Non existence of interpolatory polynomials

By A. K. VARMA* (Edmonton)

1. Recently Professor P. TURÁN and his associates have written a series of papers on Lacunary interpolation (see [1], [2], [3], [9]). Later these results are generalised in different directions by O. KIS [5, 6], G. FREUD [4], A. SHARMA, and R. B. SAXENA [7, 8] and the author [10, 11, 12, 13]. At present we are concerned with the following theorem of J. SURÁNYI, and P. TURÁN [9].

Let $n=2k+1$ and

(1.1) $$1 \geqq x_1 > x_2 > \dots > x_k > x_{k+1} = 0 > x_{k+2} > \dots > x_{2k+1} \geqq -1$$

with

(1. 2) $$x_j = -x_{2k+2-j} \qquad (j=k+2, \dots, 2k+1).$$

Theorem 1. (J. Surányi and P. Turán) *If $n=2k+1$ and the points $x_1, \dots, x_n$ satisfy* (1.1) *and* (1.2), *there is in general no polynomial $f(x)$ of degree $\leqq 2n-1$ such that for given y_ν and y_ν^**

(1. 3) $$f(x_\nu)=y_\nu,\ f''(x_\nu)=y_\nu^* \qquad (\nu=1, 2, \dots, n).$$

If there exists such a polynomial then there is an infinity of them.

Thus as remarked by the above authors in the case of an odd number of distinct symmetrical points x_ν both the problem of existence and uniqueness have a negative solution. Even they have exhibited that in the case of infinitely many solutions in theorem 1 for $n \geqq 3$ the general form of the solution is

(1. 4) $$f(x)=f_0(x)+Cf_1(x)$$

where $f_0(x)$ and $f_1(x)$ are fixed polynomials of degree $\leqq 2n-1$ and C arbitrary complex number.

But the situation changes if the number of distinct points $x_1, x_2, \dots, x_n$ is even. More precisely they they proved the following:

Theorem 2. (J. Surányi and P. Turán) *If $n=2k$, then to prescribed values y_ν and y_ν^* there is a uniquely determined polynomial $f(x)$ of degree $\leqq 2n-1$ such that*

$$f(x_\nu)=y_\nu,\ f''(x_\nu)=y_\nu^* \qquad (\nu=1, 2, \dots, n),$$

if x_ν's stand for the zeros of ultraspherical polynomials.

---

* At present the author is a Post Doctoral Fellow at the Department of Mathematics, University of Alberta, Edmonton.

The main object of this paper is to construct a simple example of a case of modified (0, 1, 3) lacunary interpolation on Tchebycheff abscissas of the first kind where the polynomial of interpolation does not exist uniquely for either n even or n odd. By modified (0, 1, 3) interpolation we mean that the value, first derivative, and third derivative of the function are prescribed at the zeros of $T_n(x)$ together with the value of $f(x)$ is also prescribed at ± 1.

More precisely we formulate the following:

Theorem 3. *Given a positive integer n and real numbers y_{k0} $(k=1, 2, \dots, n+2)$, y_{k1}, y_{k3} $(k=2, 3, \dots, n+1)$ there is, in general, no polynomial $p_{3n+1}(x)$ of degree $\leqq 3n+1$ such that*

$$(1.\,5) \qquad p_{3n+1}(x_k)=y_{k0}, \quad k=1, 2, \dots, n+2.$$

$$(1.\,6) \qquad p^{(i)}_{3n+1}(x_k)=y_{ki}, \quad k=2, 3, \dots, n+1, i=1, 3$$

and if there exists such a polynomial then there is an infinity of them.

Here x_k's are taken to be the zeros of $(1-x^2)T_n(x)$, $T_n(x)=\cos n\,\theta$, $\cos\theta=x$.

Proof. We shall show that in the case if all y_{k0}, y_{k1}, y_{k3} are zero, then there exists a polynomial $p(x)$ of degree $\leqq 3n+1$ (not identically zero) satisfying (1. 5) and (1. 6). Thus it follows from a well known theorem on the solution of a system of equations that in general there does not exist a unique polynomial $p_{3n+1}(x)$ of degree $\leqq 3n+1$ satisfying (1. 5) and (1. 6), and if they exist they are infinitely many.

