On stars of coverings and uniform spaces

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It is the aim of this note to elucidate some aspects of the interrelation existing between coverings of a set and symmetrical perfect topogenous orders defined on that set. In particular, we shall give a syntopogenous characterization of uniformities based on their definition as systems of coverings. [1])

§ 1. Coverings and symmetrical perfect topogenous orders

Let E be a nonvoid set. Coverings of E will be denoted by lower case Greek letters. A cover the members of which are pairwise disjoint will be called a partition. For α, β covers of E, we write

$$\alpha \cap \beta = \{A \cap B \mid A \in \alpha,\ B \in \beta\},$$

and $\alpha \leqq \beta$ will mean that for any $A \in \alpha$ there exists a $B \in \beta$ satisfying $A \subseteq B$. — We still need the following

Definition 1. A symmetrical perfect topogenous order on E is a relation $<$ defined on the set of all subsets of E, satisfying the following axioms:

(01) $\emptyset < \emptyset,\ E < E$;

(02) $A < B \Rightarrow A \subseteq B$;

(03) $A \subseteq A' < B' \subseteq B \Rightarrow A < B$;

(P′) $A_i < B\ (i \in I) \Rightarrow \bigcup \{A_i | i \in I\} < B$;

(S) $A < B \Rightarrow E - B < E - A.$ ▮ [2])

Remark. (03) and (P′) together imply

(P) $A_i < B_i\ (i \in I) \Rightarrow \bigcup \{A_i | i \in I\} < \bigcup \{B_i | i \in I\}$.

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[1]) Of course, the syntopogenous characterization of uniformities based on their definition as systems of coverings can be inferred from two known results, (7.33) in [1] and „Kočetkov's theorem" in [2]. Nevertheless, an explicit formulation of this result and its (nontrivial) direct proof might deserve some interest.

[2]) As usual, the sign ▮ indicates the end of a proof or of a statement.

Again, (P) and (S) together imply

(Q) $A_i < B_i \ (i \in I) \Rightarrow \bigcap \{A_i | i \in I\} < \bigcap \{B_i | i \in I\}$. ∎

Between covers and symmetrical perfect topogenous orders a close connection is established by the following

Theorem 1. (1) *If γ is a cover of E, then the relation $<_\gamma$ defined between subsets of E by*

$$A <_\gamma B \Leftrightarrow \mathrm{St}\,(A, \gamma) \subseteqq B$$

is a symmetrical perfect topogenous order [3]) *on E.*

(2) *For any symmetrical perfect topogenous order $<$ on a set E there exists a cover γ of E such that $< = <_\gamma$.*

(3) *If $<$ and $<_1$ are symmetrical perfect topogenous orders on E satisfying $<_1 \subseteqq <$, then there exist covers γ and δ of E, such that $< = <_\gamma$, $<_1 = <_\delta$ and $\gamma \leqq \delta$.*

On the other hand, $\gamma \leqq \delta$ always implies $<_\delta \subseteqq <_\gamma$.

Proof. (1) Let γ be a cover of E. Clearly,

$$xUy \Leftrightarrow (\exists G \in \gamma) x, y \in G$$

is a reflexive and symmetrical relation. Put

$$A <_\gamma B \Leftrightarrow \begin{bmatrix} x \in A \\ xUy \end{bmatrix} \rightarrow y \in B,$$

or equivalently

$$A <_\gamma B \Leftrightarrow \mathrm{St}\,(A, \gamma) \subseteqq B.$$

By [1], (5. 41), the relation $<_\gamma$ is a biperfect topogenous order. In view of the implication

$$\mathrm{St}\,(A, \gamma) \subseteqq B \Rightarrow \mathrm{St}\,(E-B, \gamma) \subseteqq E-A$$

it is even symmetrical:

$$A <_\gamma B \Rightarrow E-B <_\gamma E-A.$$

(Of course, it is also possible to check the conditions of Definition 1. directly.)

(2) For $x \in E$ put

$$U(x) = \bigcap \{V | x < V\} = \{y | x \not< E-y\}.$$

Clearly, $\{U(x) | x \in E\}$ is a cover of E, symmetrical in the sense that

$$y \in U(x) \Leftrightarrow x \in U(y).$$

Let

$$\gamma = \{H \mid x \in H \Rightarrow H \subseteqq U(x)\}.$$

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[3]) Of course, $\mathrm{St}\,(A, \gamma) = \cup \}G \in \gamma | G \cap A \neq \emptyset\}$.

