Note on Tauberian constants

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1. Introduction

We shall discuss the following problem: *there is given*

(i) an ultimately monotonic positive sequence $\tau_k \nearrow \infty$;

(ii) a *class* $U(\tau_k)$ of series Σu_k of complex terms, and with partial sums $s_n = u_0 + u_1 + \dots + u_n$, satisfying the Tauberian condition $|\tau_k u_k| = O(1)$;

(iii) a regular summability method $\{a_n(t)\}$ transforming s_k into $\sigma_t = \Sigma a_n(t) s_n$, where $\{a_n(t)\}$ may represent a matrix $a_{m,n}$ when $a_n(t) = a_n([t])$;

(iv) a positive number q (not necessarily an integer);

it is required to find the smallest constant $A(q)$, *finite or infinite* such that the inequality

$$\limsup_{t\to\infty,\, n/t\to q} |\sigma_t - s_n| \leqq A(q) \limsup_{k\to\infty} |\tau_k u_k| \tag{1. 1}$$

should be satisfied *for every series in* $U(\tau_k)$.

The constant $A(q)$ is called a Tauberian constant, in short a T-constant.

Many results are known for the Tauberian condition $|ku_k| = O(1)$. We know that for this class finite T-constants exist for every q for the Abel, Cesàro, Hausdorff, Quasi-Hausdorff and other methods [2, 4, 8, 10, 11, 14, etc.], and that $A(q)$ is unbounded as q tends to zero or to infinity.

Recent investigations of AGNEW, ANJANEYULU and MEIR [3, 5, 15] relate to the class $U(\sqrt{k})$. It has been found that with each method $a_n(t)$ a positive constant λ is associated such that $A(q)$ is finite if and only if $q = \lambda$ and n/t tends to λ „closely enough". For the Borel method Agnew found $\lambda = 1$, and for the Laurent method Anjaneyulu found $\lambda = x/(1-x)$. The „closeness' is measured by the coupling relation $\omega = (n - \lambda t)/\sqrt{t}$, and $A(q)$ is found to be finite if and only if $\Omega = \limsup |\omega|$ is finite, and $A(\lambda)$ is then a function of Ω. Obviously $A(\lambda, \Omega)$ increases with Ω.

The object of this note is to show that there are only those two types of behaviour, depending on whether τ_k tends to infinity at the same rate as k or slower, this being true for any regular summability method which has a finite T-constant.

Added 27. 3. 1967. In two recent papers BIEGERT [27, 28] has studied the T-constants for the summability methods discussed in (4. 1) of our paper. He obtained

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formulae when $\tau_k = k^p$ for all real p, and his values for the ratio λ agree with our values. (Biegert's formula (2.1) in [28] should be corrected to

$$(\gamma m/D_V - t)/(\gamma^{1-p}t^p) = \omega \quad \text{with} \quad \limsup_{t\to\infty} |\omega| < \infty,$$

as stated in recent letters. A similar correction should be made to his formula (16. 2) in [27]).

2. Notation and lemmas

We shall always assume that $n = n(\alpha)$ and $t = t(\alpha)$ are monotonic functions of a parameter $\alpha \geqq 0$, both tending to infinity with α, $n(0) = t(0) = 0$, $n(\alpha)$ being an integer for all α.

A positive sequence x_k will be called *ultimately monotonic* if it is monotonic for $k > k_0$. The suffix k will denote an integer; s_n is a step-function of α and σ_t is a function of α.

If x_k is ultimately monotonic decreasing, tending to zero, we shall write $x_k \searrow 0$. When $1/x_k \searrow 0$, we shall write $x_k \nearrow \infty$.

When x_k and y_k are positive and $x_k/y_k \searrow 0$, we shall write $x_k \ll y_k$. $U(\tau_k)$ will denote the *class* of series with complex terms Σu_k satisfying $|\tau_k u_k| = O(1)$. $A_\tau(q)$ or $A(q)$ will denote the constant in (1.1) for $U(\tau_k)$.

We shall also use the order notation „$\ll$'' for positive *functions* of α: for example $r_t \ll r_t^*$ will mean that both $r_{t(\alpha)}$ and $r^*_{t(\alpha)}$ are ultimately monotonic functions of α, and that $r_t/r_t^* \searrow 0$.

