Toeplitz matrices in quasi Hilbert algebras II.

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Dedicated to B. Barna and L. Gyarmathi at the occasion of theirs 60th birthday

§ 3. Generalized Toeplitz matrices

The concept of a quasi Hilbert algebra makes possible to give a new general foundation of the theory of Toeplitz matrices. This is the subject of the present paragraph.

1. First of all we generalize the concept of a Toeplitz matrix.

Let R be a symmetric Banach algebra in wich the norm is determined by a positive functional. Let e_k $(k=1, 2, \ldots)$ be a complete orthonormal system in the quasi Hilbert algebra H^* over R. The infinite matrix

$$M(x) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots \\ . & . & . & \cdots \end{pmatrix}$$

where

$$\alpha_{kl} = (e_k x, e_l) \in R,$$

will be called the Toeplitz matrix generated by the element $x \in H^*$. (with respect to the given complete orthonormal system).

By the sum of the Toeplitz matrizes $M(x) = (\alpha_{kl})$, $M(y) = (\beta_{kl})$ belonging to the element $x, y \in H^*$ we understand the matrix

$$M(x) + M(y) = (\alpha_{kl} + \beta_{kl}).$$

By the product of $\alpha \in R$ and of $M(x)$ we understand the matrix

$$\alpha M(x) = (\alpha \alpha_{kl}).$$

By the product of $M(x)$ and of $M(y)$ we understand the matrix

$$M(x)M(y) = \left(\sum_{\nu=1}^{\infty} \alpha_{k\nu} \beta_{\nu l} \right)$$

(by (6) this matrix always exists).

By the conjugate of $M(x)$ we understand the matrix

$$M^*(x) = (\gamma_{kl}),\ \gamma_{kl} = \alpha_{lk}^*.$$

We show that the mapping $x \to M(x)$ is a $*$-isomorphism of the quasi Hilbert algebra H^* onto the algebra of the Toeplitz matrices generated by the elements of H^*.

The properties

$$M(x+y) = M(x) + M(y),$$

$$M(\alpha x) = \alpha M(x) \quad \text{for any complex number } \alpha,$$

$$M(x^*) = M^*(x)$$

are evident, so we show only that

$$M(xy) = M(x)M(y)$$

By definition

$$\gamma_{kl} = (e_k xy,\ e_l) = (e_k x,\ e_l y^*)$$

and so, making use of the relations

$$e_k x \sim \sum_{\nu=1}^{\infty} \alpha_{k\nu} e_\nu, \quad e_k y \sim \sum_{\nu=1}^{\infty} \beta_{k\nu} e_\nu, \quad e_k xy \sim \sum_{\nu=1}^{\infty} \gamma_{k\nu} e_\nu,$$

and of the equality

$$(e_l y^*,\ e_\nu) = (e_l,\ e_\nu y) = (e_\nu y,\ e_l)^* = \beta_{\nu l}^*,$$

we get

$$\gamma_{kl} = \sum_{\nu=1}^{\infty} \alpha_{k\nu} \beta_{\nu l}. \tag{8}$$

After these considerations of algebraic character we are going to represent Toeplitz matrices by linear operators.

Let R be a symmetric Banach algebra in which the norm is determined by a positive functional f. Let V be the space of all sequences $(\alpha_1, \alpha_2, \ldots)$ of elements of R, for which $\sum_{k=1}^{\infty} |\alpha_k|^2$ exists (addition and multiplication by a complex number are defined for these sequences in the usual way, i. e. component-wise). By the inner product $\langle \mathfrak{a}, \mathfrak{b} \rangle$ of two such sequences $\mathfrak{a} = (\alpha_1, \alpha_2, \ldots)$, $\mathfrak{b} = (\beta_1, \beta_2, \ldots)$ we mean the complex number $\sum_{\nu=1}^{\infty} f(\alpha_\nu \beta_\nu^*)$. (This surely exists because $\sum_{\nu=1}^{\infty} |\alpha_\nu|^2$ and $\sum_{\nu=1}^{\infty} |\beta_\nu|^2$ both exist.) Thus V becomes a Hilbert space. Now we make correspond to the Toeplitz matrix $M(x)$ generated by the element x of the quasi Hilbert algebra H^* over R the linear mapping of the Hilbert space V, which maps the vector

$$\mathfrak{x} = (\xi_1, \xi_2, \ldots) \in V$$

to the vector

$$\mathfrak{x} M(x) = \left(\sum_{\nu=1}^{\infty} \xi_\nu \alpha_{\nu_1},\ \sum_{\nu=1}^{\infty} \xi_\nu \alpha_{\nu_2},\ \ldots \right).$$

We prove that the linear mapping thus corresponding to $M(x)$ is a bounded operator of the Hilbert space V.