For any $x \in E$, $x \in U(x)$ implies $\{x\} \in \gamma$, so γ is a cover of E. Moreover, $\{x, y\} \in \gamma$ for $x \in E$ and $y \in U(x)$. Indeed,

$$\left.\begin{array}{r} x \in U(x) \\ y \in U(x) \end{array}\right\} \Rightarrow \{x, y\} \subseteqq U(x),$$

and

$$\left.\begin{array}{r} y \in U(x) \Rightarrow x \in U(y) \\ y \in U(y) \end{array}\right\} \Rightarrow \{x, y\} \subseteqq U(y).$$

We see that γ is a cover of E, and also that

$$\mathrm{St}\,(x, \gamma) = U(x)$$

for any $x \in E$. Now each of the following conditions is equivalent to the next one:

$$A < B,$$

$$x < B \ (x \in A),$$

$$U(x) \subseteqq B \ (x \in A),$$

$$\bigcup \{U(x) | x \in A\} \subseteqq B,$$

$$\bigcup \{\mathrm{St}\,(x, \gamma) | x \in A\} \subseteqq B,$$

$$\mathrm{St}\,(A, \gamma) \subseteqq B,$$

$$A <_\gamma B.$$

(3) Put $U_1(x) = \bigcap \{V | x <_1 V\}$ and

$$\delta = \{H | x \in H \Rightarrow H \subseteqq U_1(x)\}.$$

One sees that $<_1 = <_\delta$. Moreover, $<_1 \subseteqq <$ implies $U(x) \subseteqq U_1(x)$ for $x \in E$ and so $H \in U(x)$ implies $H \in U_1(x)$. Hence we get $\gamma \subseteqq \delta \Rightarrow \gamma \leqq \delta$.

Again, if $\gamma \leqq \delta$, then $\mathrm{St}\,(A, \gamma) \subseteqq \mathrm{St}\,(A, \delta)$ for any $A \subseteqq E$. Thus we have the implication

$$\gamma \leqq \delta \Rightarrow <_\delta \subseteqq <_\gamma. \blacksquare$$

Let us now supplement the results of the previous theorem by a characterization of those orders [4] $<_\gamma$ which are generated by partitions γ. One easily sees that the many-to-one correspondence

$$\gamma \Rightarrow <_\gamma$$

becomes one-to-one, if we restrict γ to run through the partitions of E. More detailed information about partition-generated orders $<_\gamma$ is contained in the following

Theorem 2. (1) *A symmetrical perfect topogenous order* $<$ *is generated by a partition iff*

$$A < B \Rightarrow [A \subseteqq H \subseteqq B \quad \textit{for some } H \textit{ satisfying} \quad H < H].$$

[4]) Unless the contrary is explicitely stated, the words ,,order"and ,,relation" stand for ,,symmetrical perfect topogenous order".

(2) *If a relation $<$ is generated by a partition γ, then this γ is uniquely determined as the class of minimal self-preceding subsets of E:*

$$H\in\gamma \quad \textit{iff} \quad H<H \quad \textit{and} \quad \emptyset\neq G\subset H\Rightarrow G\nless G.$$

PROOF. Part (2) is clear; so is the necessity of the condition in (1). The only thing to prove is the sufficiency of that condition: Suppose it holds for an order $<$, and for $x\in E$ put $U(x)=\bigcap\{V|x<V\}$. Clearly $x<U(x)$, and in view of the condition and of the fact that $U(x)$ is the smallest set H satisfying $x<H$, we also have $U(x)<U(x)$. By what has just been said, the implication

$$x<A\Rightarrow U(x)\subseteqq A$$

also holds.

Let us now establish the implication

$$(*) \qquad U(x)\cap U(y)\neq\emptyset\Rightarrow U(x)=U(y).$$

First of all, $\left.\begin{matrix} z\in U(x)\Rightarrow x\in U(z) \\ U(z)<U(z)\end{matrix}\right\}\Rightarrow x<U(z)\Rightarrow U(x)\subseteqq U(z)$. By symmetry we get $U(z)\subseteqq U(x)$ and finally $z\in U(x)\Rightarrow U(z)=U(x)$.

