A note on a sieve method

By I. KÁTAI (Budapest)

1. Let P denote the set of primes, further Q_{+1} and Q_{-1} the set of those integers q, of which all prime divisors are $\equiv 1 \pmod 4$ and $\equiv -1 \pmod 4$, respectively. We can prove the following assertions.

1. Every sufficiently large even integer n can be represented in the form

$$n = p + q,$$

where

$$p \in P,\ q \in Q_{+1}.$$

2. Every sufficiently large odd integer n can be represented as

$$n = p + 2q,$$

where $p \in P$, $q \in Q_{+1}$.

3. There exist infinitely many solutions of the equality

$$p - 2q = 1$$

and of the equality

$$p - q = 2$$

where $p \in P$, $q \in Q_{+1}$.

We are unable to prove the analogous assertions with $q \in Q_{-1}$ instead of $q \in Q_{+1}$.

The proofs of 1., 2., 3., are based on the method of C. HOOLEY [2] and a theorem of E. BOMBIERI [3] concerning the large sieve. In the quoted paper Hooley proved on the basis of the extended Riemann hypothesis the conjecture of HARDY and LITTLEWOOD [1] for the number of representations of a number n as the sum of two squares and a prime number. Later a very powerful, ingenious method was elaborated by JU. V. LINNIK for the proof of the conjecture of Hardy and Littlewood without any hypothesis [4]. In 1965 E. Bombieri gave essential improvements of the large sieve [3]. His results allow to replace the assumption in the proof of Hooley.

Theorem 1. *The number $N(x)$ of solutions of the equality*

$$p - 1 = 2q,\ p \leqq x,\ p \in P,\ q \in Q_{+1} \tag{1. 1}$$

tends to infinity for $x \to \infty$, namely

$$N(x) \gg \frac{x}{(\log x)^3}. \tag{1. 2}$$

Theorem 2. *Let $N_1(n)$ and $N_2(n)$ denote the number of solutions of the equalities*

$$n = p + q, \tag{1. 3}$$

and

$$n = p + 2q, \tag{1. 4}$$

respectively. Then for every sufficiently large integer n we have

$$N_1(n) \gg B(n) \frac{n}{(\log n)^3}, \tag{1. 5}$$

if n is even,

$$N_2(n) \gg B(n) \frac{n}{(\log n)^3}, \tag{1. 6}$$

if n is odd, where

$$B(n) \gg (\log \log n)^{-1}.$$

Theorem 1 is an easy consequence of the following

Theorem 3. *Let $T(x)$ denote the sum*

$$T(x) = \sum_{p \leqq x} r(p-1)\,|\mu(p-1)| \tag{1. 7}$$

where $r(n)$ denotes the number of representations of n as the sum of two squares. We have

$$T(x) = A_0 \frac{x}{\log x} + O\left(\frac{x}{(\log x)^{1+\delta}}\right), \tag{1. 8}$$

where δ and A_0 are positive constants.

2. We give a short sketch of the proof of Theorem 3, and from Theorem 3 we deduce Theorem 1. The proof of Theorem 2 is very similar and can be omitted.

Lemma (E. Bombieri [3]).

$$\sum_{D \leqq Y} \max_{l \pmod D} \max_{z \leqq x} \left| \pi(x, z, l) - \frac{\operatorname{li} z}{\varphi(D)} \right| \ll \frac{x}{(\log x)^A},$$

if $Y \ll x^{\frac{1}{2}} (\log x)^{-B}$, $B \geqq 2A + 23$.

Let $T(x, k)$ denote the sum

$$T(x, k) = \sum_{p \equiv 1 \pmod k} r(p-1).$$

From (1. 7) we have

$$T(x) = \sum_{d^2 \leqq x} \mu(d)\, T(x, d^2).$$

Let $L = (\log x)^3$, $L_1 = x^{\frac{1}{2}} (\log x)^{-C}$, $L_2 = x^{\frac{1}{2}} (\log x)^C$, where $C > 0$ is a constant. It is evident that

$$T(x, k) \ll \sum_{\substack{p \leqq x \\ p \equiv 1 \pmod k}} \tau(p-1) \ll \sum_{kl \leqq x} \tau(kl) \ll \frac{\tau(k)}{k} x \log x.$$

Hence

$$\sum_{L \leqq d \leqq x^{1/2}} |\mu(d)|\, T(x, d^2) \ll x \log x \sum_{d \geqq L} \frac{\tau(d^2)}{d^2} \ll \frac{x \log x \cdot \log^3 L}{L} \ll x \cdot (\log x)^{-1,5},$$

and so

$$T(x) = \sum_{d \leqq L} \mu(d)\, T(x, d^2) + O\left(\frac{x}{(\log x)^{1,5}}\right).$$

Let now $k \leqq L^2$ and investigate the sum $T(x, k)$.

$$T(x, k) = \sum_{p \equiv 1(k)} \sum_{p-1=uv} \chi(u) = \sum_{u \leqq L_1} \chi(u)\pi(x, [k, u], 1) +$$

$$+ \sum_{L_1 < u < L_2} \chi(u)\pi(x, [k, u], 1) + \sum_{u \geqq L_2} \chi(u)\pi(x, [k, u], 1) = \Sigma_A^{(k)} + \Sigma_B^{(k)} + \Sigma_C^{(k)}.$$

Using the Lemma we have

$$\Sigma_A^{(k)} = \operatorname{li} x \sum_{u \leqq L_1} \frac{\chi(u)}{\varphi([k, u])} + O\left(\frac{x}{(\log x)^A}\right)$$

if $C \geqq 2A + 25$.