For this purpose we first consider the smallest Banach algebra $\bar{H}^*$ which contains the normed algebra H^*. Clearly, if $\mathfrak{x}=(\xi_1^*, \xi_2^*, \dots)\in V$, then $s=\sum_{\nu=1}^{\infty}\xi_\nu e_\nu\in\bar{H}^*$. It is also clear that — if we apply the usual notation for the inner product to the case of arbitrary sequences —

$$|\langle \mathfrak{x}M(x), \mathfrak{x}\rangle| = \left|f\left(\sum_{k,l=1}^{\infty}\xi_k\alpha_{kl}\xi_l^*\right)\right| = \left|f\left(\sum_{k,l=1}^{\infty}\xi_k(e_k x, e_l)\xi_l^*\right)\right| =$$

$$= \left|f\left(\sum_{k=1}^{\infty}(\xi_k e_k)x,\ \sum_{k=1}^{\infty}\xi_k e_k\right)\right| = |f((sx, s))| \leqq$$

$$\leqq \sqrt{f((sx, sx))}\cdot\sqrt{f((s, s))} = |sx|\,|s| \leqq |s|^2\,|x|,$$

where we have made use of the known fact that the positive functional f defined on R is bounded. Applying this to xx^*, we get in the case $|\mathfrak{x}|=1$ that

$$|\mathfrak{x}M(x)|^2 = \langle \mathfrak{x}M(x), \mathfrak{x}M(x)\rangle = \langle \mathfrak{x}M(x)M^*(x), \mathfrak{x}\rangle = \langle \mathfrak{x}M(xx^*), \mathfrak{x}\rangle \leqq |xx^*| \leqq |x|^2$$

and so the mapping in question is in fact a bounded operator of the Hilbert space V

We remark that if H^* is a quasi Hilbert algebra with unit element, then the spectrum of the bounded element $x\in H^*$ is contained in an interval and the spectrum of the operator belonging to $M(x)$ is also contained in the same interval.

As a matter of fact, the spectrum of the element x coincides with the spectrum of $M(x)$ with respect to the algebra of Toeplitz matrices (for this algebra is isomorphic to H^*) and the spectrum of the operator belonging to $M(x)$ cannot be larger than this, since the algebra of Toeplitz matrices is isomorphic to a subalgebra of the algebra of bounded operators of the Hilbert space V.

2. Let R be a symmetric Banach algebra in which the norm is determined by a positive functional f. Now we shall deal with the finite matrices formed by the elements of R.

We make the space V_n of the vectors $(\xi_1, \dots, \xi_n)$ consisting of elements of R to a Hilbert space (the dimension of which is not necessarily finite) by letting the inner product $\langle \mathfrak{a}, \mathfrak{b}\rangle$ of the vectors $\mathfrak{a}=(\alpha_1, \dots, \alpha_n)\in V_n$, and $\mathfrak{b}=(\beta_1, \dots, \beta_n)\in V_n$ be equal to $f(\alpha_1\,\beta_1^*)+\dots+f(\alpha_n\beta_n^*)$. By the trace of the linear operator L of the Hilbert space V_n we understand the number

$$\operatorname{tr} L = \langle \mathfrak{e}_1 L, \mathfrak{e}_1\rangle + \dots + \langle \mathfrak{e}_n L, \mathfrak{e}_n\rangle,$$

where $\mathfrak{e}_k\in V_n$ is the vector the k-th component of which is the unit element of R, while all its other components are equal to zero of R $(k=1, 2, \dots, n)$.

Let us make correspond to the matrix A of order n, consisting of elements of R the linear operator of the Hilbert space V_n, which maps the vector $\mathfrak{x}\in V_n$ to the vector $\mathfrak{x}A$. Clearly, the operator thus corresponding to A is bounded. One easily sees that if we make correspond to any matrix of order n, consisting of elements of R, the operator just defined, we get a $*$-isomorphism of the algebra of the matrices considered onto the Banach algebra of those bounded operators L of the Hilbert space V_n, which have the property that for any element $\alpha \in R$ and any vector $\mathfrak{x} \in V_n$ the relation $(\alpha \mathfrak{x})L = \alpha(\mathfrak{x}L)$ holds.