Let now be $z\in U(x)$ and $z\in U(y)$. Then

$$U(z)=U(x)=U(y)$$

and $(*)$ results proved.

Thus $\gamma=\{U(x)|x\in E\}$ is a partition of E and in the same way as in the proof of Theorem 1. we get $A<B$ iff $\bigcup\{U(x)|x\in A\}\subseteqq B$, and this in turn iff $\mathrm{St}\,(A,\gamma)\subseteqq B$. (In establishing the second „iff" we have to make use of the implication

$$a\in U(x)\Rightarrow U(a)=U(x).) \blacksquare$$

§ 2. A characterization of uniformities defined as systems of coverings

We start with the following well-known (see e.g. [2] or [3])

Definition 2. A system Σ of coverings of a set E is a uniformity on E if the following conditions are satisfied:

$$\text{(C1)} \qquad \left.\begin{matrix}\alpha\in\Sigma \\ \alpha\leqq\beta\end{matrix}\right\}\Rightarrow\beta\in\Sigma;$$

$$\text{(C2)} \qquad \alpha,\beta\in\Sigma\Rightarrow\alpha\cap\beta\in\Sigma;$$

$$\text{(C3)} \qquad \alpha\in\Sigma\Rightarrow(\exists\beta\in\Sigma)\big[\{\mathrm{St}\,(x,\beta)|x\in E\}\leqq\alpha\big]. \blacksquare$$

The uniformities of a given set are partially ordered in a natural way:

$$\Sigma_1\subseteqq\Sigma_2\Leftrightarrow(\alpha\in\Sigma_1\rightarrow\alpha\in\Sigma_2).$$

Remark. [5]) Condition (C3) can be replaced by the following one:

$$(\quad) \qquad \alpha\in\Sigma\Rightarrow(\exists\beta\in\Sigma)[\{\mathrm{St}\,(B,\beta)|B\in\beta\}\leqq\alpha].$$

[5]) See e.g. [2], p. 563, bottom. For the readers convenience, we expose in detail the proof outlined there.

PROOF. (C3a) implies (C3), since from $x \in B$ there follows $\mathrm{St}(x, \beta) \subseteqq \mathrm{St}(B, \beta)$.
On the other hand, (C3) implies (C3a). As a matter of fact, if to a given $\alpha \in \Sigma$ a cover $\gamma \in \Sigma$ is chosen by (C3), and to this γ a cover $\beta \in \Sigma$ is chosen again by (C3), then β satisfies the requirement of (C3a) with respect to α: If $\{\mathrm{St}(x, \gamma) | x \in E\} \leqq \alpha$, and $\{\mathrm{St}(x, \beta) | x \in E\} \leqq \gamma$, then for $B \in \beta$ we have

$$\mathrm{St}(B, \beta) = \cup \{\mathrm{St}(x, \beta) | x \in B\},$$

and for any $x \in B$,

$$B \subseteqq \mathrm{St}(x, \beta) \subseteqq G_x \in \gamma.$$

Thus, for any fixed $x_0 \in B$, we can infer from

$$x_0 \in B \subseteqq \cap \{G_x | x \in B\}$$

that

$$\mathrm{St}(B, \beta) \subseteqq \cup \{G_x | x \in B\} \subseteqq \mathrm{St}(x_0, \gamma). \blacksquare$$

Definition 3. A nonvoid family $\mathscr{S} = \{< | < \in \mathscr{S}\}$ of symmetrical perfect topogenous orders on a set E is a symmetrical perfect syntopogenous structure on E, if it satisfies the following conditions:

(S1) $$<_1, <_2 \in \mathscr{S} \Rightarrow (\exists < \in \mathscr{S})(<_1 \subseteqq < \ \& <_2 \subseteqq <);$$

(S2) $$< \in \mathscr{S} \Rightarrow (\exists <' \in \mathscr{S})(< \subseteqq <'^2). \blacksquare$$