From the conditions of the theorem, we have

$$(1.\,7) \qquad p_{3n+1}(x)=(1-x^2)T_n^2(x)q_{n-1}(x)$$

where $q_{n-1}(x)$ is a polynomial in x of degree $\leqq n-1$. As $p'''_{3n+1}(x_k)=0$, $k=2, 3, \dots, n+1$ we have $-x_k\,q_{n-1}(x_k)+(1-x_k^2)q'_{n-1}(x_k)=0$, for $k=2, 3, \dots, n+1$. But this means that

$$(1.\,8) \qquad (1-x^2)q'_{n-1}(x)-xq_{n-1}(x)=CT_n(x)$$

with a numerical C. Let

$$(1.\,9) \qquad q_{n-1}(x)=\sum_{i=0}^{n-1} C_i\cos i\theta, \quad \cos\theta=x$$

then

$$(1.\,10) \qquad q'_{n-1}(x)=\sum_{i=1}^{n-1} iC_i\frac{\sin i\theta}{\sin\theta}.$$

Therefore (1. 8) becomes

$$(1.\,11) \quad -2C_0\cos\theta+\sum_{i=1}^{n-1} C_i[(i-1)\cos(i-1)\theta-(i+1)\cos(i+1)\theta]=2C\cos n\theta.$$

Now comparing the co-efficients of $\cos n\theta, \cos(n-1)\theta, \dots$ and constant term on both sides of (1. 11) we obtain.

$$C_{n-2} = C_{n-4} = \ldots = C_4 = C_2 = C_0 = 0 \quad (n \text{ even}) \tag{1. 12}$$

$$C_{n-1} = C_{n-3} = \ldots = C_1 = -\frac{2C}{n} \quad (n \text{ even}) \tag{1. 13}$$

$$C_{n-2} = C_{n-4} = \ldots = C_3 = C_1 = 0 \quad (n \text{ odd}) \tag{1. 14}$$

$$C_{n-1} = C_{n-3} = \ldots = C_4 = C_2 = 2C_0 \quad (n \text{ odd}) \tag{1. 15}$$

Therefore using above relations we obtain

$$p_{3n+1}(x) = C(1-x^2)T_n^2(x) \sum_{k=1}^{\frac{n}{2}} \cos(2k-1)\theta, \qquad n \text{ even}, \tag{1. 16}$$

$$p_{3n+1}(x) = C(1-x^2)T_n^2(x) \left[-\frac{1}{2} + \sum_{k=1}^{\frac{n-1}{2}} \cos 2k\theta \right], \qquad n \text{ odd}. \tag{1. 17}$$

This shows that when n is even or odd there is in general no polynomial $p_{3n+1}(x)$ of degree $\leqq 3n+1$ which satisfies (1. 5) and (1. 6). In other words, there is a universal relationship between the values $\pi_{3n+1}(x_k)$ $(k=1, 2, \ldots, n+2)$, $\pi'_{3n+1}(x_k)$ and $\pi'''_{3n+1}(x_k)$ $(k=2, 3, \ldots, n+1$, for an arbitrary polynomial $\pi_{3n+1}(x)$ of degree $\leqq 3n+1$, the co-efficients being independent of $\pi_{3n+1}(x)$.

The author expresses his deep gratitude to Professor P. Turán for some valuable suggestions.

This paper was read at the meeting of the Indian Mathematical Congress, 1965, held at Jaipur.

References

[1] J. Balázs and P. Turán, Notes on interpolation. II, *Acta Math. Acad. Sci. Hung.* **8** (1957), 201—215.

[2] J. Balázs and P. Turán, Notes on interpolation. III, *ibid.* **9** (1958), 195—214.

[3] J. Balázs and P. Turán, Notes on interpolation. IV, *ibid.* **9** (1958), 243—258.

[4] G. Freud, Bemerkung über die Konvergenz eines Interpolationsfahrens von P. Turán, *ibid.* **9** (1958), 337—341.

[5] O. Kis, On trigonometric (0,2)-interpolation (Russian), *ibid.* **11** (1960), 256—276.

[6] O. Kis, Remarks on interpolation (Russian), *ibid.* **11** (1960), pp. 49—64.

[7] R. B. Saxena and A. Sharma, On some interpolatory properties of Legendre polynomials *ibid.* **9** (1958), 345—358.

[8] R. B. Saxena and A. Sharma, Convergence of interpolatory polynomials, *ibid.* **10** (1959), 157—175.

[9] J. Surányi and P. Turán, Notes on interpolation, I, *ibid.* **6** (1955), 67—79.

[10] A. K. Varma and A. Sharma, Some interpolatory properties of Tchebycheff polynomials (0,2) case modified. *Publ. Math. (Debrecen)* **8** (1961), 336—349.

[11] A. K. Varma and A. Sharma, Trigonometric Interpolation, *Duke Math.* **32** (1965), 341—358,

[12] A. K. Varma, On some interpolatory properties of Tchebycheff polynomials, *ibid.* **28** (1961). 449—462.

[13] A. K. Varma, On a problem of P. Turán on lacunary interpolation. *Canad. Math. Bull.* **10** (1967) 531—557.

(Received February 9, 1967.)