# On the total curvature of closed curves in Riemannian manifolds

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Several proofs are known for the following theorem of W. FENCHEL: If $\varkappa(G)$ is the total curvature of a closed curve $G$ in a euclidean space, then $\varkappa(G) \geqq 2\pi$, and the equality holds if and only if $G$ is a convex plane curve. [1]) In this paper we show that shat the basic idea of one of the proofs of the above theorem [2]) can be applied to solve the corresponding problem for curves in Riemannian manifolds. The following theorem is proved: *If $M$ is a complete simply connected Riemannian manifold with everywhere nonpositive sectional curvature, and $\varkappa(G)$ is the total curvature of a closed curve $G$ in $M$, then $\varkappa(G) \geqq 2\pi$; the equality holds if and only if $G$ is the boundary of a 2-dimensional totally geodesic submanifold isometric with a convex domain of the euclidean plane.* Dropping either of the assumptions that $M$ is simply connected and that it has everywhere nonpositive sectional curvature would not leave the lower bound $2\pi$ valid, as a closed geodesic of a cylinder and a sufficiently small geodesic triangle in case of positive sectional curvature show. For convenience $M$ is assumed to be of class $C^\infty$ throughout the whole paper.

## 1. Riemannian manifolds with everywhere nonpositive sectional curvature

Let $M$ be a complete simply connected Riemannian manifold with everywhere nonpositive sectional curvature. The following facts are wellknown:

1. If $m \in M$, then $\exp_m: M_m \to M$ is a diffeomorphism of the tangent space $M_m$ onto $M$.

2. If $p, q \in M$, then there is exactly one geodesic $\sigma_{pq}: [0, \alpha] \to M$ with $\sigma_{pq}(0) = p$, $\sigma_{pq}(\alpha) = q$.

3. If $\sigma: [0, \alpha] \to M$ is a geodesic and $v \in M_{\sigma(\alpha)}$, then there is exactly one Jacobi field $V: [0, \alpha] \to TM$ along $\sigma$ with $V(0) = 0$ and $V(\alpha) = v$.

The following lemma is a special case of Rauch's comparison theorem. [3])

**Lemma 1. 1.** *Let $M$ be a complete Riemannian manifold with everywhere nonpositive sectional curvature, $\sigma: [0, \alpha] \to M$ a geodesic, $\sigma(0) = m$, $v \in M_{\sigma(\alpha)}$, $x \in T(M_m)$*

[1]) FENCHEL [3].

[2]) BORSUK [2], MILNOR [4].

[3]) BISHOP—CRITTENDEN [1], 177—179. Conditions as to the equality are not explicitely stated there, but they are implicitely contained in the proof.

*and* $v = d\exp_m x$. *Let further* $V\colon [0, \alpha] \to TM$ *be the Jacobi field along* $\sigma$ *with* $V(0) = 0$ *and* $V(\alpha) = v$ *and* $X\colon [0, \alpha] \to T(M_m)$ *the field defined by* $V(t) = d\exp_m X(t)$, $t \in [0, \alpha]$. *Then* $\|v\| \geqq \|x\|$, *and in order the equality* $\|v\| = \|x\|$ *to be valid the following conditions are necessary if* $v \neq 0$:

a) $\|V(t)\| = \|X(t)\|$, $t \in [0, \alpha]$.

b) $\dfrac{V(t)}{\|V(t)\|}$, $t \in (0, \alpha]$ *is a parallel field along* $\sigma$.

c) *The Riemannian curvature of the plane section spanned by the tangent vector* $\sigma_*(t)$ *and* $V(t)$ *is zero for* $t \in (0, \alpha]$.