Remark. Condition (S2) is capable of the following more explicit formulation: If $< \in \mathscr{S}$, then there exists a $<' \in \mathscr{S}$ such that

$$A < B \Rightarrow (\exists C)(A <' C <' B). \blacksquare$$

Definition 4. A symmetrical perfect syntopogenous structure $\mathscr{S}$ on a set E is said to be descending, if the implication

(D) $$\left.\begin{matrix} < \in \mathscr{S} \\ <_1 \subseteqq < \end{matrix}\right\} \Rightarrow <_1 \in \mathscr{S}$$

holds for any symmetrical perfect topogenous order $<_1$ over E. $\blacksquare$

The symmetrical perfect syntopogenous structures on a given set E can be partially ordered by the following convention:

$$\mathscr{S}_1 \leqq \mathscr{S}_2 \Leftrightarrow [<_1 \in \mathscr{S}_1 \Rightarrow (\exists <_2 \in \mathscr{S}_2)(<_1 \subseteqq <_2)].$$

For a given order $<$ over E and $x \in E$, write now $U_<(x) = \cap \{V | x < V\}$ and put $\gamma_< = \{U_<(x) | x \in E\}$. Clearly, $\gamma_<$ is a cover of E. (As a matter of fact, $\gamma_<$ will be a cover of E, as soon as $<$ has property (02).) With these notations, there results the following useful

Lemma. $\mathrm{St}(x, \gamma_<) = U_{<^2}(x)$ for $x \in E$, and consequently

$$\{\mathrm{St}(x, \gamma_<) | x \in E\} = \gamma_{<^2}$$

for any symmetrical perfect topogenous order $<$ on E.

Remark. $<^2$ is a symmetrical perfect topogenous order on E, whenever $<$ is. (See [1], (2. 16), (3. 53) and (4. 23).)

Proof.

$$U_<(x) = \bigcap\{V \mid x < V\} = \{y \mid x \not< E - y\},$$

and by the symmetry of $<$, $x \not< E - y \Leftrightarrow y \not< E - x$, i.e.

$$y \in U_<(x) \Leftrightarrow x \in U_<(y).$$

Also, $x < U_<(x)$ by the perfectness of $<$, and

$$x < H \Leftrightarrow U_<(x) \subseteqq H.$$

Now, each of the following conditions is equivalent to the next one:

$$x <^2 H;$$
$$x < K < H \quad \text{for some} \quad K \subseteqq E;$$
$$x < U_<(x) \subseteqq K < H \quad \text{for some} \quad K \subseteqq E;$$
$$x < U_<(x) < H;$$
$$U_<(x) < H;$$
$$y < H \quad \text{for any} \quad y \in U_<(x);$$
$$U_<(y) \subseteqq H \quad \text{for any} \quad y \in U_<(x);$$
$$\bigcup\{U_<(y) \mid y \in U_<(x)\} \subseteqq H;$$
$$\bigcup\{U_<(y) \mid x \in U_<(y)\} \subseteqq H;$$
$$\mathrm{St}\,(x, \gamma_<) \subseteqq H.$$

Thus we have proved

$$x <^2 H \Leftrightarrow \mathrm{St}\,(x, \gamma_<) \subseteqq H,$$

establishing thereby the lemma. ■

Now we are able to characterize 6) uniformities defined as systems of coverings by the following

Theorem 3. (1) *Let Σ be a uniformity on E. For $\gamma \in \Sigma$ and A, $B \subseteqq E$ put $A <_\gamma B \Leftrightarrow \Leftrightarrow \mathrm{St}\,(A, \gamma) \subseteqq B$, and $\mathscr{S}_\Sigma = \{<_\gamma \mid \gamma \in \Sigma\}$.*

The set $\mathscr{S}_\Sigma$ is a descending, symmetrical and perfect syntopogenous structure on E.

(2) *Let $\mathscr{S}$ be a descending, symmetrical and perfect syntopogenous structure on E. For $< \in \mathscr{S}$ and $x \in E$ put $U_<(x) = \bigcap\{V \mid x < V\}$, and let*

$$\gamma_< = \{U_<(x) \mid x \in E\}.$$

6) This characterization of uniformities differs slightly from that given in [1], Chapter 7.: We need a „descending condition" absent in [1]. This difference is due to the fact that here we are dealing with „whole uniformities" and not with (symmetrical) bases for them as is the case in [1].