When $r_t \ll t$ and $n/t \to \lambda$ so closely that $n/t = \lambda + O(r_t/t)$, then the function $\omega(\alpha)$ defined by $\omega = (n - \lambda t)/r_t$ is bounded as $\alpha \to \infty$. Hence the non-negative constant $\Omega = \lim \sup |\omega(\alpha)|$ and the order of smallness of r_t are both measures of the closeness of approach of the ratio n/t to the limit λ. The smaller is r_t and the smaller is Ω, the closer comes n/t to λ. In lemma 2 we shall show that r_t can be replaced by another function ϱ_n of α.

Lemma 1. *If $t/t^* \to 1$ as $\alpha \to \infty$, and if for some $c \geqq 0$, $t^{-c} \ll r_t \ll t^c$, then $r_t/r_t^* \to 1$.*

PROOF. For $\alpha > \alpha_0$, $t - 1 < t^* < t + 1$, hence if r_t is ultimately increasing, $(t-1)^c \leqq r_{t-1}/r_t < r_{t^*}/r_t \leqq (t+1)^c/t^c$. If r_t is ultimately decreasing, then the same argument can be applied to $1/r_t$.

Lemma 2. If, for some positive c, $n^{-c} \ll \varrho_n \ll n$, and if for a positive q we define $r_t = \varrho_{[qt]}$, $\omega = (n - qt)/\varrho_n$, and $\omega^* = (n - qt)/r_t$, then as $\alpha \to \infty$, $\lim \sup |\omega| = \lim \sup |\omega^*|$.

PROOF. Let $p = p(\alpha) = (|n - qt| + n^{-c})/\varrho_n$, $p^* = (|n - qt| + n^{-c})/r_t$, then p and p^* are positive, and $p/p^* = r_t/\varrho_n = \varrho_{[qt]}/\varrho_n \to 1$ by lemma 1, since $qt/n = 1 - \omega\varrho_n/n \to 1$ when ω is bounded. Also $p = |\omega| + n^{-c}/\varrho_n$, hence $\lim \sup p = \lim \sup |\omega|$, and similarly $\lim \sup p^* = \lim \sup |\omega^*|$. Again when ω is unbounded, so is $\omega^* = \omega\varrho_n/r_t$.

Lemma 3. If $b_k > 0$, $c_k \nearrow \infty$, then for $n_0 < n < p$,

$$c_n \sum_{k=n}^{p} b_k \leqq \sum_{k=n}^{p} b_k c_k \leqq c_p \sum_{k=n}^{p} b_k.$$

This is trivial, but will be often used.

In what follows we shall *define* for a positive number q, and for a positive sequence x_n such that $x_n \ll n$,

$$\omega = \omega(\alpha) = \frac{n(\alpha) - qt(\alpha)}{x_{n(\alpha)}}, \quad \text{and} \quad \Omega = \limsup_{\alpha\to\infty} |\omega(\alpha)|. \tag{2. 1}$$

When $\tau_k \ll k$, the smallest constant A, finite or infinite, for which the inequality

$$\limsup_{\limsup|\omega|=\Omega} |\sigma_t - s_n| \leqq A \limsup_{k\to\infty} |\tau_k u_k| \tag{2. 2}$$

is satisfied for a *fixed* Ω and *for every series in* $U(\tau_k)$ will be denoted by

$$A(q, \Omega, x_n). \tag{2. 3}$$

When $x_n = 1$ for $n > n_0$, we shall write $A(q, \Omega, 1_n)$.

3. Theorems and proofs

Theorem 3. I. *If a regular summability method $\{a_n(t)\}$ has for the class $U(k)$ a finite T-constant for some positive q, then the constant $A(q)$ is finite for every positive q,* and $A(q) \to \infty$ as $q \to 0$ and as $q \to \infty$.*

Theorem 3. II. *If $1_k \ll \tau_k \ll k$, $1_n \ll x_n \ll n$, and if for the class $U(\tau_k)$ and for the regular method $\{a_n(t)\}$ the T-constant $A(q, \Omega, x_n)$ is finite for one positive value λ of q, and for one non-negative value Ω_0 of Ω, then*

