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Second order parallel tensors on P-Sasakian manifolds

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Dedicated to the memory of Professor K. Yano

Abstract. The object of the present paper is to study the symmetric and skew-symmetric properties of a second order parallel tensor in a P-Sasakian manifold.

Introduction. In 1926 H. Levy ([1]) proved that a second order symmetric parallel non-singular tensor on a space of constant curvature is a constant multiple of the metric tensor. In recent papers ([2]) R. Sharma generalized Levy's result and also studied a second order parallel tensor on Kähler space of constant holomorphic sectional curvature as well as on contact manifolds ([3]), ([4]).

In this paper it is shown that in a P-Sasakian manifold a second order symmetric parallel tensor is a constant multiple of the associated metric tensor. Further, it is shown that on a P-Sasakian manifold there is no non-zero parallel 2-form.

1. Preliminaries. Let (M,g) be an n-dimensional Riemannian manifold admitting a 1-form η which satisfies the conditions

$$(\nabla_X\eta)(Y)-(\nabla_Y\eta)(X)=0, \tag{1}$$

$$\begin{aligned}(\nabla_X\nabla_Y\eta)(Z)=&-g(X,Z)\eta(Y)-g(X,Y)\eta(Z)\\&+2\eta(X)\eta(Y)\eta(Z),\end{aligned} \tag{2}$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. If moreover (M, g) admits a vector field ξ and a (1,1) tensor field φ such that

$$g(X, \xi) = \eta(X), \tag{3}$$

$$\eta(\xi) = 1, \tag{4}$$

$$\nabla_X \xi = \varphi X, \tag{5}$$

then such a manifold is called a para-Sasakian manifold or briefly a P-Sasakian manifold by T. Adati and K. Matsumoto ([5]) which are considered as special cases of an almost paracontact manifold introduced by I. Sato ([6]).

It is known that in a P-Sasakian manifold the following relations hold ([5], [6]) :

$$\varphi^2 X = X - \eta(X)\xi, \tag{6}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \text{ where } R \text{ denotes the curvature tensor} \tag{7}$$

$$R(\xi, X)\xi = X - \eta(X)\xi, \tag{8}$$

$$\eta(\varphi X) = 0. \tag{9}$$

The above result will be used in the next section.

Definition. A tensor T of second order is said to be a second order parallel tensor if $\nabla T = 0$ where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.

2. Let α denotes a $(0, 2)$-symmetric tensor field on a P-Sasakian manifold M such that $\nabla \alpha = 0$. Then it follows that

$$\alpha(R(W, X)Y, Z) + \alpha(Y, R(W, X)Z) = 0 \tag{2.1}$$

for arbitrary vector fields W, X, Y, Z on M.

Substitution of $W = Y = Z = \xi$ in (2.1) gives us

$$\alpha(\xi, R(\xi, Y)\xi) = 0 \text{ (because } \alpha \text{ is symmetric).}$$

As the manifold is P-Sasakian, using (7) in the above equation we get

$$g(X, \xi)\alpha(\xi, \xi) - \alpha(X, \xi) = 0. \tag{2.2}$$

Differentiating (2.2) covariantly along Y, we get

$$\begin{aligned}[g(\nabla_Y X, \xi) + g(X, \nabla_Y \xi)]\, \alpha(\xi, \xi) + 2g(X, \xi)\alpha(\nabla_Y \xi, \xi) \\ - [\alpha(\nabla_Y X, \xi) + \alpha(X, \nabla_Y \xi)] = 0.\end{aligned} \tag{2.3}$$

Putting $X = \nabla_Y X$ in (2.2), we get

$$g(\nabla_Y X, \xi)\alpha(\xi, \xi) - \alpha(\nabla_Y X, \xi) = 0. \tag{2.4}$$

From (2.3) and (2.4) we get

$$g(X, \varphi Y)\alpha(\xi, \xi) + 2g(X, \xi)\alpha(\varphi Y, \xi) - \alpha(X, \varphi Y) = 0. \tag{2.5}$$

Replacing X by φY in (2.2) and using (9) gives

$$\alpha(\varphi Y, \xi) = 0. \tag{2.6}$$

From (2.5) and (2.6) it follows that

$$g(X, \varphi Y)\alpha(\xi, \xi) - \alpha(X, \varphi Y) = 0. \tag{2.7}$$

Replacing Y by φY in (2.7) and using (3), (6) and (2.2) we get

$$\alpha(X, Y) = \alpha(\xi, \xi)g(X, Y). \tag{2.8}$$

Differentiating (2.8) covariantly along any vector field on M, it can be easily seen that $\alpha(\xi, \xi)$ is constant. Hence we can state the following theorem:

Theorem 1. *On a P-Sasakian manifold a second order symmetric parallel tensor is a constant multiple of the associated metric tensor.*

As an immediate corollary of Theorem 1, we have the following result:

Corollary. *If the Ricci tensor field is parallel in a P-Sasakian manifold, then it is an Einstein manifold.*

The above corollary is proved by T. Adati and T. Miyazawa ([7]) in another way.

