)) =$$

$$= \sum_{i=1}^{k} a_i(x, t) \sum_{|p|=m} (\varphi'_{i_1}(t))^{p_1} \ldots (\varphi'_{i_n}(t))^{p_n} D^p f(x+\varphi_i(t)) +$$

$$+ \sum_{i=1}^{k} a_i(x, t) \sum_{|p|<m} \bar{c}_{ip}(t) D^p f(x+\varphi_i(t)) + \sum_{i=1}^{k} \sum_{l=1}^{m} \binom{m}{l} \frac{\partial^l}{\partial t^l} a_i(x, t) \frac{\partial^{m-l}}{\partial t^{m-l}} f(x+\varphi_i(t)),$$

where $\bar{c}_{ip}(t)$ depend on $\varphi_{ij}^{(l)}(t)$ $(j = 1, \ldots, k,\ l = 1, \ldots, m)$.

Now, it is easy to see that the principal part of equation (3)

$$\sum_{i=1}^{k} a_i(x, t) \sum_{|p|=m} (\varphi'_{i_1}(t))^{p_1} \dots (\varphi'_{i_n}(t))^{p_n} D^p f(x+\varphi_i(t))$$

has for $t=\alpha$ also form (4).

This finishes the proof.

Remark 4. If the assumptions of our theorems are satisfied, every continuous solution f of equation (1) is a function of class C^∞ everywhere. Adding some assumptions on the functions $\varphi_i(t)$ it is possible to prove that also every locally integrable solution f of equation (1) is a function of class C^∞ everywhere.

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