On a theorem of G. Birkhoff

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In this note we are going to give a short proof of the following classical theorem of G. BIRKHOFF [1].

Theorem. *A class $\mathfrak{U}$ of Ω-algebras forms a variety if and only if it is closed under forming*

(1) *subalgebras,*

(2) *homomorphic images,*

(3) *direct products.*

Our terminology is that of [2]. However, we mention that in the following *by law we mean an arbitrary fully invariant congruence $\varkappa$ of the countably generated Ω-word algebra F. Moreover we say the law $\varkappa$ to hold in the Ω-algebra A if for any congruence φ corresponding to an arbitrary homomorphism of F into A, we have $\varphi \geqq \varkappa$.*

Since the necessity is obvious, we prove only the sufficiency.

If $\mathfrak{U}$ consists only of the one-element algebra, then $\mathfrak{U}$ is the class of all algebras satisfying the law F^2. If $\mathfrak{U}$ contains an algebra A having more than one element, then A has a subalgebra generated by at most countably many elements of it which also belongs to $\mathfrak{U}$ by (1). Hence there exists a congruence θ in such that $F/\theta \in \mathfrak{U}$. If $\varkappa$ denotes the intersection of all such congruences θ, we shall show that $\varkappa$ is a law.

In fact, if ε is an arbitrary endomorphism of F, then $F\varepsilon$ is a subalgebra of F.

Now consider the union F', of all classes of the congruence $\varkappa$ which have non-empty intersection with $F\varepsilon$. Then F' is obviously a subalgebra of F, moreover, making use of the second isomorphism theorem, we have

$$F'/\varkappa_{F'} \cong F\varepsilon/\varkappa_{F\varepsilon} \tag{i}$$

where the set in the subscript indicates the restriction of $\varkappa$ to this set. Now we define the following relation ϱ in F:

$$a\varrho b \quad \text{if and only if} \quad a\varepsilon\varkappa b\varepsilon;$$

it is easily seen that ϱ is a congruence in F and

$$F\varepsilon/\varkappa_{F\varepsilon} \cong F/\varrho. \tag{ii}$$

By (1) and (3) we have $F/\varkappa \in \mathfrak{U}$. Moreover (1) implies $F'/\varkappa_{F'} \in \mathfrak{U}$ and thus using (i) and (ii) we obtain $F/\varrho \in \mathfrak{U}$. In view of the definition of $\varkappa$ we have $\varkappa \leqq \varrho$, hence for any two elements $a, b \in F$ the relation $a\varkappa b$ implies $a\varrho b$ i. e. $a\varepsilon\varkappa b\varepsilon$ (recall the definition of ϱ). This fact shows that $\varkappa$ is fully invariant, i. e. it is a law.

Now we shall prove that an arbitrary Ω-algebra A belongs to $\mathfrak{U}$ if and only if the law $\varkappa$ holds in it.

Indeed, first assume $A \in \mathfrak{U}$. Let B be an arbitrary subalgebra of A such that $B \cong F/\theta$ holds with a suitable θ. (1) implies $B \in \mathfrak{U}$, so that we have $\theta \geqq \varkappa$.

Conversely, assume that the law $\varkappa$ holds in A. First, assuming that A is finitely generated, we have $A \cong F/\theta$. By the assumption $\theta \geqq \varkappa$, hence it follows from the third isomorphism theorem that A is a homomorphic image of $F/\varkappa$ and so (2) implies $A \in \mathfrak{U}$.

In the general case $\varkappa$ holds obviously for any finitely generated subalgebra of A, thus, as it was proved in the preceding paragraph, any such algebra belongs to $\mathfrak{U}$. The finitely generated subalgebras of A form a local system in A, so, making use of a theorem of P. M. Cohn [2] (p. 101.), asserting that any class of algebras closed with respect to forming subdirect products and homomorphic images is local, we are led to $A \in \mathfrak{U}$, which completes the proof.

References

[1] G. Birkhoff, On the structure of abstract algebras, *Proc. Cambridge Philos. Soc.* **31** (1935), 433—454.

[2] P. M. Cohn, Universal algebra, New York, 1965.

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