The length of a curve $\varphi$ is denoted by $L(\varphi)$ and the set of its points by $|\varphi|$. Let $p, q, r$ be points of a complete simply connected Riemannian manifold with everywhere nonpositive sectional curvature and $\sigma_{pq}, \sigma_{qr}, \sigma_{rp}$ the geodesics joining them. The set $T_{pqr} = |\sigma_{pq}| \cup |\sigma_{qr}| \cup |\sigma_{rp}|$ is called a *geodesic triangle* and the set $\Delta^p_{qr} = \{x\colon x \in |\sigma_{pz}|, z \in |\sigma_{qr}|\}$ a *geodesic triangular cell*. In general $\Delta^q_{pr}, \Delta^p_{qr}, \Delta^r_{pq}$ are different. By a *euclidean representation* of $T_{pqr}$ a point triple $\{P, Q, R\}$ of a euclidean plane is meant for which $\overline{PQ} = L(\sigma_{pq})$, $\overline{QR} = L(\sigma_{qr})$, $\overline{RP} = L(\sigma_{rp})$ hold.

The following lemma is a special case of one of A. D. Alexandrov's comparison theorems.

**Lemma 1. 2.** *Let* $M$ *be a complete simply connected Riemannian manifold with everywhere nonpositive sectional curvature. If* $p, q, r \in M$ *are different and* $\{P, Q, R\}$ *is a euclidean representation of* $T_{pqr}$, *then the angle of the geodesics* $\sigma_{pq}, \sigma_{pr}$ *is not greater than* $QPR\sphericalangle$; *the two angles are equal if and only if* $\Delta^p_{qr} = \Delta^q_{pr} = \Delta^r_{pq}$ *and this geodesic triangular cell is totally geodesic and isometric with the triangular domain* $PQR\,\Delta$.

A proof of this lemma can evidently be given by the method of Y. Tsukamoto [4]) on the basis of Lemma 1.1.

## 2. Concepts for the total curvature of curves in Riemannian manifolds

Let $a_0, a_1, \ldots, a_m$, $m \geqq 2$ be such points of a Riemannian manifold $M$ that $a_i, a_{i+1}$ are different and can be joined by a unique minimal geodesic $\sigma_i$, $i = 0, 1, \ldots, m-1$; then the ordered set $P = \{a_0, a_1, \ldots, a_m\}$ is called a *geodesic polygon*. The points $a_i$ are called the *vertices* of $P$ and the geodesics $\sigma_i$ its *sides*. The polygon is said to be *closed* or *open* according to whether $a_m = a_0$, or $a_m \neq a_0$. In case of a closed polygon $a_{-1} = a_{m-1}$, $a_{m+1} = a_1$ are defined too. The angle $\gamma_i$ of the geodesics $\sigma_{i-1}$, $\sigma_i$ is called the *angle* of $P$ at $a_i$, where $i = 1, \ldots, m-1$ if $P$ is open, and $i = 1, \ldots, m$ when it is closed. By the *total curvature* $\varkappa(P)$ of $P$ the sum of its angles is meant.

Let $\varphi\colon [0, \alpha] \to M$ be a representation of a continuous curve $G$ of $M$ and $0 = \tau_0 < \tau_1 < \ldots < \tau_m = \alpha$, $m \geqq 2$ such a subdivision of $[0, \alpha]$ that $\{\varphi(\tau_0), \varphi(\tau_1), \ldots, \varphi(\tau_m)\}$ is a geodesic polygon $P$, then $P$ is said to be *inscribed* in $G$. If $\{P_l\}_{l=1,2,\ldots}$ is such a sequence of geodesic polygons inscribed in $G$ that the maximal length

[4]) Tsukamoto [5].

of the sides of $P_l$ tends to zero as $l\to\infty$, then it is called an *approximating sequence*. If for any approximating sequence $\{P_l\}_{l=1,2,\dots}$ of geodesic polygons inscribed in $G$ the corresponding sequence $\{\varkappa(P_l)\}_{l=1,2,\dots}$ is convergent, then $G$ is said to have a total curvature, and $\lim_{l\to\infty}\varkappa(P_l)$, which is then the same value for all approximating sequences, is called the *total curvature* $\varkappa(G)$ of $G$.

The total curvature of curves is usually defined as follows: Let $G$ be a curve of class $C^2$ in $M$, $\varphi\colon[0,\alpha]\to M$ the representation of $G$ in terms of arc length and $\varkappa(s)$ the first curvature of $G$ in the point $\varphi(s)$, $0\leqq s\leqq\alpha$, then $\bar{\varkappa}(G)=\int_0^\alpha\varkappa(s)ds$ is called the total curvature of $G$. A proof is required therefore that if $G$ is of class $C^2$ then $\varkappa(G)$ exists and is equal to $\bar{\varkappa}(G)$.

