# On the distribution of digits

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## 1. Introduction

Let $K(>1)$ be a fixed positive integer. Then any integer $n$ can be uniquely represented as follows:

$$n = a_1 K^{n_1} + \ldots + a_t K^{n_t}$$

where $n_1 > n_2 > n_3 > \ldots > n_t \geqq 0$ are integers; $1 \leqq a_i \leqq K-1$ $(i=1, \ldots, t)$. We set $\alpha(n) = \sum_{i=1}^{t} a_i$.

R. BELLMAN and H. SHAPIRO proved the relation

$$\sum_{n \leqq x} \alpha(n) = \frac{x \log x}{2 \log 2} + O(x \log \log x)$$

in the case $K=2$ [1]. S. C. TANG extended this result to the general case and discovered a better alternative residual, namely he proved [2], that for any positive integer

$$\sum_{n \leqq x} \alpha(n) = \frac{K-1}{2} \frac{x \log x}{\log K} + O(x).$$

The first named author proved in [3], that assuming the validity of the density hypothesis concerning the Riemann zeta function

$$\sum \alpha(p) = \frac{K-1}{2} \frac{x}{\log K} + O\left(\frac{x}{(\log \log x)^{1/3}}\right)$$

holds, where $p$ in the sum runs over all of the primes not exceeding $x$.

In the present paper we shall investigate the limit distribution of $\alpha(n)$ and of $\alpha(p)$, assuming the validity of the density hypothesis in the second case.

In what follows we use the following notations:

(1. 1)—(1. 2) $$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2}\, dt, \quad \operatorname{li} x = \int_{2}^{x} \frac{du}{\log u}.$$

Let us put

(1. 3)—(1. 4) $$M_x = \frac{K-1}{2} \frac{\log x}{\log K}, \quad D_x^2 = \frac{K^2-1}{12} \frac{\log x}{\log K}$$

and let $N_x(y)$ be the number of those $n$-s for which

$$n \leqq x \quad \text{and} \quad \alpha(n) < M_x + yD_x, \tag{1.5}$$

and similarly let $M_x(y)$ be the number of those primes $p$ for which

$$p \leqq x \quad \text{and} \quad \alpha(p) < M_x + yD_x \tag{1.6}$$

holds.

Let $\zeta(s)$ be the zeta function of Riemann and let $N(\sigma_0, T)$ denote the number of the zeros of $\zeta(s)$ in the rectangle $\sigma_0 \leqq \sigma \leqq 1$, $|t| \leqq T$, $s = \sigma + it$.

We shall prove the following statements.

**Theorem 1.** *For every fixed K*

$$\frac{N_x(y)}{x} = \Phi(y) + O\left(\frac{\log\log x}{(\log x)^{1/2}}\right) \tag{1.7}$$

*holds uniformly in y as x tends to infinity.*

**Theorem 2.** *Assuming that*

$$N(\sigma, T) < cT^{2(1-\sigma)} \log^2 T, \quad \text{if } \frac{1}{2} \leqq \sigma \leqq 1, \ 1 \leqq T < \infty$$

*for a suitable constant c we have*

$$\frac{M_x(y)}{\operatorname{li} x} = \Phi(y) + O\left(\frac{1}{(\log\log x)^{1/3}}\right)$$

*uniformly in y as x tends to infinity.*

The proof of Theorem 1 goes with the application of the limit distribution theory for the sums of independent random variables.

For the proof of Theorem 2 we need a lemma concerning the distribution of prime numbers in small intervals.

## 2. Formulation and proof of Lemma 1.

In the sequel $p$ denotes prime numbers, $c$ a positive constant, not the same at every place.

Let $\Lambda(n)$ be Mangoldt's function i.e. $\Lambda(n) = \log p$, if $n$ is a power of $p$ and $\Lambda(n) = 0$ if $n$ has two different prime divisors. Let

$$\psi(x) = \sum_{n \leqq x} \Lambda(n); \quad \pi(x) = \sum_{p \leqq x} 1, \tag{2.1)—(2.2}$$

i.e. $\pi(x)$ denotes the number of primes not exceeding $x$.