(i) *$A(q, \Omega, x_n) = \infty$ when $q \neq \lambda$ for every Ω and for any x_n $(1_n \ll x_n \ll n)$;*

(ii) *x_n/τ_n is bounded;*

(iii) *$A(\lambda, \Omega, y_n)$ is finite for every y_n such that y_n/τ_n is bounded and for every Ω, it has its minimum at $\Omega = 0$, and tends to infinity with Ω;*

(iv) *if the same method is applied to another class $U(\tau_k^*)$, where $\tau_k \ll \tau_k^* \ll k$, then finite T-constants exist for the same λ, for all Ω and for all y_n such that y_n/τ_n^* is bounded.*

Theorem 3. III. *There are regular methods*

(i) *which have no finite T-constants for $U(k)$;*

(ii) *which have finite T-constants for $U(k)$ but none for any class $U(\tau_k)$ such that $\tau_k \ll k$.*

Proof of 3. I. Let q and q_0 be distinct positive numbers, and let $A(q_0)$ be finite. Let $n/t \to q_0$, and $n/t' \to q$. For each t' we define n' as $n' = [q_0 t']$ so that $n'/t' \to q_0$ and $n'/n \to q_0/q$. Let Σu_k be any series in $U(k)$ so that $\limsup |ku_k| = L$ is finite. Here, and later we shall use the trivial inequalities:

$$|\sigma_{t'} - s_n| \leqq |s_{n'} - s_n| + |\sigma_{t'} - s_{n'}|, \tag{3. 1}$$

$$|\sigma_{t'} - s_n| \geqq |s_{n'} - s_n| - |\sigma_{t'} - s_{n'}|. \tag{3. 2}$$

As $\alpha \to \infty$, $\limsup |\sigma_{t'} - s_{n'}| \leqq A(q_0)L$, and for $\alpha > \alpha_0$

$$|s_{n'} - s_n| \leqq (L + \varepsilon) \left| \sum_{k=n}^{n'} 1/k \right| \leqq (L + \varepsilon) |\log (q_0/q)|,$$

so that, by (3. 1) and (3. 2),

$$|\log (q_0/q)| - A(q_0) \leqq A(q) \leqq |\log (q_0/q)| + A(q_0).$$

This proves the theorem.

PROOF OF 3. II. To prove (i), we first observe that

(3. 3) *if* $\Omega < \Omega_0$ *then* $A(q, \Omega, x_n) \leqq A(q, \Omega_0, x_n)$ *for any* q *and any* x_n,

since the left term refers to a closer approach. Let $q \neq \lambda$. For each n we define first n' as $n' = [\lambda n/q]$ and the $t' = n'/\lambda$ so that for the series $\Sigma 1/\tau_k$ we obtain $\limsup |\sigma_{t'} - s_{n'}| \leqq A(\lambda, 0, x_n)$, and this is finite by (3. 3). Again, for $n > n_0$, $n'/n = [\lambda n/q]/n > C > 0$, where C is a constant different from 1. Applying lemma 3 we obtain when $n' > n$

$$|s_{n'} - s_n| = \left|\sum_{k=n}^{n'} \frac{1}{k}\frac{k}{\tau_k}\right| > \left|\log \frac{n'}{n}\right| \frac{n}{\tau_n} > |\log C| \frac{n}{\tau_n}, \tag{3. 4}$$

and when $n' < n$, the last term is replaced by $|\log C| n'/\tau_{n'}$. Hence (i) follows from (3. 2).

To prove (ii) and (iii) we consider any bounded function $\omega(\alpha)$, such that $\limsup |\omega| = \Omega$, and any given sequence such that $1_n \ll x_n \ll n$. For each n we define n' as $n' = [|n - \omega x_n|]$ and t' as $t' = n'/\lambda$, so that for any series in $U(\tau_k)$ again $\limsup |\sigma_{t'} - s_{n'}| \leqq A(\lambda, 0, x_n)L$. Let $\omega' = (n - \lambda t')/x_n = (n - n')/x_n$, then $\omega' = \omega + \eta_n/x_n$, where for $n > n_0$ $|\eta_n| < 1$, so that $\limsup |\omega'/ = \limsup |\omega|$. Again, for $n > n_1$,