Let $G$ be a curve of class $C^1$ in $M$ and $\varphi\colon[0,\alpha]\to M$ the representation of $G$ in terms of arc length. Denote by $\varphi_*(t',t'')$ the vector obtained by parallel translation of the tangent vector $\varphi_*(t')$ along $\varphi$ to the point $\varphi(t'')$ for $t',t''\in[0,\alpha]$. Let $\gamma(t',t'')$ be the angle of the vectors $\varphi_*(t',t'')$, $\varphi_*(t'')$, then $\gamma(t',t'')$, $0\leqq t'\leqq t''\leqq\alpha$ defines an interval function; if it is integrable, then $\int_0^\alpha\gamma(t',t'')$ is denoted by $\varkappa^0(G)$.

**Lemma 2. 1.** *Let $G$ be a curve of class $C^2$ in a Riemannian manifold, then $\varkappa^0(G)=\bar{\varkappa}(G)$.*

Since

$$\lim_{t',t''\to t}\frac{\gamma(t',t'')}{t''-t'}=\lim_{t',t''\to t}\frac{\|\varphi_*(t',t'')-\varphi_*(t'')\|}{t''-t'}=\|\nabla_{\varphi^*(t)}\varphi_{*(t)}\|=\varkappa(t)$$

holds, the interval function $\gamma(t',t'')$ is differentiable and $\varkappa(t)$, $0\leqq t\leqq\alpha$ is its derivative. Consequently the integrability of $\gamma(t',t'')$ and $\varkappa^0(G)=\bar{\varkappa}(G)$ are implied by standard theorems.

**Lemma 2. 2.** *If $G$ is a curve of class $C^2$ in a Riemannian manifold $M$, then $\varkappa(G)$ exists and it is equal to $\varkappa^0(G)$.*

Let $\varphi\colon[0,\alpha]\to M$ be the representation of $G$ in terms of arc length, and $\delta>0$, then $0=\tau_0<\tau_1<\dots<\tau_m=\alpha$ is called a *δ-subdivision* of $[0,\alpha]$, if $\tau_k-\tau_{k-1}\leqq\delta$ for $k=1,\dots,m$. To prove the lemma it suffices to show that to any $\varepsilon>0$ there is such a $\delta>0$, that for any $\delta$-subdivision $0=\tau_0<\tau_1<\dots<\tau_m=\alpha$, $m\geqq2$ of $[0,\alpha]$ the geodesic polygon $P=\{\varphi(\tau_0),\varphi(\tau_1),\dots,\varphi(\tau_m)\}$ exists and

$$\left|\varkappa(P)-\sum_{k=1}^{m-1}\gamma(\tau_{k-1}^0,\ \tau_k^0)\right|<\varepsilon$$

holds, where $\tau_k^0=\frac{\tau_k+\tau_{k+1}}{2}$, $k=0,1,\dots,m-1$.

It is enough to consider the case when $G$ has no multiple points. A coordinate system $v\colon U\to R^n$ of class $C^2$ exists therefore, which is defined on a compact neighborhood $U$ of $|\varphi|$ and such that the coefficients of the Riemannian connexion of $M$ all vanish when calculated in $v$ for points of $G$. Let $g_{ij}(x)$, $i,j=1,\dots,n$, $x\in U$ be the components of the fundamental tensor in $v$ and $S=\sup\{|g_{ij}(x)|\colon i,j=$

$=1, \dots, n, x \in U\}$. Let $\varphi^i(t)$ $i=1, \dots, n$, $t \in [0, \alpha]$ denote the coordinates of $\varphi(t)$ in $v$. Since $G$ is of class $C^2$, to any $\varepsilon > 0$ there is such a $\delta_1 > 0$, that

$$\left| \frac{\varphi^i(t'') - \varphi^i(t')}{t'' - t'} - \dot{\varphi}^i(t^0) \right| < \frac{\varepsilon}{8\alpha n \sqrt{S}} |t'' - t'|,$$

if

$$0 \leqq t', t'' \leqq \alpha, \; 0 < |t'' - t'| \leqq \delta_1, \; t^0 = \frac{t' + t''}{2}.$$