Let further

$$\Delta_U(x) = \Delta(x) = \psi(x) - \psi\big(x - (U+1)\big) - (U+1), \tag{2.3}$$

$$\varrho_U(x) = \pi(x+U) - \pi(x) - \frac{U}{\log x}, \tag{2.4}$$

where $U$ is a positive number.

Let $h(x)$ be a monotonous non-decreasing function of $x$, which tends to infinity as $x \to +\infty$ and which satisfies the relation

$$1 \leqq h(x) < c \log x. \tag{2.5}$$

**Lemma 1.** If

$$N(\sigma, T) < cT^{2(1-\sigma)} \log^2 T \quad \text{in} \quad \frac{1}{2} \leqq \sigma \leqq 1, \tag{2.6}$$

*and*

$$U = [(\log x)^{7,5} h(x)], \tag{2.7}$$

*then*

$$\sum_{n \leqq x} \Delta_U^2(n) < c \frac{U^2 x}{h(x)}, \tag{2.8}$$

*and*

$$\sum_{n \leqq x} \varrho_U^2(n) < c \frac{U^2 x}{h(x) \log^2 x}. \tag{2.9}$$

We remark that a similar, stronger result was discovered by H. Cramer [4] and improved by A. Selberg [5] on the basis of the Riemann conjecture.

Proof. First we prove that (2. 9) follows from (2. 8). We have

$$\varrho_U(n) = (\log x)^{-1} \{ \sum_{n \leqq p < n+U} \log x - U \} = (\log x)^{-1} \{ \sum_{n \leqq \nu < n+U} \Lambda(\nu) - U \} +$$

$$+ (\log x)^{-1} \sum_{n \leqq p < n+U} \log \frac{x}{p} + (\log x)^{-1} \sum_{n \leqq p^k < n+U} \log p, \quad k \geqq 2$$

and so

$$|\varrho_U(n)| \leqq \frac{1}{\log x} |\Delta_U(n)| + \frac{\log \frac{x}{n}}{\log x} \left\{ |\varrho_U(n)| + \frac{U}{\log x} \right\} + 2 \sum_{n < p^k < n+U} 1, \quad k \geqq 2$$

holds. Assuming that $n \geqq x^{\frac{1}{2}}$, i.e. $\log \frac{x}{n} < \frac{1}{2} \log x$, we have

$$\frac{1}{2} |\varrho_U(n)| < \frac{|\Delta_U(n)|}{\log x} + \frac{U}{\log^2 x} + 2 \sum_{n < p^k < n+U}, \quad k \geqq 2.$$

Hence

$$\sum_{x^{1/2} \leqq n < x} |\varrho_U(n)|^2 < \frac{c}{\log^2 x} \sum_{n \leqq x} \Delta_U^2(n) + \frac{cxU^2}{\log^4 x} + c \sum_{n \leqq x} \{ \sum_{n < p^k < n+U} 1 \}^2, \quad k \geqq 2 \tag{2.10}$$

follows. Using the evident inequality

$$\{ \sum_{n \leqq p^k < n+U} 1 \}^2 < U \sum_{n \leqq p^k < n+U} 1$$

we obtain that the last sum on the right hand side of (2. 10) is smaller than

$$U \sum_{n \leqq x} \sum_{n \leqq p^k < n+U} 1 < U^2 \sum_{p^k < x} 1 \leqq U^2 \big( \pi(x^{1/2}) + \pi(x^{1/3}) + \dots \} \quad k \geqq 2$$

because of (2. 5). Further using the inequalities

$$\frac{xU^2}{\log^4 x} < \frac{xU^2}{h(x)\log^2 x}, \quad \sum_{n \leq x^{1/2}} |\varrho_U(n)|^2 < U^2 x^{1/2} < \frac{xU^2}{h(x)\log^2 x}$$

we obtain that

$$\sum_{n \leq x} |\varrho_U(n)|^2 < \frac{c}{\log^2 x} \sum_{n \leq x} |\Delta_U(n)|^2 + c\,\frac{U^2 x}{h(x)\log^2 x}$$

and hence it follows that (2. 9) is a consequence of (2. 8).