Let now $\Sigma_{\mathscr{S}}$ be the set of covers of E with some $\gamma_<$ inscribed:

$$\Sigma_{\mathscr{S}} = \{\gamma \mid \gamma \geqq \gamma_< \quad \text{for some} \quad < \in \mathscr{S}\}.$$

The set $\Sigma_{\mathscr{S}}$ is a uniformity on E.

(3) *The mappings $\Sigma \to \mathscr{S}_\Sigma$ and $\mathscr{S} \to \Sigma_{\mathscr{S}}$ are one-to-one correspondences, inverse to each other, between the sets of all uniformities and all descending symmetrical and perfect syntopogenous structures on E, which preserve the respective partial orders.*

PROOF. (1) By Theorem 1. part (1) each relation $<_\gamma$ is a symmetrical perfect topogenous order on E. Moreover, the set $\mathscr{S}_\Sigma$ is descending. As a matter of fact, if γ is a cover of E and $<_1 \subseteqq <_\gamma$, then there exists [7]) a cover δ such that $<_1 = <_\delta$ and $\gamma \leqq \delta$.

Let δ be defined as in the proof of part (3) of Theorem 1. Then $<_1 = <_\delta$, and $G \in \gamma$ implies $G \in \delta$. Indeed, if $x \in G$ then $G \subseteqq U_1(x)$, and this because in view of $G \subseteqq \mathrm{St}(x, \gamma)$ we have the implications

$$y \in G \Rightarrow x \nless_\gamma E - y \Rightarrow x \nless_1 E - y \Leftrightarrow x \nless_\delta E - y \Rightarrow y \in U_1(x).$$

Thus we have established $\gamma \subseteqq \delta$, and thereby also $\gamma \leqq \delta$.

We still have to prove that $\mathscr{S}_\Sigma$ satisfies the two conditions laid down in Definition 3.

(S1): Let $\gamma, \delta \in \Sigma$. If $\gamma \leqq \delta$, then $<_\delta \subseteqq <_\gamma$ by Theorem 1. part (3).

Now let $<_{\gamma_1}, <_{\gamma_2} \in \mathscr{S}_\Sigma$. Then $\gamma_1, \gamma_2 \in \Sigma$, and this in turn implies $\gamma_1 \cap \gamma_2 \in \Sigma$. Put $\gamma = \gamma_1 \cap \gamma_2$. Then

$$\gamma \leqq \gamma_1 \Rightarrow <_{\gamma_1} \subseteqq <_\gamma,$$

and

$$\gamma \leqq \gamma_2 \Rightarrow <_{\gamma_2} \subseteqq <_\gamma,$$

i.e. (S1) holds with $<_\gamma = <_{\gamma_1 \cap \gamma_2} \in \mathscr{S}_\Sigma$.

(S2): Let $< \in \mathscr{S}_\Sigma$, i.e. $< = <_\gamma$ for some $\gamma \in \Sigma$. — Choose $\delta \in \Sigma$ so as to have

$$\{\mathrm{St}(K, \delta) \mid K \in \delta\} \leqq \gamma.$$

Now let us show that

$$A <_\gamma B \Rightarrow A <_\delta \mathrm{St}(A, \delta) <_\delta B.$$

We clearly have $A <_\delta \mathrm{St}(A, \delta)$. At the same time, we also have $\mathrm{St}(A, \delta) <_\delta B$, i.e. we have

$$\left.\begin{array}{l} x \in \mathrm{St}(A, \delta) \\ x, y \in K \in \delta \end{array}\right\} \Rightarrow y \in B.$$

Indeed, $x \in \mathrm{St}(A, \delta)$ implies the existence of a $K_1 \in \delta$ such that $x \in K_1$ and $K_1 \cap A \neq \emptyset$.

Also, we have

$$x, y \in K \subseteqq \mathrm{St}(K, \delta) \subseteqq G \in \gamma$$

and in view of $x \in K \cap K_1$, $K_1 \subseteqq \mathrm{St}(K, \delta)$.

