Hereditary subradicals of the lower Baer radical

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1. A (non-empty) class M of (associative) rings is a *radical class* if it satisfies the two conditions:

(a) Every homomorphic image of a ring in M is again in M.

(b) If the ring K has the property that every nonzero homomorphic image of K has a nonzero ideal in M, K is in M.

(see [2] for this and related concepts).

If the class M satisfies the condition that every ideal of a ring in M is also in M, then M is said to be *hereditary*.

A class M which satisfies only (a) can always be embedded in a minimal containing radical class as follows [4]; [5]:

Let $M_1 = M$; if M_α is defined for all ordinals $1 \leqq \alpha < \beta$, define M_β to consist of all rings K such that every nonzero homomorphic image of K contains a nonzero ideal in M_α for some ordinal $\alpha < \beta$. The class LM consisting of all rings which belong to some M_α is the desired class, called the *(Kurosh) lower radical* class.

When M consists of all zero rings (i.e., rings R for which $R^2 = 0$) then $LM = M_2$ [4], and LM is called the *lower Baer radical* class, which we denote by B. If we let Z_0 be the zero ring on the infinite cyclic group and let P consists of all homomorphic images of Z_0, then $LP = B$ [4]. Thus the lower Baer radical is generated by a single ring, and we write $L(Z_0) = B$. Our purpose is to show that every hereditary radical subclass of B enjoys this property.

2. For a prime p, let Z_p denote the zero ring on the cyclic group of order p.

Theorem 1. *Let $R \subseteqq B$ a hereditary radical class. Then*

(i) *$R = B$ if and only if R contains a zero ring having a nonzero element of infinite additive order*; *hence $R = L(Z_0)$.*

(ii) *If no zero ring in R has elements of infinite additive order, then $R = L(K)$ where $K = \oplus \sum Z_p$, $Z_p \in R$.*

Conversely, for any non-empty set Q of (distinct) primes, if $K = \oplus \sum_{p \in Q} Z_p$, then LH is a hereditary radical subclass of B, where H consists of all homomorphic images of K.

PROOF. (i) If $R = B$, then $Z_0 \in R$. Conversely, suppose $A \in R$ is a zero ring having an element a of infinite additive order. Since A is a zero ring, every additive subgroup of A is an ideal of A. In particular the cyclic group (a) generated by a is an ideal of A, and $(a) \cong Z_0$. Thus $Z_0 \in R$ so that $B = L(Z_0) \subseteqq R$.

(ii) Assume no zero ring in R has elements of infinite additive order. Note that R contains a zero ring, since if $K \in R$ then $K \in M_2$, so that K contains an ideal I which is a zero ring; hence $I \in R$, since R is hereditary. Then R contains Z_p for some prime p, as follows. If $0 \neq x \in I$, then $px = 0$ for some minimal integer $p > 0$ and, in fact, we can assume p is a prime. Then $(x) \cong Z_p$, (x) is an ideal of R, so $Z_p \in R$. Let Q consist of all distinct primes p for which $Z_p \in R$ and let $K = \oplus \sum_{p \in Q} Z_p$ (we choose one representative from each isomorphism class determined by a Z_p). If $\emptyset(K)$ is a nonzero image of K, then $\emptyset(Z_p) \neq 0$ for some p, since we are dealing with the direct *sum* (restricted). Now Z_p is simple so $\emptyset(Z_p) \cong Z_p$. Thus $K \in R$ by (b). Letting H consist of all homomorphic images of K, we have $L(K) = LH \subseteq R$. On the other hand if $A \in R$ and $\emptyset(A)$ is a nonzero image of R, then $\emptyset(A)$ has a nonzero ideal I, with $I^2 = 0$. But then $I \in R$, and as above, every nonzero image of I has an ideal isomorphic to Z_p for some p. Since each $Z_p \in H$, $I \in H_2$ and hence $A \in H_3 \subseteq L(K)$. Thus $R = L(K)$.