A map $\psi: [0, \xi] \to U$ is called a *v-coordinate segment* if $v \circ \psi$ is linear. The sequence $\psi_k: [0, \xi_k] \to U$, $k=0, 1, \dots, m-1$, $m \geqq 2$ of $v$-coordinate segments is called a *v-coordinate polygon* if $\psi_{k-1}(\xi_{k-1}) = \psi_k(0)$ for $k=1, \dots, m-1$. The $v$-coordinate segments $\psi_k$, $k=0, 1, \dots, m-1$ are called *sides* and their endpoints *vertices* of the polygon. There exists such a $\delta_2 > 0$, that for any $\delta_2$-subdivision $0 = \tau_0 < \tau_1 < \dots < \tau_m = \alpha$, $m \geqq 2$ the $v$-coordinate polygon with the consecutive vertices $\varphi(\tau_0)$, $\varphi(\tau_1), \dots, \varphi(\tau_m)$ exists. Consider a $\zeta$-subdivision $0 = \tau_0 < \tau_1 < \dots < \tau_m = \alpha$, $m \geqq 2$ with $0 < \zeta < \delta_1, \delta_2$ and the corresponding $v$-coordinate polygon $\psi_k: [0, 1] \to U$, $k=0, 1, \dots, m-1$ with the consecutive vertices $\varphi(\tau_0), \varphi(\tau_1), \dots, \varphi(\tau_m)$. Let $\bar{\gamma}_k$ be the angle of the sides $\psi_{k-1}, \psi_k$, as well as $\gamma'_k$ the angle of the vectors $\psi_{k-1*}(1)$, $\varphi_*(\tau^0_{k-1}, \tau_k)$ and $\gamma''_k$ the angle of the vectors $\psi_{k*}(0)$, $\varphi_*(\tau^0_k, \tau_k)$ for $k=1, \dots, m-1$. Put $\lambda^i = \dfrac{\varphi^i(\tau_k) - \varphi^i(\tau_{k-1})}{\tau_k - \tau_{k-1}}$ for $i=1, \dots, n$, $k=1, \dots, m$. In consequence to assertions above

$$\sum_{k=1}^{m-1} (\sin \gamma'_k + \sin \gamma''_k) \leqq \sum_{k=1}^{m-1} \left( \sqrt{\sum_{i,j=1}^{n} g_{ij}(\varphi(\tau_k)) (\lambda^i_k - \dot{\varphi}^i(\tau^0_{k-1})) (\lambda^j_k - \dot{\varphi}^j(\tau^0_{k-1}))} + \right.$$

$$\left. + \sqrt{\sum_{i,j=1}^{n} g_{ij}(\varphi(\tau_k)) (\lambda^i_{k+1} - \dot{\varphi}^i(\tau^0_k)) (\lambda^j_{k+1} - \dot{\varphi}^j(\tau^0_k))} \right) \leqq$$

$$\leqq \sum_{k=1}^{m-1} \left( \sqrt{\sum_{i,j=1}^{n} |S| \left( \frac{\varepsilon}{8\alpha n \sqrt{S}} (\tau_k - \tau_{k-1}) \right)^2} + \sqrt{\sum_{i,j=1}^{n} |S| \left( \frac{\varepsilon}{8\alpha n \sqrt{S}} (\tau_{k+1} - \tau_k) \right)^2} \right) < \frac{\varepsilon}{4}.$$

By the assumptions of the lemma there is such a $\delta_3 > 0$, that for any $\delta_3$-subdivision of $[0, \alpha]$ the inequalities $\gamma'_k, \gamma''_k < \pi/2$, $k=1, \dots, m-1$ hold. Therefore

$$\left| \sum_{k=1}^{m-1} \bar{\gamma}_k - \sum_{k=1}^{m-1} \gamma(\tau^0_{k-1}, \tau^0_k) \right| \leqq \sum_{k=1}^{m-1} |\bar{\gamma}_k - \gamma(\tau^0_{k-1}, \tau^0_k)| \leqq \sum_{k=1}^{m-1} (\gamma'_k + \gamma''_k) \leqq$$

$$\leqq 2 \sum_{k=1}^{m-1} (\sin \gamma'_k + \sin \gamma''_k) < \frac{\varepsilon}{2}$$

for any $\zeta$-subdivision of $[0, \alpha]$ with $0 < \zeta < \delta_1, \delta_2, \delta_3$.