$$1 - (\Omega + 1)x_n/n < n'/n < 1 + (\Omega + 1)x_n/n, \tag{3. 5}$$

hence, for $n > n_2$

$$(\Omega + 1)x_n/2n < |\log (n'/n)| < 2(\Omega + 1)x_n/n. \tag{3. 6}$$

Taking $u_k = 1/\tau_k$, we obtain, as in (3. 4), for $n > n_3$ and $n' > n$,

$$|s_{n'} - s_n| > |\log (n'/n)| n/\tau_n > (\Omega + 1)x_n/2\tau_n, \tag{3. 7}$$

and (ii) follows from (3. 2) and (3. 7) when $\Omega = \Omega_0$.

For any u_k such that $\limsup |\tau_k u_k| = L$ is finite, using (3. 6) and lemma 3, and with y_n replacing x_n, and assuming that $n' > n$, we obtain

$$\begin{aligned} \limsup |s_{n'} - s_n| &< 2L(\Omega + 1)(y_n/n)(n'/\tau_{n'}) = \\ &= 2L(\Omega + 1)(n'/n)(\tau_n/\tau_{n'})(y_n/\tau_n). \end{aligned} \tag{3. 8}$$

Now (3. 5) shows that $n'/n \to 1$, and by lemma 1, $\tau_n/\tau_{n'} \to 1$ and y_n/τ_n is bounded, hence by (3.1) $A(\lambda, \Omega, y_n)$ is finite for any Ω. The same result holds when $n' < n$. The T-constant decreases with Ω, and exists at $\Omega = 0$, hence it attains its minimum there, and (3. 7) shows that it tends to infinity with Ω. This proves (iii).

To prove (iv) we observe that $U(\tau_k^*)$ is a subclass of $U(\tau_k)$ so that $A_{\tau^*}(\lambda, \Omega_0, x_n)$ is finite, hence by (iii) so is $A_{\tau^*}(\lambda, \Omega, y_n)$ for every Ω and for every y_n such that y_n/τ_n^* is bounded.

PROOF OF 3. III. (i) follows from the example:

Let $\{a_n(t)\}$ be defined by $\sigma_t = s_{[t^2]}$, then for the series $\Sigma 1/k$ and for $q > 0$ we have $\sigma_t - s_n = s_{[n^2/q^2]} - s_n = 0 (\log n)$ which is unbounded.

To prove (ii), let $0<x<y<1$, and let $L(x)$, $L(y)$ be the corresponding Laurent matrices. Anjaneyulu ([5]) proved that for $L(x)$ the value of λ is $x/(1-x)$, hence for $L(y)$ it is $\lambda'=y/(1-y)$. Let $\{a_{m,n}\}$ be the matrix formed by taking all rows in succession alternately from $L(x)$ and $L(y)$. Then for any $U(\tau_k)$ with $1_k \ll \tau_k \ll k$, theorem 3. II (i) would require that for odd m, $(m+1)/2n$ should tend to λ, and for even m, $m/2n$ should tend to λ'. This shows that no finite T-constant exists for such a class. But for $U(k)$ both $L(x)$ and $L(y)$ have finite T-constants for any positive q, $A(q)$ and $A'(q)$ say. Hence for our „mixed" matrix the T-constant is the larger of $A(q/2)$ and $A'(q/2)$.

4. Examples

Theorem 3. II has shown how the existence of a finite T-constant depends for a given method on the class $U(\tau_k)$ to which the method is applied. If $\tau_k \ll k$, there is at most one value of q, $q=\lambda$, to which the ratio n/t must tend, and the smaller is the order of τ_k, the closer must be the approach. To find finite T-constants to any given regular method, the first problem is to find the (unique) number λ (if any) belonging to the method, the next problem is to find the best possible, hence the smallest, order of τ_k, and the last problem is to find the values of the T-constants for all admissible values of q, Ω and x_n. Here we shall show how λ can be found for a wide class of methods; we do not attempt to give general rules for solving the other two more difficult problems

(4. 1) *Sonnenschein methods* [20, 21].