For the proof of the inequality (2. 9) let

$$f(z) = \sum_{n=1}^{\infty} \Lambda(n) e^{-nz}; \quad z = u+iy,\ 0 < u < 1,\ -\pi \leq u \leq \pi. \tag{2. 11}$$

Let

$$T(z) = \sum_{\varrho} z^{-\varrho}\Gamma(\varrho), \tag{2. 12}$$

where the sum is extended over all the non-trivial zeros of $\zeta(s)$. Let further

$$\Delta = \left(\log \frac{1}{U}\right)^{-7}, \tag{2. 13}$$

$$g(z) \overset{\text{def}}{=} f(z)(1+e^{-z}+\ldots+e^{-Uz}) - (U+1)\frac{e^{-z}}{1-e^{-z}}. \tag{2. 14}$$

From (2. 14) it follows that

$$g(z) = \sum_{n=1}^{\infty} \Delta_U(n) e^{-nz}, \tag{2. 15}$$

and by the Parseval formula we have

$$\sum_{n=1}^{\infty} \Delta^2(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(z)|^2\, dy.$$

Let now assume that $uU$ tends to zero for $u \to 0$. Then

$$1+e^{-z}+\ldots+e^{-Uz} = O\left(\frac{1}{|z|}\right) = O\left(\frac{1}{\Delta}\right), \quad \frac{e^{-z}}{1-e^{-z}} = O\left(\frac{1}{y}\right),$$

if $|y| \geq \Delta$. Hence

$$\int_{|y| \geq \Delta} |g(z)|^2\, dy \leq 2 \int_{|y| \geq \Delta} |f(z)|^2 |1+e^{-z}+\ldots+e^{-Uz}|^2\, dy + cU^2 \int_{|y| \geq \Delta} y^{-2}\, dy \leq$$

$$\leq \frac{c}{\Delta^2} \int_{-\pi}^{\pi} |f(z)|^2\, dy + c\,\frac{U^2}{\Delta}$$

follows. Using the prime number theorem we have

$$\int_{-\pi}^{\pi} |f(z)|^2\, dy < c\,\frac{1}{u}\log\frac{1}{u},$$

and so

$$\int\limits_{|y|\geqq\Delta} |g(z)|^2\, dy < \frac{c}{\Delta^2 u}\log\frac{1}{u} + c\,\frac{U^2}{\Delta} \tag{2. 16}$$

holds.

We shall investigate the integral $\int\limits_{|y|\leqq\Delta} |g(z)|^2 dy$. JU. V. LINNIK proved [6] that

$$f(z) = \frac{1}{z} - T(z) + O\left(\log^3\frac{1}{u}\right), \tag{2. 17}$$

and that under the assumptions of Lemma 1

$$\int_{-\Delta}^{\Delta} |T(z)|^2\, dy < \frac{c}{u}\left(\log\frac{1}{u}\right)^{-1} \tag{2. 18}$$

holds [7].

From (2. 17) we obtain

$$\int\limits_{|y|\leqq\Delta} |g(z)|^2\, dy \leqq 3\int_{-\Delta}^{\Delta}\left|\frac{1+\dots+e^{-Uz}}{z} - (U+1)\frac{e^{-z}}{1-e^{-z}}\right|^2 dy +$$
$$+3\int_{-\Delta}^{\Delta} |T(z)|^2\,|1+\dots+e^{-Uz}|^2\, dy + O\left(\log^6\frac{1}{u}\right)\int_{-\Delta}^{\Delta} |1+\dots+e^{-Uz}|^2\, dy.$$