Let $\varrho$ be the distance function of $M$. There is a neighbourhood $V \subset U$ of $|\varphi|$ and an $\eta > 0$ such that if $x, y \in V$ and $\varrho(x, y) = \beta < \eta$, then a unique minimal geodesic $\sigma: [0, \beta] \to U$ and the $v$-coordinate segment $\psi: [0, \beta] \to U$ both joining $x, y$ exist. [5])

[5]) Bishop—Crittenden [1], 246—250.

Let $\dot{\sigma}^i(t)$ and $\dot{\psi}^i(t)$ denote for $i=1, \ldots, n$, $t \in [0, \beta]$ the coordinates of the vectors $\sigma_*(t)$, $\psi_*(t)$ respectively, then

$$|\dot{\sigma}^i(z)-\dot{\psi}^i(z)-[\dot{\sigma}^i(t)-\dot{\psi}^i(t)]| \leqq \int_0^\beta \sum_{k,l=1}^n |\Gamma^i_{kl}\dot{\sigma}^k(\tau)\dot{\sigma}^l(\tau)|\,d\tau$$

for $0 \leqq t \leqq \beta$, $z=0, \beta$. There is a $t_i$ with $0 \leqq t_i \leqq \beta$ for every $i=1, \ldots, n$, such that $\dot{\sigma}^i(t_i)-\dot{\psi}^i(t_i)=0$. The lengths of the geodesic polygons inscribed in $G$ have an upper bound $L<\infty$. The absolute values of coordinates of tangent vectors to geodesics in points of $U$ have an upper bound $N<\infty$. There is a neighbourhood $W \subset V$ of $|\varphi|$ such that for points in $W$ the inequalities $|\Gamma^i_{kl}| \leqq \dfrac{\varepsilon}{8n^3\sqrt{S}LN^2}$, $i, k, l=1, \ldots n$ hold. Let $x, y \in |\varphi|$ be such points that $\varrho(x, y)=\beta<\eta$, and $|\sigma| \subset W$, then by the preceeding assertions $|\dot{\sigma}^i(z)-\dot{\psi}(z^i)| \leqq \dfrac{\varepsilon}{8n\sqrt{S}L}\beta$, $i=1, \ldots, n$. There is a $\delta_4>0$ such that for any $\delta_4$-subdivision of $[0, \alpha]$ the corresponding geodesic and $\nu$-coordinate polygons lie in $W$ and the distances of the consecutive vertices are less than $\eta$. Denote $\sigma_k\colon [0, \beta_k] \to W$, $\psi_k\colon [0, \beta_k] \to W$, $k=0, 1, \ldots, m-1$ the sides of these polygons. Let $\bar{\gamma}_k'$ be the angle of the vectors $\sigma_{k-1*}(\beta_{k-1})$, $\psi_{k-1*}(\beta_{k-1})$ and $\bar{\gamma}_k''$ that of $\sigma_{k*}(0)$, $\psi_{k*}(0)$ for $k=1, \ldots, m-1$. By the assertions above

$$\sum_{k=1}^{m-1} (\sin\bar{\gamma}_k'+\sin\bar{\gamma}_k'') \leqq$$

$$\leqq \sum_{k=1}^{m-1}\left(\sqrt{\sum_{i,j=1}^n g_{ij}(\varphi(\tau_k))\big(\dot{\sigma}^i_{k-1}(\beta_{k-1})-\dot{\psi}^i_{k-1}(\beta_{k-1})\big)\big(\dot{\sigma}^j_{k-1}(\beta_{k-1})-\dot{\psi}^j_{k-1}(\beta_{k-1})\big)}\,+\right.$$