These are generated by a non-constant function $a(z)=\Sigma a_n z^n$ such that $a_n \geqq 0$, and $\Sigma a_n = 1$. If, for $t \geqq 0$, $(a(z))^t = \Sigma\, a_n(t) z^t$, then the sequence to function transformation $\sigma_t = \Sigma_n\, a_n(t) z^n$ is regular. A Sonnenschein *matrix* is given by $a_{m,n}=a_n(m)$ for $n, m = 0, 1, 2, \ldots$. For example:

e^{z-1} generates the *Borel method* $\{e^{-t} t^n/n!\}$ [6, 12];

$1-p+pz$, $0<p<1$, the *Euler matrix* $\left\{\binom{m}{n}(1-p)^{m-n}p^n\right\}$ [1, 16, 22, 25];

$\dfrac{pz}{1-(1-p)z}$, $0<p<1$, the *Hardy—Littlewood—Fekete (circle) matrix*, also called *Taylor matrix* $\left\{\binom{n-1}{m-1}p^m(1-p)^{n-m}\right\}$ [7, 9, 12, 13, 16, 23, 24, 25, 26];

$\dfrac{p}{1-(1-p)z}$, $0<p<1$, the *Laurent matrix* $\left\{\left(\dfrac{m+n-1}{m-1}\right)p^m(1-p)^n\right\}$ [16, 25].

When $\Sigma n a_n$ converges so that its sum is $a'(1)$ (as in all the above examples), then for $t \geqq 1$ and $|z| \leqq 1$, $\Sigma n a_n(t) z^n = zd(a(z))^t/dz = tz(a(z))^{t-1}a'(z)$ so that $\Sigma n a_n(t) = ta'(1)$. Hence the method $\{a_n(t)\}$ transforms the sequence $s_n = n$ into the function

$a'(1)t$. Hence if $n/t \to q$, then $\limsup |\sigma_t - s_n| = \limsup |a'(1)t - qt|$ is finite for this particular sequence if and only if $q = a'(1)$, so that $a'(1)$ *is the value of* λ. Hence for the: Borel method $\lambda = 1$, Euler matrix $\lambda = p$, Taylor matrix $\lambda = 1/p$, Laurent matrix $\lambda = (1-p)/p$, the same as given by Anjaneyulu ([5]), his parameter x being our parameter $1-p$.

(4. 2) *Hausdorff and allied matrices.*

The *Hausdorff matrices* are generated by a mass-function $\mu(x)$ of bounded variation in the closed interval [0, 1], and such that $\mu(x)$ is continuous at $0+$, and $\mu(0) = 0$, $\mu(1) = 1$. The matrix is defined by the integral

$$a_{m,n} = \int_0^1 \binom{m}{n} (1-x)^{m-n} x^n \, d\mu(x), \quad [6, 12],$$

and the integrand is analogue to the Euler matrix.

The *Quasi-Hausdorff matrix* is generated by the same type of mass function, and

$$a_{m,n} = \int_0^1 \binom{n-1}{m-1} x^m (1-x)^{n-m} \, d\mu(x) \quad [12, 17, 18],$$

the integrand being analogue to the Taylor matrix.

The *Laurent—Hausdorff matrix* is generated by the same mass function

$$a_{m,n} = \int_0^1 \binom{m+n-1}{m-1} x^m (1-x)^n \, d\mu(x) \quad \text{[19] analogue to the Laurent matrix.}$$

Hence by an argument similar to that in (4. 1) we obtain:

$$\text{for the Hausdorff matrix } \lambda = \int_0^1 x \, d\mu(x),$$

$$\text{Quasi-Hausdorff matrix } \lambda = \int_0^1 \frac{d\mu(x)}{x},$$

$$\text{Laurent—Hausdorff matrix } \lambda = \int_0^1 \frac{(1-x)\, d\mu(x)}{x}.$$

(4. 3) *Abel method.*

This is a series to function transformation, defined by $\sigma_x = \Sigma u_k x^k$, $0 < x < 1$, and $x \nearrow 1$. Putting $x = e^{-1/t}$, we obtain $\sigma_t = \Sigma u_k e^{-n/t}$, $t \to \infty$. When $s_n = n+1$, so that $u_k = 1$, we have $\sigma_t = 1/(1 - e^{-1/t})$, hence $|\sigma_{n/q} - s_n|$ is bounded as $n \to \infty$ only if $q = 1$. Thus the only possible value for λ is 1.

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