Now let $Q = \{p_i\}$ be any non-empty set of distinct primes, enumerated so that $p_i < p_{i+1}$, and let $K = \oplus \sum_{p_i \in Q} Z_{p_i}$. We first show that S is a homomorphic image of K if and only if S is isomorphic to an ideal of K. Thus let I be a nonzero ideal of K and let $0 \neq x \in I$. Then x is an infinite-tuple $x = (x_i)$ where $x_i \in Z_{p_i}$ and $x_i = 0$ for all but a finite number of x_i. Choose j minimal so that $x_k = 0$ for all $k > j$. Then $x = (x_1, \ldots, x_j, 0, \ldots)$ and so $p_1 \ldots p_{j-1} x = (0, \ldots, p_1 \ldots p_{j-1} x_j, \ldots) \neq 0$. Now any nonzero element of Z_p generates Z_p additively for any p, so we get $Z_{p_j} \subseteq I$. If $x_{j-1} \neq 0$, after subtraction of a suitable member of Z_{p_j} we repeat to get $Z_{p_{j-1}} \subseteq I$. Thus $Z_{pi} \subseteq I$ for any $i \leqq j$ for which $x_i \neq 0$. Let $A = \{p_j \in Q$: the j^{th} component of every $x \in I$ is zero$\}$. Then for $p_t \notin A$, $Z_{p_t} \subseteq I$, so $\oplus \sum_{p_t \in A} Z_{p_t} \subseteq I$. However I is obviously contained in this sum so that $K \cong I \oplus J$, where J is the sum taken over those $p_t \in A$. Thus $K/I \cong J$ and $K/J \cong I$, and this suffices to establish our claim. In addition, if S is a homomorphic image of K, then since S is a direct sum of Z_p's , a similar argument shows that any ideal of S is a direct sum of Z_p's. Thus H consisting of all homomorphic images of K is hereditary. By ([3], Theorem 1) we can assert that $L(K) \subseteq B$ is a hereditary radical class and this completes the proof.

We remark that in case $Q = \{p\}$ then $L(Z_p) = B \cap G_p$, where G_p is the hereditary radical class consisting of all rings whose additive group is p-primary ([1], Lemma 1). We will make use of this for our next result, as well as the following fact from the general theory, which can be found in [2]. If P is a radical class, in any ring A, the sum of all P-ideals of A (i.e., ideals of A belonging to the class P), gives a unique maximal P-ideal, called the *P-radical* of A, which we denote by $p(A)$.

Let Q be a non-empty set of primes, K, H as in Theorem 1. For a ring A let $h_a(A)$ denote the LH-radical of A, and $h_p(A)$ the $L(Z_p)$-radical of A for any $p \in Q$. Since $h_p(A) \in G_p$, it follows that for any $p \in Q$, $h_p(A) \cap \left(\sum_{p \neq q} h_p(A) \right) = 0$.

Theorem 2. *For any ring* A, $h_Q(A) = \oplus \sum_{p \in Q} h_p(A)$.

PROOF. The above observation implies that the sum of the $h_p(A)$ is direct. Further for each $p \in Q$, $h_p(A) \in L(Z_p) \subseteq LH$, so $h_p(A) \subseteq h_Q(A)$. Then $I = \oplus \sum_{p \in Q} h_p(A) \subseteq$ $\subseteq h_Q(A)$. For the opposite inclusion we first show that if $R \in LH$, then the additive group of R is torsion. Thus if $t(R)$ is the torsion subgroup of R, then $t(R)$ is an ideal of R and $R/t(R)$ has no nonzero torsion elements. In [1], it was shown that if a class M is hereditary and homomorphically closed then $LM=M_3$. Thus if $R/t(R)\neq 0$, then, since $R/t(R)\in H_3$, it contains an ideal $0\neq I_2\in H_2$, and in turn I_2 contains an ideal $0\neq I_1\in H_1=H$. Hence I_1, being a homomorphic image of $\oplus\Sigma Z_p$, contains at least one Z_p, and this contradicts $t(R/t(R))=0$. It now follows that $h_Q(A)$ is a group direct sum of its p-primary components A_p, each of which is an ideal of $h_Q(A)$. It is easily verified that $A_p=h_p(A)\cap h_Q(A)$, since $h_p(A)\in G_p$ and hence each A_p is an ideal of A contained in $h_p(A)$. Thus $h_Q(A)\subseteqq I$.

References

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