$$\left.+\sqrt{\sum_{i,j=1}^n g_{ij}(\varphi(\tau_k))\big(\dot{\sigma}^i_k(0)-\dot{\psi}^i_k(0)\big)\big(\dot{\sigma}^j_k(0)-\dot{\psi}^j_k(0)\big)}\right) \leqq$$

$$\leqq \sum_{k=1}^{m-1}\left(\sqrt{\sum_{i,j=1}^n |S|\left(\frac{\varepsilon}{8n\sqrt{S}L}\varrho\big(\varphi(\tau_{k-1}), \varphi(\tau_k)\big)\right)^2}\,+\right.$$

$$\left.+\sqrt{\sum_{i,j=1}^n |S|\left(\frac{\varepsilon}{8n\sqrt{S}L}\varrho\big(\varphi(\tau_k), \varphi(\tau_{k+1})\big)\right)^2}\right) < \frac{\varepsilon}{4}.$$

By the assumptions of the lemma there is a $\delta_5>0$ such that for any $\delta_5$-subdivision of $[0, \alpha]$ the inequalities $\bar{\gamma}_k', \bar{\gamma}_k''<\dfrac{\pi}{2}$, $k=1, \ldots, m-1$ hold. Consequently for any $\zeta$-subdivision, with $0<\zeta<\delta_4, \delta_5$ the inequalities

$$\left|\sum_{k=1}^{m-1}(\bar{\gamma}_k-\gamma_k)\right| \leqq \sum_{k=1}^{m-1}|\bar{\gamma}_k-\gamma_k| \leqq \sum_{k=1}^{m-1}(\bar{\gamma}_k'+\bar{\gamma}_k'') \leqq 2\sum_{k=1}^{m-1}(\sin\bar{\gamma}_k'+\sin\bar{\gamma}_k'') < \frac{\varepsilon}{2}$$

are valid. Let $0<\delta<\delta_j$ be for $j=1, \ldots, 5$, then

$$\left|\varkappa(P)-\sum_{k=1}^{m-1}\gamma(\tau^0_{k-1}, \tau^0_k)\right| \leqq \left|\sum_{k=1}^{m-1}\gamma_k-\bar{\gamma}_k\right|+\left|\sum_{k=1}^{m-1}\bar{\gamma}_k-\sum_{k=1}^{m-1}\gamma(\tau^0_{k-1}, \tau^0_k)\right| < \varepsilon$$

for any $\delta$-subdivision of $[0, \alpha]$.

### 3. The total curvature of closed curves in Riemannian manifolds with everywhere nonpositive sectional curvature

In what follows $M$ is assumed to be a complete simply connected Riemannian manifold with everywhere nonpositive sectional curvature.

**Lemma 3. 1.** *Let* $P=\{a_0, \dots, a_{i-1}, a_i, \dots, a_m\}$ *and* $P'=\{a_0, \dots, a_{i-1}, a, a_i, \dots, a_m\}$ $m \geqq 2$ *be closed geodesic polygons of* $M$ *with sides* $\sigma_k\colon [0, \alpha_k] \to M$, $k=0, 1, \dots, m-1$ *and* $\sigma_k\colon [0, \alpha_k] \to M$, $k=0, 1, \dots, i-2$, $\sigma'_{i-1}\colon [0, \alpha'_{i-1}] \to M$, $\sigma\colon [0, \alpha] \to M$, $\sigma_k$: $[0, \alpha_k] \to M$, $k=i, \dots, m-1$, *then* $\varkappa(P) \leqq \varkappa(P')$; *the equality holds if and only if* $\Delta^{a}_{a_{i-1}a_i}$ *is a totally geodesic triangular cell isometric with one in the euclidean plane, and* $\sigma'_{i-1*}(0) = u\sigma_{i-2*}(\alpha_{i-2}) + v\sigma_{i-1*}(0)$, $\sigma_*(\alpha) = u'\sigma_{i-1*}(\alpha_{i-1}) + v'\sigma_{i*}(0)$ *with* $0 \leqq u, v, u' v'$.

