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On the subdirectly irreducible semi-De Morgan algebras

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Abstract. In this note, we present a characterization of the subdirectly irreducible algebras of the class of semi-De Morgan algebras and we show that the coherent semi-De Morgan algebras are the coherent De Morgan algebras.

1. Introduction

The equational class of semi-De Morgan algebras was introduced by H. P. Sankappanavar in [7] as a common abstraction of De Morgan algebras and distributive pseudocomplemented lattices. Our work gives an answer to Problem 10 raised in the last paragraph of that paper.

A special congruence ϕ, that in the case of pseudocomplemented lattices is the filter congruence associated with the filter of dense elements, has been very useful in the study of semi-De Morgan algebras in [7] and [9]. In this note we prove that if a semi-De Morgan algebra $\mathcal{L}$ which is not a De Morgan algebra is subdirectly irreducible, then ϕ is the minimum congruence in $\mathrm{Con}(\mathcal{L})\setminus\{\Delta\}$, lattice of congruences of $\mathcal{L}$, without Δ, the zero element (Theorem 2.10).

Demi-pseudocomplemented lattices (also called demi-p-lattices) form a subvariety of semi-De Morgan algebras that has deserved special attention (see [8] and [9]). This class is closer to the class of pseudocomplemented lattices and H. P. Sankappanavar has proved that it maintains some of the very interesting properties of pseudocomplemented lattices such as the congruence extension property. We determine the coherent

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semi-De Morgan algebras and the coherent demi-p-lattices. The notion of coherent algebra was introduced in D. Geiger [5] and the coherent De Morgan algebras were determined in [3] by R. Beazer.

2. Subdirectly irreducible semi-De Morgan algebras

We start by recalling some definitions and essential results.

Definition 2.1. An Ockham algebra is an algebra $(A; \vee, \wedge, ', 0, 1)$ for which $(A; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and satisfies the identities

$$(x \vee y)' \approx x' \wedge y', \quad (x \wedge y)' \approx x' \vee y', \quad 0' \approx 1 \quad \text{and} \quad 1' \approx 0.$$

The subvariety $K_{1,1}$ of the variety of Ockham algebras, first considered by J. Berman in [4], is the class of Ockham algebras which satisfy $x' = x'''$.

Definition 2.2. An algebra $\mathcal{L} = (\mathcal{L}, \vee, \wedge, ', 0, 1)$ is a semi-De Morgan algebra if the following five conditions hold $(x, y \in \mathcal{L})$:

(1) $(\mathcal{L}, \vee, \wedge, 0, 1)$ is a distributive lattice with $0, 1$.

(2) $0' \approx 1$ and $1' \approx 0$.

(3) $(x \vee y)' \approx x' \wedge y'$.

(4) $(x \wedge y)'' \approx x'' \wedge y''$.

(5) $x''' \approx x'$.

We will denote by $\boldsymbol{SDM}$ this equational class of algebras. The following rules hold in $\boldsymbol{SDM}$:

(6) $(x \wedge y)' \approx (x'' \wedge y'')' \approx (x' \vee y')''$.

(7) $(x \wedge y)'' \approx (x' \vee y')'$.

(8) $x \leq y$ implies $y' \leq x'$.

(9) $x \wedge (x \wedge y)' \geq x \wedge y'$.

(10) $(x \vee y)'' \approx (x' \wedge y')'$.

Clearly the intersection of the variety $\boldsymbol{SDM}$ with the variety of Ockham algebras is the variety $K_{1,1}$, so semi-De Morgan algebras are also a generalization of $K_{1,1}$ algebras. A semi-De Morgan algebra is a De Morgan algebra if and only if it satisfies the identity $x'' \approx x$. In what follows $\boldsymbol{M}$ will denote the class of De Morgan algebras. Indeed $\boldsymbol{M}$ is a subvariety of $K_{1,1}$.

If $\mathcal{L} \in \boldsymbol{SDM}$ and it satisfies the equation $x' \wedge x'' \approx 0$, then $\mathcal{L}$ is called a demi-p-lattice.

If $\mathcal{L} \in \boldsymbol{SDM}$ the set of the elements $a \in \mathcal{L}$ such that $a = a''$ will be denoted by $DM(\mathcal{L})$. Let us define, for $a, b \in DM(\mathcal{L})$, $a \dot{\vee} b = (a' \wedge b')'$, then $(DM(\mathcal{L}), \dot{\vee}, \wedge, ', 0, 1)$ is a De Morgan algebra.

The function $'' : \mathcal{L} \to \mathcal{L}$, $x \to x''$, is a homomorphism onto $DM(\mathcal{L})$ the kernel of which is $\phi = \{(x, y) \in \mathcal{L} \times \mathcal{L} \mid x' = y'\}$. Furthermore $\mathcal{L}/\phi \cong DM(\mathcal{L})$. These results were proved by H. P. Sankappanavar in [7].

