Pairwise compactness

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1. Introduction

KELLY [4] introduced the notion of a *bitopological space* (i.e. a triple (X, τ_1, τ_2) where X is a set and τ_1, τ_2 are two topologies on X), defined pairwise Hausdorff, pairwise regular, pairwise normal spaces and obtained generalizations of several standard results such as Urysohn's lemma, Tietze's extension theorem, Urysohn's metrization theorem and the Baire category theorem. FLETCHER [2] defined pairwise completely regular and pairwise uniform spaces and proved their equivalence. In a private communication Fletcher proposed the problem of suitably defining *pairwise compactness* for a bitopological space (X, τ_1, τ_2), with the property that the space can be pairwise uniform and pairwise compact, without τ_1 being equal to τ_2. In addition to the above requirement, pairwise compact spaces should satisfy some obvious results e.g. with the addition of pairwise Hausdorff property, such spaces be pairwise normal.

In this paper we give a definition of pairwise compact bitopological spaces and show that it solves Fletcher's problem. We also derive some related results.

In order to make this paper complete, we recall known definitions and theorems which are used in the sequel.

1.1. Definition. (WESTON [7]). In a bitopological space (X, τ_1, τ_2), we say that τ_1 *is coupled to* τ_2 iff for all $G \in \tau_1$, $\bar{G} \subset \bar{\bar{G}}$, where $\bar{G}, \bar{\bar{G}}$ denote the closures of G in τ_1, τ_2 respectively.

The following definition was given by Weston [7], who used the term *consistent*.

1.2. Definition. A bitopological space (X, τ_1, τ_2) is *pairwise Hausdorff* iff $x, y \in X$ and $x \neq y$ implies there exist $U \in \tau_1$, $V \in \tau_2$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

We will use p- to denote pairwise e.g. p-Hausdorff stands for pairwise Hausdorff.

1.3. **Lemma.** (Weston [7]) *If (X, τ_1, τ_2) is p-Hausdorff and τ_1 is coupled to τ_2, then τ_1 is Hausdorff.*

1.4. Definition. (Kelly [4]) A bitopological space (X, τ_1, τ_2) is (1. 2)-*regular* iff for each $x \in X$ there exists a τ_1-neighbourhood base of τ_2-closed sets. If in addition, it is (2.1)-regular then it is *p-regular*.

Kelly [4] defined p-normal bitopological spaces in an analogous manner: (X, τ_1, τ_2) is p-normal iff, given a τ_1-closed set A and a τ_2-closed set B with $A \cap B = \emptyset$, there exist a τ_2-open set U and a τ_1-open set V such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

1. 5. Definition. (Fletcher [2]) (X, τ_1, τ_2) is *p-completely regular* iff for any τ_i-closed set F and $x \in X-F$, there exists a τ_i-upper semicontinuous, τ_j-lower semi-continuous function $f: X \to [0, 1]$ $(i, j=1, 2;\ i \neq j)$ such that $f(x)=0$, $f(F)=1$.

CSÁSZÁR [1] first proved that every topological space is quasi-uniformizable. Subsequently, PERVIN [6] gave a direct proof of this result. Reference [5] contains a systematic exposition of quasi-uniform spaces.

1. 6. Definition. (Fletcher [2]) (X, τ_1, τ_2) is *p-uniform* iff there exists a quasi-uniformity $\mathcal{U}$ such that $\tau_1 = T(\mathcal{U})$ and $\tau_2 = T(\mathcal{U}^{-1})$, where $T(\mathcal{U})$ denotes the topology on X induced by $\mathcal{U}$.

1. 7. **Theorem.** (Fletcher [2]) *(X, τ_1, τ_2) is p-uniform iff it is p-complete ly regular.*

The above result is a generalization of the classical result of A. Weil namely, a topological space is uniformizable iff it is completely regular.

2. Pairwise compactness

2. 1. Definition. Let (X, τ_1, τ_2) be a bitopological space. Let V be a non-empty element of τ_2. Then $\tau_1(V) = \{\emptyset, X, \{U \cup V \text{ for } U \in \tau_1\}\}$ is a topology on X and is called *adjoint topology of τ_1 with respect to V.*

2. 2. Definition. (X, τ_1, τ_2) is (1, 2)-compact iff $\tau_1(V)$ is compact for every non-empty element V of τ_2. If in addition it is (2, 1)-compact we say that (X, τ_1, τ_2) is *p-compact*. In a (1, 2)-compact space, $\tau_1 \subset \tau_2$ implies (X, τ_1) is compact.

We may similarly define (1, 2)-compact and p-compact subsets of X.

The following examples show that our definition of p-compactness satisfies the requirement mentioned in Section 1.

2. 3. Example: Let X be the set of all real numbers and τ_i $(i=1, 2)$ be the topologies generated by sets of the form $\{x|x<a\}$ and $\{x|x>a\}$ respectively. Then (X, τ_1, τ_2) is p-compact but neither τ_1 nor τ_2 is compact.

2. 4. Example: Let N denote the natural numbers and $X = N \cup Ni$ where $i^2 = -1$. Let τ_1 be generated by $\{N \cup Fi\}$ where F is a finite subset of N and $\tau_2 = \{\emptyset, X, N, Ni\}$. Here (X, τ_2) is compact but (X, τ_1, τ_2) is not (1, 2)-compact.