For the estimation of $\Sigma_C^{(k)}$ we have

$$\Sigma_C^{(k)} = \sum_{v < L_1} \Sigma_v,$$

where

$$\Sigma_v = \sum_{\substack{p-1=uv \\ p-1 \equiv O \pmod{k} \\ vL_2 < uv \leqq x}} \chi(u).$$

In other words Σ_v is the difference between the number of the primes for which

$$p \leqq x,\ p \equiv 1 \pmod{k},$$
$$p \equiv 1 + v \pmod{4v}$$

and those for which

$$p \leqq x,\ \ p \equiv 1 \pmod{k}$$
$$p \equiv 1 - v \pmod{4v}$$

holds.

So we have

$$\Sigma_C^{(k)} \ll \sum_{v < L_1} \max_{l_1, l_2 \pmod{[4v, k]}} |\pi(x, [4v, k], l_1) - \pi(x, [4v, k], l_2)| +$$

$$+ \sum_{v < L_1} \max_{l_1, l_2 \pmod{[4v, k]}} |\pi(vL_2, [4v, k], l_1) - \pi(vL_2, [4v, k], l_2)| \ll \frac{x}{(\log x)^A}.$$

The investigation of $\Sigma_B^{(k)}$.

Let

$$D(m) = \sum_{\substack{L_1 < u < L_2 \\ u | m}} 1, \quad F(m) = \sum_{\substack{u | m \\ L_1 < u < L_2}} \chi(u), \quad K(n, L) = \sum_{\substack{d \leqq L \\ d^2 | n}} 1.$$

$$\Sigma_B^{(k)} = \sum_{\substack{p \equiv 1 \pmod{k} \\ p \leqq x}} F(p-1).$$

Let

$$\Sigma_B = \sum_{d \leqq L} \Sigma_B^{(d^2)}.$$

So we have

$$\Sigma_B \ll \sum_{p \leqq x} |F(p-1)|\, K(p-1, L) \ll \Big(\sum_{\substack{p \leqq x \\ D(p-1) \neq 0}} K^2(p-1, L)\Big)^{1/2} \Big(\sum_{p \leqq x} F^2(p-1)\Big)^{1/2} =$$

$$= \Sigma_U^{1/2} \cdot \Sigma_E^{1/2}.$$

Further

$$\Sigma_U \ll \Big(\sum_{p \leqq x} K^4(p-1, L)\Big)^{1/2} \Big(\sum_{\substack{p \leqq x \\ D(p-1) \neq 0}} 1\Big)^{1/2} = \Sigma_V^{1/2} \cdot \Sigma_D^{1/2},$$

and hence

$$\Sigma_B \ll \Sigma_V^{1/4} \cdot \Sigma_D^{1/4} \cdot \Sigma_E^{1/2}.$$

Further we have

$$\Sigma_V \ll \sum_{\substack{d_i \leqq L \\ i = 1, 2, 3, 4}} \pi(x, [d_1^2, d_2^2, d_3^2, d_4^2], 1) \ll \frac{x}{\log x}.$$

Using the method of Hooley (see [2], p. 204—209) we have

$$\Sigma_D \ll \frac{x}{\log x} (\log x)^{-\gamma} (\log \log x)^{c_1}, \quad \Sigma_E \ll \frac{x}{\log x} (\log \log x)^{c_2},$$

where c_1 ; c_2 depend on C only.

Hence

$$\Sigma_B \ll \frac{x}{\log x} (\log x)^{-\gamma/4} (\log \log x)^{c_3}$$

follows.

Further

$$\Sigma_C \ll \sum_{d \leqq L} \Sigma_C^{(d^2)} \ll \frac{x}{(\log x)^{A-2}},$$

$$\Sigma_A = \sum_{d \leqq L} \mu(d)\, \Sigma_A^{(d^2)} = \mathrm{li}\, x \sum_{d \leqq L} \mu(d) \sum_{u \leqq L_1} \frac{\chi(u)}{\varphi([d^2, u])} + O\left(\frac{x}{(\log x)^{A-2}}\right).$$

For the double sum on the right hand side

$$\sum_{d \leqq L} \mu(d) \sum_{u \leqq L_1} \frac{\chi(u)}{\varphi([d^2, u])} = A_0 + O\left(\frac{1}{\log x}\right)$$

holds and choosing $A=4$ in our Lemma we have

$$\Sigma_A = A_0\, \mathrm{li}\, x + O\left(\frac{x}{\log^2 x}\right).$$

Hence (1. 8) immediately follows.

Since

$$T^2(x) \ll N(x) \sum_{p \leqq x} r^2(p-1) \ll N(x) \cdot \sum_{n \leqq x} r^2(n),$$

and

$$\sum_{n \leqq x} r^2(n) \ll x \log x,$$

so from Theorem 3

$$N(x) \gg \frac{x}{\log^3 x},$$

i.e. (1. 2) holds.

Using the large sieve we can prove a better lower estimation for $N(x)$.

References

[1] G. H. Hardy—J. E. Littlewood, Some problems of partitio numerorum III: On the expression of a number as a sum of primes, *Acta Math.*, **44** (1923), 1—70.

[2] C. Hooley, On the representation of numbers as the sum of two squares and a prime, *Acta Math.*, **97** (1957), 189—210.

[3] E. Bombieri, On the large sieve, Mathematika, **12** (1965), 201—225.

[4] Ю. В. Линник, Дисперсонный метод в бинарных аддитивных задачах, *Ленинград* 1961.

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