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[7]) This is a somewhat stronger result than the one contained in part (3) of Theorem 1. As a matter of fact, here we have to find a δ corresponding to a fixed γ, whereas in the earlier result only the relation $<$ was fixed but not the cover γ satisfying $< = <_\gamma$. (The correspondence $\gamma \to <_\gamma$ is, of course, many-to-one!)

Now, if $x_1 \in K_1 \cap A$, then $x_1 \in A$ and $x_1, y \in G \in \gamma$.

From $A <_\gamma B$ we now get $y \in B$, and $\mathrm{St}(A, \delta) <_\delta B$ results proved.

(2) We have to check the three conditions of Definition 2.

(C1): Clear.

(C2): We see that for $<_1, <_2 \in \mathscr{S}$ the implication

$$<_1 \subseteqq <_2 \Rightarrow [U_{<_2}(x) \subseteqq U_{<_1}(x)\ (x \in E)]$$

holds, and consequently $<_1 \subseteqq <_2 \Rightarrow \gamma_{<_2} \leqq \gamma_{<_1}$.

Let now be $\gamma, \delta \in \Sigma_{\mathscr{S}}$. Then $\gamma_{<_1} \leqq \gamma$ and $\gamma_{<_2} \leqq \delta$ for some $<_1, <_2 \in \mathscr{S}$. By (S1) there is a $< \in \mathscr{S}$ such that $<_1 \subseteqq <$ and $<_2 \subseteqq <$. Thus however

$$\left.\begin{array}{l}\gamma_< \leqq \gamma_{<_1}\\ \gamma_< \leqq \gamma_{<_2}\end{array}\right\} \Rightarrow \gamma_< \leqq \gamma_{<_1} \cap \gamma_{<_2} \leqq \gamma \cap \delta.$$

This shows that $\gamma \cap \delta \in \Sigma_{\mathscr{S}}$.

(C3): Let $\gamma \in \Sigma_{\mathscr{S}}$, i.e. let $\gamma_< \leqq \gamma$ for some $< \in \mathscr{S}$.

Choose now $<_1 \in \mathscr{S}$ in accordance with (S2), i.e. let $< \subseteqq <_1^2$. (Of course, $<_1^2 \subseteqq <_1 \in \mathscr{S}$ implies $<_1^2 \in \mathscr{S}$, but we do not use this fact.) We see that

$$< \subseteqq <_1^2 \Rightarrow \gamma_{<_1^2} \leqq \gamma_<,$$

and so, by the Lemma, we obtain

$$\gamma_{<_1^2} = \{\mathrm{St}(x, \gamma_{<_1}) | x \in E\} \leqq \gamma_<.$$

(3) Let $\Sigma \to \mathscr{S}_\Sigma$ and $\mathscr{S} \to \Sigma_{\mathscr{S}}$. If $\mathscr{S} = \mathscr{S}_\Sigma$, then $\Sigma_{\mathscr{S}} = \Sigma$. Indeed,

$$\Sigma_{\mathscr{S}} = \{\gamma |\ \gamma \geqq \gamma_< \quad \text{for some} \quad < \in \mathscr{S}\},$$

i.e. in our case

$$\Sigma_{\mathscr{S}} = \{\gamma | \gamma \geqq \gamma_{<_\varrho} \quad \text{for some} \quad <_\varrho \in \mathscr{S}_\Sigma\} = \{\gamma | \gamma \geqq \gamma_{<_\varrho} \quad \text{for some} \quad \varrho \in \Sigma\}.$$

Now $\gamma_{<_\varrho} = \{U_{<_\varrho}(x) | x \in E\}$, and by virtue of

$$U_{<_\varrho}(x) = \cap \{V | x <_\varrho V\} = \cap \{V | \mathrm{St}(x, \varrho) \subseteqq V\} = \mathrm{St}(x, \varrho)$$

we get

$$\gamma_{<_\varrho} = \{\mathrm{St}(x, \varrho) | x \in E\}.$$

We see that $\gamma \in \Sigma_{\mathscr{S}}$ for $\mathscr{S} = \mathscr{S}_\Sigma$ iff $\{\mathrm{St}(x, \varrho) | x \in E\} \leqq \gamma$ for some $\varrho \in \Sigma$. This, however, means that $\Sigma_{\mathscr{S}} = \Sigma$. As a matter of fact, for any uniformity Σ we have

$$\alpha \in \Sigma \Leftrightarrow (\exists \beta \in \Sigma)[\{\mathrm{St}(x, \beta) | x \in E\} \leqq \alpha],$$

the implication $\Rightarrow$ being simply condition (C3), and the reverse implication $\Leftarrow$ being true because by (C1) $\beta \leqq \{\mathrm{St}(x, \beta) | x \in E\} \leqq \alpha$ implies $\alpha \in \Sigma$ for any $\beta \in \Sigma$.