2. 5. Example: Let X be the set of Example 2. 4. Let τ_1 be generated by $(N-F) \cup G$ where F is a finite subset of N and G an arbitrary subset of Ni. Let τ_2 be generated by $H \cup (Ni - Fi)$ where F is as before and H an arbitrary subset of N. Here (X, τ_1, τ_2) is p-Hausdorff and p-compact (hence p-uniform by 2. 19) but (X, τ_i) are not compact for $i=1, 2$. This answers Fletcher's question.

2. 6. Definition. A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is *p-continuous* iff $f: (X, \tau_1) \to (Y, \sigma_1)$ *and* $f: (X, \tau_2) \to (Y, \sigma_2)$ are continuous.

The following result is easily proved.

2. 7. **Lemma.** *(1, 2)-compactness and p-compactness are p-continuous invariants.*

2. 8. **Lemma** *In a (1, 2)-compact bitopological space (X, τ_1, τ_2) a τ_1-closed subset is (1, 2)-compact and a proper τ_2-closed subset is τ_1-compact and (1, 2)-compact.*

PROOF. Let C be a τ_1-closed subset of X and let $C \subset \bigcup_{i \in I} \{U_i \cup V | U_i \in \tau_1, V \in \tau_2\}$. Then $X = (X-C) \cup \bigcup_{i \in I} \{U_i \cup V\}$. $X - C \in \tau_1$ and X is (1, 2)-compact implies $X = (X-C) \cup \bigcup_{i=1}^{n} \{U_i \cup V\}$. This shows that $C \subset \bigcup_{i=1}^{n} \{U_i \cup V\}$ i.e. C is (1, 2)-compact. The second part of the statement is similarly proved.

2. 9. **Theorem.** *If (X, τ_1, τ_2) is p-compact then a proper τ_j-closed subset is p-compact and τ_i-compact $(i \neq j, 1, 2)$.*

2. 10. **Lemma.** *In a p-Hausdorff space (X, τ_1, τ_2) τ_i Hausdorff implies a (1, 2)-compact subset of X is τ_i-closed, $i = 1, 2$.*

2. 11. **Lemma.** *In a p-Hausdorff bitopological space a τ_i-compact subset is τ_j-closed, $(i \neq j)$.*

2. 12. Corollary. In a p-Hausdorff bitopological space (X, τ_1, τ_2), τ_j is coupled to τ_i implies that an (i, j)-compact subset of X is τ_j-closed, $(i \neq j)$.

2. 13. Definition. A subset C of X is (1, 2)-*separated* iff for $x \in C$, $y \in X - C$ there exist $U_x \in \tau_1$, $V_y \in \tau_2$ such that $x \in U_x$, $y \in V_y$ and $U_x \cap V_y = \emptyset$. In Example 2. 3 every τ_2-closed set is (1, 2)-separated. In a p-Hausdorff bitopological space every subset is p-separated.

2. 14. **Lemma.** *If a τ_j-compact subset is (j, i)-separated then it is τ_i-closed $(i \neq j)$.*

2. 15. **Theorem.** *If (X, τ_1, τ_2) is (1, 2)-compact and every τ_2-closed set is (1, 2)-separated, then (X, τ_1, τ_2) is (2, 1)-regular.*

PROOF. Let C be a τ_2-closed subset of X. Then C is τ_1-compact by Lemma 2. 8. If $p \notin C$, then by hypothesis there exist $V_{y_i} \in \tau_2$ and $U_{y_i} \in \tau_1$ for each $y_i \in C$ such that $y_i \in U_{y_i}$, $p \in V_{y_i}$, $U_{y_i} \cap V_{y_i} = \emptyset$.Since C is τ_1-compact, there exists $n \in N$, such that $C \subset \bigcup_{i=1}^{n} U_{y_i}$. Let $V_p = \bigcap_{i=1}^{n} V_{y_i}$. Then $V_p \cap U_{y_i} = \emptyset$ which proves the result.

2. 16. Corollary. If (X, τ_1, τ_2) is p-compact and a τ_i-closed set is (j, i)-separated then (X, τ_1, τ_2) is p-regular, $i \neq j$ $j, i = 1, 2$.

2.17. Corollary. A p-compact, p-Hausdorff bitopological space is p-regular. The following theorem is proved by using a familiar technique.

2.18. **Theorem.** *A p-compact, p-Hausdorff bitopological space is p-normal and hence is p-completely regular.* *)

2. 19. Corollary. A p-compact, p-Hausdorff bitopological space is p-uniform.

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*) REFEREE'S NOTE. The fact that a p-normal p-Hausdorff bitopological space is p-completely regular is capable of the following simple proof:

For A, B subsets of (X, τ_1, τ_2) put $A \prec B$ iff there exists a τ_1-closed F and a τ_2-open G such that $A \subset F \subset G \subset B$.

One easily sees that the relation $\prec$ so defined is a topogenous order, which forms by itself a topogenous structure. If we now apply to this topogenous structure the generalized Uryson's lemma of [1] (12.41), there results the existence of a function with the required properties.

References

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