The first integral on the right hand side has the order $K^4\Delta$, because we have

$$\frac{e^{-z}}{1-e^{-z}} = \frac{1}{z} + O(1), \quad 1+\dots+e^{-Uz} = U+1+O(U^2|z|).$$

The second term is smaller than $U^2\,\frac{1}{u}\left(\log\frac{1}{u}\right)^{-1}$ (see 2. 18)) and finally

$$\int_{-\Delta}^{\Delta} |1+\dots+e^{-Uz}|^2\, dy \leqq (U+1)^2\, 2\pi.$$

Hence

$$\int_{-\pi}^{\pi} |g(z)|^2\, dy < c\left\{U^4\Delta + \frac{U^2}{u}\left(\log\frac{1}{u}\right)^{-1} + U\log^6\frac{1}{u} + \frac{1}{\Delta^2}\,\frac{1}{u}\left(\log\frac{1}{u}\right)^{-1} + \frac{U^2}{\Delta}\right\}.$$

Let now $\frac{1}{u} = x$, $U = (\log x)^{7,5} h(x)$, $1 \leqq h(x) \leqq \log x$, then the right hand side is smaller then $c\,\frac{U^2 x}{h(x)}$.

Now our inequality (2. 8) rapidly follows from the relation

$$\sum_{n\leqq x} \Delta^2(n) < c\sum_{n=1}^{\infty} \Delta^2(n) e^{-2n/x}.$$

## 3. The proof of Theorem 1.

Let $\xi_1, \ldots, \xi_n, \ldots$ be independent random variables assuming the values $\nu = 0, 1, 2, \ldots, K-1$ with probability

$$P(\xi_i = \nu) = \frac{1}{K}, \qquad i = 1, 2, \ldots. \tag{3.1}$$

Let $M(\xi)$ and $D(\xi)$ denote the mean-value and the variance of a random variable $\xi$ respectively, further let for the sake of brevity

$$M = M(\xi_i) = \frac{K-1}{2}, \quad D^2 = D^2(\xi_i) = \frac{K^2-1}{12} \qquad (i = 1, 2, \ldots). \tag{3.2}$$

Let $\eta_n = \xi_1 + \ldots + \xi_n$, so $M(\eta_n) = nM$, $D^2(\eta_n) = nD^2$. Let $F_n(y)$ be a distribution function defined by

$$F_n(y) = P(\eta_n < nM + y\sqrt{n}D). \tag{3.3}$$

In the theory of probability the following assertion is well known.

**Lemma 2.**

$$|F_n(y) - \Phi(y)| < \frac{c}{\sqrt{n}} \tag{3.4}$$

*uniformly in* $-\infty \leqq y \leqq \infty$.

For the proof see the book of B. V. Gnedenko—A. N. Kolmogorov [8] (Theorem 1, p. 201).

Let $B_m(x)$ denote the number of those $n$-s, for which $n < x$, $\alpha(n) = m$ is satisfied.

It is evident, that for an integer $l \geqq 1$

$$B_m(K^l) = K^l P(\eta_l = m). \tag{3.5}$$

Further, if $1 \leqq A \leqq K-1$, then

$$B_m(AK^l) = \sum_{n<K^l} 1 + \ldots + \sum_{(i-1)K^l \leqq n < iK^l} 1 + \ldots + \sum_{(A-1)K^l \leqq n < AK^l} 1 =$$
$$= B_m(K^l) + B_{m-1}(K^l) + \ldots + B_{m-A}(K^l), \quad \alpha(n) = m.$$

Now we assume:

$$x = A_1 K^{n_1} + \ldots + A_t K^{n_t}, \qquad n_1 > n_2 > \ldots > n_t \geqq 0,$$
$$1 \leqq A_i \leqq K-1, \qquad i = 1, \ldots, t.$$

Then

$$B_m(x) = \sum_{i=0}^{A_1-1} B_{m-i}(K^{n_1}) + B_m(x - A_1 K^{n_1}) = \tag{3.6}$$

$$= \sum_{i=0}^{A_1-1} B_{m-i}(K^{n_1}) + \sum_{i=A_1}^{A_1+A_2-1} B_{m-i}(K^{n_2}) + \ldots + \sum_{i=A_1+\ldots+A_{t-1}}^{A_1+\ldots+A_t} B_{m-i}(K^{n_t}).$$

Let now suppose that

(3. 7) $$n_1 - n_t > c \log\log x = \mathrm{L}$$

with a sufficiently large constant $c$.