Again, let $\mathscr{S} \to \Sigma_{\mathscr{S}}$ and $\Sigma \to \mathscr{S}_\Sigma$.

If $\Sigma = \Sigma_{\mathscr{S}}$ then $\mathscr{S}_\Sigma = \mathscr{S}$.

First of all,

$$\mathscr{S}_\Sigma = \{<_\gamma | \gamma \in \Sigma\} = \{<_\gamma | \gamma \in \Sigma_{\mathscr{S}}\} = \{<_\gamma | \gamma \geqq \gamma_< \quad \text{for some} \quad < \in \mathscr{S}\}.$$

Consider now the subset

$$\{<_\gamma | \gamma = \gamma_< \quad \text{for some} \quad < \in \mathscr{S}\}$$

of $\mathscr{S}_\Sigma$, and let $<_1$ be an arbitrary element of this subset, i.e. let there be a $< \in \mathscr{S}$ such that $A <_1 B$ iff $\mathrm{St}(A, \gamma_<) \subseteq B$.

Now, each of the following statements is equivalent to the next one:

$$\mathrm{St}(A, \gamma_<) \subseteq B,$$

$$\mathrm{St}(a, \gamma_<) \subseteq B \quad \text{for} \quad a \in A,$$

$$U_{<^2}(a) \subseteq B \quad \text{for} \quad a \in A,$$

$$a <^2 B \ (a \in A),$$

$$A <^2 B.$$

This shows that $<_1 = <^2$, and also that

$$\{<_\gamma | \gamma = \gamma_<\} = \{<^2 | < \in \mathscr{S}\}.$$

Now let $\gamma \geqq \gamma_<$. In view of $\alpha \geqq \beta \Rightarrow <_\alpha \subseteq <_\beta$ we have

$$\mathscr{S}_\Sigma = \{<_\gamma | \gamma \geqq \gamma_< \quad \text{for some} \quad < \in \mathscr{S}\} = \{<' | <' \subseteq <^2 \quad \text{for some} \quad < \in \mathscr{S}\} = \mathscr{S}.$$

Indeed, if $<' \in \mathscr{S}$ then $<' \subseteq <^2$ for some $< \in \mathscr{S}$ by condition (S2) from Definition 3. — On the other hand, $<' \subseteq <^2 \subseteq < \in \mathscr{S}$ implies $<' \in \mathscr{S}$ by the descending condition.

Let us still show that the mappings just considered are order-preserving: The implications

$$\Sigma \subseteq \Sigma_1 \Rightarrow \mathscr{S}_\Sigma \subseteq \mathscr{S}_{\Sigma_1} \Rightarrow \mathscr{S}_\Sigma \leqq \mathscr{S}_{\Sigma_1}$$

are evident.

On the other hand, $\mathscr{S} \leqq \mathscr{S}_1 \Rightarrow \Sigma_\mathscr{S} \subseteq \Sigma_{\mathscr{S}_1}$. Indeed, let $\gamma \in \Sigma_\mathscr{S}$, i.e. let $\gamma \geqq \gamma_<$ for some $< \in \mathscr{S}$.

By $\mathscr{S} \leqq \mathscr{S}_1$ we have $< \subseteq <_1$ for some $<_1 \in \mathscr{S}_1$, and $< \subseteq <_1 \Rightarrow \gamma_{<_1} \leqq \gamma_<$ now yields

$$\gamma \geqq \gamma_< \geqq \gamma_{<_1} \Rightarrow \gamma \geqq \gamma_{<_1}$$

for a $<_1 \in \mathscr{S}_1$, i. e. $\gamma \in \Sigma_{\mathscr{S}_1}$. ∎

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