Let $\gamma_0, \gamma_1, \dots, \gamma_{i-1}, \gamma_i, \dots, \gamma_{m-1}$ denote the angles of $P$ and $\gamma_0, \gamma_1, \dots, \gamma'_{i-1}$, $\gamma, \gamma'_i, \dots, \gamma_{m-1}$ those of $P'$. Let $\xi$ be the angle of the vectors $\sigma'_{i-1*}(0)$, $\sigma_{i-1*}(0)$ and $\eta$ that of the vectors $\sigma_*(\alpha)$, $\sigma_{i-1*}(\alpha_{i-1})$, then $\varkappa(P') = \varkappa(P) + \gamma'_{i-1} - \gamma_{i-1} + \gamma + {}$ ${}+ \gamma'_i - \gamma_i \geqq \varkappa(P) + \xi + \gamma'_{i-1} - \gamma'_{i-1} + \eta + \gamma'_i - \gamma_i \geqq \varkappa(P)$ by Lemma. 1. 2. It can be verified by the same lemma that the above condition of equality is necessary and sufficient.

**Lemma 3. 2.** *Let* $P=\{a_0, a_1, \dots, a_m\}$, $m \geqq 2$ *be a closed geodesic polygon of* $M$, *then* $\varkappa(P) \geqq 2\pi$; *the equality holds if and only if* $P$ *is the boundary of a totally geodesic submanifold isometric with a convex polygonal domain of the euclidean plane.*

Consider the closed geodesic polygons $P_1=\{a_0, a_{m-2}, a_{m-1}, a_m\}$, $P_2=\{a_0, a_{m-3}, a_{m-2}, a_{m-1}, a_m\}$, $\dots$, $P_m = P$, then by Lemmas 1. 2 and 3.1 $2\pi \leqq \varkappa(P_1) \leqq \varkappa(P_2) \leqq \dots \leqq$ $\dots \leqq \varkappa(P)$. Assume that $\varkappa(P)=2\pi$, then by an obvious mathematical induction and by the same lemmas it can be shown that

$$[P] = \Delta^{a_{m-2}}_{a_0 a_{m-1}} \cup \Delta^{a_{m-3}}_{a_0 a_{m-2}} \cup \dots \cup \Delta^{a_1}_{a_0 a_2}$$

is a totally geodesic 2-dimensional submanifold with boundary $P$ and mapped isometrically by $\exp_{a_0}^{-1}$ onto a convex polygonal domain in $M_{a_0}$. The submanifold $[P]$ is unique.

**Theorem.** *Let* $G$ *be a closed curve of class* $C^2$ *in a complete simply connected Riemannian manifold* $M$ *with everywhere nonpositive sectional curvature, and* $\varkappa(G)$ *the total curvature of* $G$, *then* $\varkappa(G) \geqq 2\pi$; *equality holds if and only if* $G$ *is the boundary of a totally geodesic submanifold of* $M$ *isometric with a convex domain of the euclidean plane.*

There is an approximating sequence $\{P_l\}_{l=1,2,\dots}$ of closed geodesic polygons inscribed in $G$, such that $P_l$, $P_{l+1}$ $l=1, 2, \dots$ are in the same relation as $P$, $P'$ of Lemma 3. 1, and $a \in |\varphi|$ is a common vertex of all of them. Consequently $2\pi \leqq \varkappa(P_l) \leqq$ $\leqq \varkappa(P_{l+1})$, and $\varkappa(G) = \lim_{l\to\infty} \varkappa(P_l) \geqq 2\pi$. Assume that $\varkappa(K)=2\pi$, then $\varkappa(P_l)=2\pi$, $l=1, 2, \dots$. Let $[P_l]$ denote the totally geodesic submanifold bounded by $P_l$. By Lemmas 1. 2, 3.1, 3. 2 one can show that $\exp_a^{-1}(\bigcup_{l=1}[P_l])$ is a convex plane domain mapped isometrically by $\exp_a$ onto a totally geodesic submanifold bounded by $G$.

## References

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