Let for the sake of brevity

(3. 8) $$T_y = M_x + yD_x = \frac{\log x}{\log K}\cdot M + y\left(\frac{\log x}{\log K}\right)^{1/2} D.$$

Using the relations (3. 5), (3. 6) we have

(3. 9) $$N_x(y) = \sum_{\substack{\alpha(n)<T_y\\ n\leq x}} 1 = K^{n_1}\sum_{i=0}^{A_1-1} P(\eta_{n_1} < T_y - i) + K^{n_2}\sum_{i=A_1}^{A_1+A_2-1} P(\eta_{n_2} < T_y - i) +$$
$$+ \dots + K^{n_t}\sum_{i=A_1+\dots+A_{t-1}}^{A_1+\dots+A_t} P(\eta_{n_t} < T_y - i).$$

Using the inequality (3. 4) we have that for every pair of real numbers $y_1, y_2$ the inequality

(3. 10) $$|F_n(y_1) - F_n(y_2)| \leq |F_n(y_1) - \Phi(y_1)| + |\Phi(y_1) - \Phi(y_2)| + |\Phi(y_2) - F_n(y_2)| \leq$$
$$\leq \frac{2c}{\sqrt{n}} + \int_{y_1}^{y_2} e^{-\frac{u^2}{2}}\,du < \frac{2c}{\sqrt{n}} + |y_1 - y_2|$$

is satisfied.

Let $i$ be an arbitrary integer value in the interval $0 \leq i \leq A_1 + \dots + A_t \leq Kt < cK \log\log x$ and let $\Theta$ be defined by

(3. 11) $$T_y - i = Mn_l + (y+\Theta)\sqrt{n_l}\,D,$$

$l$ is an arbitrary integer in $1 \leq l \leq t$.

Using the inequalities $n_1 \leq \dfrac{\log x}{\log K} \leq n_1 + 1$ and (3. 7) we have that

$$|\Theta| \leq \frac{M\left|\dfrac{\log x}{\log K} - n_l\right| + yD\left|\sqrt{n_l} - \left(\dfrac{\log x}{\log K}\right)^{1/2}\right|}{\sqrt{n_l}\,D} \leq$$
$$\leq c(\log x)^{-1/2}\log\log x\left(1 + \frac{|y|}{(\log x)^{1/2}}\right).$$

Hence for every term on the right hand side of (3. 9)

$$|P(\eta_{n_l} < T_y - i) - F_{n_l}(y)| < c(\log x)^{-1/2}\log\log x\left(1 + \frac{|y|}{(\log x)^{1/2}}\right)$$

and so from (3. 4)

(3. 12) $$|P(\eta_{n_l} < T_y - i) - \Phi(y)| < c(\log x)^{-1/2}\log\log x\left(1 + \frac{|y|}{(\log x)^{1/2}}\right)$$

follows.

So from (3. 9) we have

(3. 13) $$N_x(y)=x\Phi(y)+O(x(\log x)^{-\frac{1}{2}}\log\log x(1+|y|(\log x)^{-\frac{1}{2}})).$$

Let now $x^*$ be an arbitrary integer, the $K$-adical representation of which is

$$x^*=A_1K^{n_1}+\dots+A_uK^{n_u}.\qquad \text{Let}\quad n_1\geqq n_2\geqq\dots\geqq n_t\geqq L,\ L>n_{t+1}>\dots>n_u,$$

$$x=A_1K^{n_1}+\dots+A_tK^{n_t},\ x=x^*+x_1.$$

So we have

(3. 14) $$x_1<K^{n_{t+1}+1}\leqq K^{n_1}K^{-L}<x^*e^{-L\log K}<x^*/(\log x),$$

further from (3. 14)

(3. 15) $$|N_{x^*}(y)-\Phi(y)x^*|\leqq|N_x(y)-x\Phi(y)|+2|x^*-x|\leqq$$

$$\leqq c(\log x^*)^{-1/2}\log\log x^*\left(1+\frac{|y|}{(\log x^*)^{1/2}}\right)$$

follows.