Now we are going to investigate the sections of the Toeplitz matrix $M(x)$ generated by the element x of the algebra H^*. By the n-th section of the matrix $M(x)$ we mean the matrix

$$M_n(x) = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \cdot & \cdots & \cdot \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix}.$$

We remark that if H^* is a quasi Hilbert algebra with unit element and the spectrum of the bounded element $x \in H^*$ is contained in a finite interval, then the spectrum of the operator belonging to $M_n(x)$ is contained in the same interval.

As a matter of fact, the spectrum of the operator belonging to $M_n(x)$ lies in the interval bounded by the numbers

$$k_n(x) = \inf_{|\mathfrak{x}|=1} \langle \mathfrak{x} M_n(x), \mathfrak{x} \rangle \quad \text{and} \quad K_n(x) = \sup_{|\mathfrak{x}|=1} \langle \mathfrak{x} M_n(x), \mathfrak{x} \rangle$$

whereas the endpoints of the least interval containing the spectrum of the operator belonging to $M(x)$ are

$$k(x) = \inf_{|\mathfrak{x}|=1} \langle \mathfrak{x} M(x), \mathfrak{x} \rangle \quad \text{and} \quad K(x) = \sup_{|\mathfrak{x}|=1} \langle \mathfrak{x} M(x), \mathfrak{x} \rangle.$$

So, in view of the evident inequalities $k(x) \leqq k_n(x)$ and $K_n(x) \leqq K(x)$ our assertion follows from our previous remark concerning the spectrum of $M(x)$.

Now, on the basis of what has said above, it is easy to define continuous functions of the n-th section of the Toeplitz matrix belonging to the bounded element x of the algebra H^* with unit element. Suppose that the spectrum of the bounded element $x \in H^*$ is contained in the finite interval $[a, b]$ and let $F(\lambda)$ be a continuous real function defined on the interval $[a, b]$. We have just seen that in this case $[a, b]$ contains also the spectrum of the operator belonging to $M_n(x)$. Therefore we can define $F(M_n(x))$ by the formula

$$F(M_n(x)) = \int_a^b F(\lambda)\, dE_n(\lambda),$$

where $E_n(\lambda)$ is the spectral function of the operator belonging to $M_n(x)$.

Finally we remark that our above results remain valid without modification also in the case of complete orthonormal system for which the indices run through all integers and not only through the positive integers. Of course, we then have matrices which are infinite in four directions, instead of matrices infinite in only two directions. The case when the complete orthonormal system under consideration consists of the powers with integer exponents of a unitary element u of the algebra H^* with unit element, is of particular importance. In this case, if $x \in H^*$ is a Hermitian element which commutes with u, then it is easy to see that α_{kl} depends only on the difference $l-k$, for

$$\alpha_{kl} = (u^k x, u^l) = (x, u^l (u^k)^*) = (x, u^{l-k}),$$

therefore we can put $\alpha_{kl}=\alpha_{l-k}$. So in this case the Toeplitz matrix generated by the element x can be written in the form

$$T(x)=\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdot \\ \cdot & \alpha_{-1} & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdot \\ \cdot & \alpha_{-2} & \alpha_{-1} & \alpha_0 & \alpha_1 & \alpha_2 & \cdot \\ \cdot & \alpha_{-3} & \alpha_{-2} & \alpha_{-1} & \alpha_0 & \alpha_1 & \cdot \\ \cdot & \alpha_{-4} & \alpha_{-3} & \alpha_{-2} & \alpha_{-1} & \alpha_0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

By the n-th section of the matrix $T(x)$ we mean the matrix

$$T_n(x)=\begin{pmatrix} \alpha_0 & \alpha_1 & \dots\, \alpha_n \\ \alpha_{-1} & \alpha_0 & \dots\, \alpha_{n-1} \\ \cdot & \cdot & \dots\, \cdot \\ \alpha_{-n} & \alpha_{-n+1} & \dots\, \alpha_0 \end{pmatrix}.$$

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