Next, let M be a P-Sasakian manifold and α a parallel 2-form. Putting $Y = W = \xi$ in (2.1) and using (7) and (8), we obtain

$$\alpha(X, Z) = \eta(X)\alpha(\xi, Z) - \eta(Z)\alpha(\xi, X) + g(X, Z)\alpha(\xi, \xi). \tag{2.9}$$

Since α is a 2-form, that is, α is a (0,2) skew-symmetric tensor therefore $\alpha(\xi, \xi) = 0$. Hence (2.9) reduces to

$$\alpha(X, Z) = \eta(X)\alpha(\xi, Z) - \eta(Z)\alpha(\xi, X). \tag{2.10}$$

Now, let A be a (1,1) tensor field which is metrically equivalent to α, i.e., $\alpha(X, Y) = g(AX, Y)$. Then, from (2.10) we have

$$g(AX, Z) = \eta(X)g(A\xi, Z) - \eta(Z)g(A\xi, X),$$

and thus,

$$AX = \eta(X)A\xi - g(A\xi, X)\xi. \tag{2.11}$$

Since α is parallel, then A is parallel. Hence, using that $\nabla_X \xi = \varphi X$, it follows that

$$\nabla_X(A\xi) = A(\nabla_X \xi) = A(\varphi X).$$

Thus

$$\nabla_{\varphi X}(A\xi) = A\left(\varphi^2 X\right) = AX - \eta(X)A\xi. \tag{2.12}$$

Therefore, we have from (2.11) and (2.12)

$$\nabla_{\varphi X}(A\xi) = -g(A\xi, X)\xi. \tag{2.13}$$

Now, from (2.11) we get

$$g(A\xi, \xi) = 0. \tag{2.14}$$

From (2.13) and (2.14) it follows that

$$g\left(\nabla_{\varphi X}(A\xi), A\xi\right) = 0. \tag{2.15}$$

Replacing X by φX in (2.15) and since $\nabla_\xi \xi = 0$, it follows that

$$g\left(\nabla_X(A\xi), A\xi\right) = 0, \tag{2.16}$$

for any X and thus $\|A\xi\| =$ constant on M.

From (2.16) we deduce

$$g\left(A\left(\nabla_X \xi\right), A\xi\right) = -g\left(\nabla_X \xi, A^2\xi\right) = 0.$$

Replacing X by φX in the above equation, it follows

$$g\left(\nabla_{\varphi X}\xi, A^2\xi\right) = g\left(\varphi^2 X, A^2\xi\right) = g\left(X - \eta(X)\xi, A^2\xi\right) = 0.$$

Thus, $g\left(X, A^2\xi\right) = g\left(\eta(X)\xi, A^2\xi\right)$.

Hence

$$A^2\xi = -\|A\xi\|^2\xi. \tag{2.17}$$

Differentiating the above equation covariantly along X, it follows that

$$\begin{aligned}\nabla_X(A^2\xi) = A^2(\nabla_X\xi) = A^2(\varphi X) &= -\|A\xi\|^2(\nabla_X\xi) \\ &= -\|A\xi\|^2(\varphi X).\end{aligned}$$

Hence $A^2(\varphi X) = -\|A\xi\|^2(\varphi X)$.

Replacing X by φX, we have (2.17)

$$A^2X = -\|A\xi\|^2 X.$$

Now, if $\|A\xi\| \neq 0$, then $J = \frac{1}{\|A\xi\|}A$ is an almost complex structure on M. In fact, (J, g) is a Kähler structure on M. The fundamental 2-form is $g(JX, Y) = \lambda g(AX, Y) = \lambda\alpha(X, Y)$, with $\lambda = 1/\|A\xi\| =$ constant. But, (2.11) means

$$\alpha(X, Z) = \eta(X)\alpha(\xi, Z) - \eta(Z)\alpha(\xi, X),$$

and thus α is degenerate, which is a contradiction. Therefore $\|A\xi\| = 0$ and hence $\alpha = 0$.

Hence we can state the following theorem:

Theorem 2. *On a P-Sasakian manifold there is no non-zero parallel 2-form.*

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