So the relation (1. 7) is satisfied uniformly on the interval $|y|\leqq c(\log x)^{\frac{1}{2}}$. Further, if $y>c(\log x)^{\frac{1}{2}}$, or $y<-c(\log x)^{\frac{1}{2}}$, then $N_x(y)=x$ or $N_x(y)=0$, respectively, and in these cases the inequalities

$$0\leqq 1-\Phi(y)\leqq c\int_y^\infty e^{-u^2/2}\,du<(\log x)^{-1/2},\quad y\geqq c(\log x)^{1/2}$$

$$0\leqq\Phi(y)<\int_{-\infty}^y e^{-u^2/2}\,du<(\log x)^{-1/2},\quad y<-c(\log x)^{1/2}$$

hold.

So Theorem 1 is proved.

## 4. Proof of Theorem 2.

Let

(4. 1) $$U=[(\log x)^{7,5}h(x)],$$

where $h(x)$ is a function of $x$ tending monotonically to infinity as $x\to+\infty$, and satisfying the relation $h(x)=O(\log x)$. Let $l$ be a natural number satisfying the inequality

(4. 2) $$U\leqq K^l<KU.$$

Hence

(4. 3) $$(0<)c_1<\frac{l}{\log\log x}<c_2$$

follows.

Let further $A_j$ denote the set of integers in the interval

(4. 4) $$[K^l j,\ K^l(j+1)] \quad \text{for} \quad j=0, 1, \ldots, j_0,$$

where

(4. 5) $$j_0 = \left[\frac{x}{K^l}\right].$$

It is evident that these sets are disjoint and their union contains any natural number smaller than $x$.

Further, if $n \in A_j$, then

(4. 6) $$\alpha(j) \leqq \alpha(n) \leqq \alpha(j) + (K-1)l.$$

Let $\lambda(n, y)$ be defined as follows:

(4. 7) $$\lambda(n, y) = \begin{cases} 1, & \text{if } \alpha(n) < T_y, \\ 0, & \text{if } \alpha(n) \geqq T_y. \end{cases}$$

From the definition of $M_x(y)$

(4. 8) $$M_x(y) = \sum_{p \leqq x} \lambda(p, y),$$

further from (4. 4)

$$M_x(y) = \sum_{j=0}^{j_0} \sum_{p \in A_j} \lambda(p, y) - \sum_{\substack{x<p \\ p \in A_{j_0}}} \lambda(p, y)$$

follows.

Using the inequality

(4. 9) $$\pi(x+y) - \pi(x) < c\frac{y}{\log y} \qquad (x \geqq 1,\ y > 1),$$

which can be obtained by the sieve-method of Selberg [9], we have

(4. 10) $$\sum_{x<p} \lambda(p, y) < c\frac{U}{\log U} < c\frac{U}{\log\log x}, \quad p \in A_{j_0}.$$

Further

$$\lambda(p, y) = \lambda(j, y)$$

if

(4. 11) $$T_y - lK \leqq j < T_y.$$

So

(4. 12) $$M_x(y) = \sum_{j=0}^{j_0} \lambda(j, y)\big(\pi(K^l(j+1)) - \pi(K^l j)\big) + O(\Sigma_1) + O(\Sigma_2),$$

where

(4. 13) $$\Sigma_1 = \sum_{j \leqq j_0}{}' \big(\pi(K^l(j+1)) - \pi(K^l j)\big),$$

(4. 14) $$|\Sigma_2| \leqq \frac{U}{\log\log x},$$

and the dash in (4. 13) means that we sum over those $j$-s for which (4. 11) is satisfied.

Let now

$$V = \sum_{j \leqq j_0} 1, \quad T_y - lK < \alpha(j) < T_y. \tag{4.15}$$

Then

$$\Sigma_1 \leqq V \frac{U}{\log x} + O(\Sigma_3), \tag{4.16}$$

where

$$\Sigma_3 \overset{\text{def}}{=} \sum_{j \leqq j_0} |\varrho_{K^l}(K^l j)|. \tag{4.17}$$

Further we have

$$\sum_{j \leqq j_0} \lambda(j, y)\big(\pi(K^l(j+1)) - \pi(K^l j)\big) = \frac{K^l}{\log x} \sum_{j \leqq j_0} \lambda(j, y) + O(\Sigma_3), \tag{4.18}$$

and so from (4. 12) we obtain

$$M_x(y) = \frac{K^l}{\log x} \sum_{j \leqq j_0} \lambda(j, y) + O\left(V \frac{U}{\log x}\right) + O\left(\frac{U}{\log \log x}\right) + O(\Sigma_3). \tag{4.19}$$

First we shall estimate $\Sigma_3$.

Let $A$ be a natural number $< K^l$. For the integers $u$ in $1 \leqq u \leqq A$ we have

$$|\varrho_{K^l}(n+u) - \varrho_{K^l}(n)| \leqq \pi(n + K^l + A) - \pi(n + K^l) + \pi(n + A) - \pi(n) \leqq c \frac{A}{\log A}$$

if $n \leqq x$ (see (4. 9)).
Hence the inequality

$$|\varrho_{K^l}(K^l j)| \leqq \frac{1}{A+1} \sum_{u=0}^{A} |\varrho_{K^l}(K^l j + u)| + c \frac{A}{\log A}$$

follows, and so

$$\Sigma_3 \leqq \frac{1}{A+1} \sum_{j=0}^{j_0} \sum_{u=0}^{A} |\varrho_{K^l}(K^l j + u)| + c j_0 \frac{A}{\log A} \tag{4.20}$$

holds.

Since $A < K^l$, any natural number $n$ can be represented in the form $n = K^l j + u$ ($j = 0, \ldots, j_0$; $u = 0, \ldots, A$) at most once, and the number of the represented numbers equals $(A+1)(j_0+1)$. Using the Hölder-inequality and Lemma 1 we have that the double sum is smaller than

$$\frac{1}{A+1} \Big\{ \sum_{n = K^l j + u} 1 \Big\}^{1/2} \Big\{ \sum_{n \leqq 2x} |\varrho_{K^l}(n)|^2 \Big\}^{1/2} \leqq \frac{c}{A} \left(\frac{xA}{K^l}\right)^{1/2} \left(\frac{U^2 x}{h(x) \log^2 x}\right)^{1/2} \leqq \tag{4.21}$$

$$\leqq c \frac{x}{\log x} \left(\frac{U}{A h(x)}\right)^{1/2}.$$

Further

$$j_0 \frac{A}{\log A} \leqq \frac{x}{U} \frac{A}{\log A}. \tag{4.22}$$

Let us now choose $h(x)=\log x$, $A=\dfrac{U(\log\log x)^{2/3}}{\log x}$, so the right hand sides of the inequalities (4. 21), (4. 22) are smaller than

$$\frac{cx}{\log x\,(\log\log x)^{1/3}},$$

and so

$$\Sigma_3<\frac{cx}{\log x\,(\log\log x)^{1/3}} \tag{4. 23}$$

holds.

We shall now estimate V.

Let

$$T_y=M\frac{\log\frac{x}{K^l}}{\log K}+y_1D\left(\frac{\log\frac{x}{K^l}}{\log K}\right)^{1/2}\overset{\text{def}}{=}T^*_{y_1}$$

$$T_y-lK=M\frac{\log\frac{x}{K^l}}{\log K}+y_2D\left(\frac{\log\frac{x}{K^l}}{\log K}\right)^{1/2}\overset{\text{def}}{=}T^*_{y_2}.$$

Hence

$$|y_1-y_2|\leqq c\frac{lK}{(\log x)^{1/2}}\leqq c\frac{\log\log x}{(\log x)^{1/2}} \tag{4. 24}$$

follows.

Further using Theorem 1 we have

$$V=\sum_{T_{y_1}\leqq\alpha(j)<T_{y_2}}N_{x/K^l}(y_2)-N_{x/K^l}(y_1)\leqq \tag{4. 25}$$

$$\leqq\frac{x}{K^l}|\Phi(y_1)-\Phi(y_2)|+\left|N_{x/K^l}(y_1)-\frac{x}{K^l}\Phi(y_1)\right|+\left|N_{x/K^l}(y_2)-\frac{x}{K^l}\Phi(y_2)\right|\leqq$$

$$\leqq c\frac{x}{K^l}(\log\log x)(\log x)^{-1/2},\quad j\leqq j_0.$$

Further

$$T_y=M\frac{\log\frac{x}{K^l}}{\log K}+(y+\Theta)D\left(\frac{\log\frac{x}{K^l}}{\log x}\right)^{1/2},$$

where

$$|\Theta|\leqq\frac{\log\log x}{(\log x)^{1/2}}\left(1+\frac{|y|}{(\log x)^{1/2}}\right).$$

So, using similar argumentation as in the estimation of $V$, we have

$$\sum_{\alpha(j)<T_y}1=N_{x/K^l}(y+\Theta)=\frac{x}{K^l}\{\Phi(y)+O((\log x)^{-1/2}\log\log x)\},\quad j\leqq j_0$$

uniformly in $|y|<c(\log x)^{\frac{1}{2}}$.

To deal with the case $|y|>c(\log x)^{1/2}$ one has to repeat the argument used for the proof of Theorem 1.

Taking into account the relations (4. 19), (4. 23), (4. 25) the proof is completed.

## 5. Remarks

Let $\beta_\nu(n)$ denote the number of $\nu$-s among the digits of the $K$-adical representation of $n$, i.e. when

$$K^U \leqq n < K^{U+1}, \quad n = \sum_{j=0}^{u} \varepsilon_j(n) K^j, \qquad (\varepsilon_j(n) = 0, 1, \dots, K-1),$$

then

$$\beta_\nu(n) = \sum_{j=0}^{u} 1, \quad \varepsilon_j(n) = \nu.$$

Let further $U_\nu(x, y)$ be the number of those $n$-s for which

$$n \leqq x, \quad \beta_\nu(n) < \frac{\log x}{K \log K} + \frac{(K-1)^{1/2}}{K} \left(\frac{\log x}{\log K}\right)^{1/2} y$$

is satisfied, and similarly let $V_\nu(x, y)$ be the number of those primes $p$ for which

$$p \leqq x, \quad \beta_\nu(p) < \frac{\log x}{K \log K} + \frac{(K-1)^{1/2}}{K} \left(\frac{\log x}{\log K}\right)^{1/2} y$$

holds.

Using the same methods as used for the proof of Theorems 1, 2 we can prove the following assertions.

**Theorem 3.** For every $\nu = 0, 1, \dots, K-1$

$$\frac{1}{x} U_\nu(x, y) = \Phi(y) + O\left(\frac{\log \log x}{(\log x)^{1/2}}\right)$$

uniformly in $y$ as $x$ tends to infinity.

**Theorem 4.** Assuming the conditions of Theorem 2 we have

$$\frac{1}{\operatorname{li} x} V_\nu(x, y) = \Phi(y) + O\left(\frac{1}{(\log \log x)^{1/3}}\right), \qquad \nu = 0, 1, \dots, K-1$$

uniformly in $y$ as $x$ tends to infinity.

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