Proposition 2.3. *Let $\mathcal{L} \in \boldsymbol{SDM}$. Each class of the congruence ϕ has one and only one element in $DM(\mathcal{L})$; more precisely, for each $x \in \mathcal{L}$, $x/\phi \cap DM(\mathcal{L}) = \{x''\}$.*

Proof. Clearly $x'' \in DM(\mathcal{L}) \cap x/\phi$. But, if $y \in DM(\mathcal{L}) \cap x/\phi$ there exists $z \in \mathcal{L}$ such that $z' = y$. Since $y' = x'$ we have $z'' = x'$, so $y = x''$. □

We shall write the subscript "lat" together with the letters that denote the relation, whenever we deal with lattice congruences. The notation $x \ll y$ will be used to indicate that the element y covers the element x.

The following proposition was first proved for $\mathcal{L} \in K_{1,1}$ (see [6], Lemma 2.1).

Proposition 2.4. *Let $\mathcal{L} \in \boldsymbol{SDM}$ and $a, b \in \mathcal{L}$ such that $a' = b'$. Then $\theta(a, b) = \theta_{\text{lat}}(a, b)$.*

Proof. Since $a' = b'$, we have $(a, b) \in \phi$, thus $\theta_{\text{lat}}(a, b) \leq \phi$. So $\theta_{\text{lat}}(a, b) \in \text{Con}(\mathcal{L})$ and $\theta_{\text{lat}}(a, b) = \theta(a, b)$. □

Proposition 2.5. *Let $\mathcal{L} \in \boldsymbol{SDM}$ be a finitely subdirectly irreducible algebra. Then for each $x, y \in \mathcal{L}$,*

(i) $|x/\phi| \leq 2$;

(ii) $(x, y) \in \phi$ *implies* $x = y$ *or* $x = y''$ *or* $y = x''$.

Proof. (i) Firstly we will prove that when $\mathcal{L}$ is a finitely subdirectly irreducible semi-De Morgan algebra there does not exist a chain of length greater than or equal to 2 in the same class of ϕ. Suppose that there exists $a < b < c$ such that $a' = b' = c'$. Let $(x, y) \in \theta(a, b) \cap \theta(b, c)$, by Proposition 2.4, we have $x \vee b = y \vee b$ and $x \wedge b = y \wedge b$. It, therefore, follows that $x = y$ and so $\theta(a, b) \cap \theta(b, c) = \Delta$, a contradiction.

As a congruence class of a lattice congruence is always a sublattice there can not exist $(x, y) \in \phi$ such that x and y are incomparable. Then $x/\phi = \{x\}$ or $x/\phi = \{x, y\}$ where $x \ll y$ or $y \ll x$.

(ii) If $(x, y) \in \phi$ and $x \neq y$, by Proposition 2.3, either $x \in DM(\mathcal{L})$ or $y \in DM(\mathcal{L})$. If $x \notin DM(\mathcal{L})$ then, since $|x/\phi| = 2$ and $x, x'' \in x/\phi$, we have $y = x''$. □

Corollary 2.6. *Let $\mathcal{L} \in \boldsymbol{SDM}$ be a finitely subdirectly irreducible algebra. Then for each $a \in \mathcal{L}$ we have $a = a''$ or $a \ll a''$ or $a'' \ll a$.*

To reach the characterization of the subdirectly irreducible semi-De Morgan algebras, in terms of the congruence ϕ, we present the three lemmas that follow.

Lemma 2.7. *Let $\mathcal{L} \in \boldsymbol{SDM}$ be a finitely subdirectly irreducible algebra. Then two distinct pairs of elements $c \neq c''$ and $d \neq d''$ can not be on the same chain.*

Proof. Suppose $c \ll c''$, $d \ll d''$ and $c'' \leq d$. If $(x, y) \in \theta(c, c'') \cap \theta(d, d'')$, by Proposition 2.4, we have $x \vee c'' = y \vee c''$ and $x \wedge d = y \wedge d$. But as $c'' \leq d$ it follows that $x \vee d = y \vee d$. The distributivity of $\mathcal{L}$ implies $x = y$, which is a contradiction. If we consider $d'' \leq c$ we arrive at a contradiction too.

In the cases $c \ll c''$ and $d'' \ll d$ or $c'' \ll c$ and $d \ll d''$ or $c'' \ll c$ and $d'' \ll d$ we can argue in the same way. □

Remark. Recall that, in a distributive lattice $\mathcal{L}$, if $a, b, c \in \mathcal{L}$ and $a \ll b$ then, $a \wedge c = b \wedge c$ or $a \vee c = b \vee c$. If $a \wedge c = b \wedge c$, then $a \vee c \ll b \vee c$, and if $a \vee c = b \vee c$, then $a \wedge c \ll b \wedge c$.

Lemma 2.8. *Let $\mathcal{L} \in \boldsymbol{SDM}$ be a finitely subdirectly irreducible algebra. If $a, b \in \mathcal{L}$, $a \neq a''$ and $b \neq b''$, then $\theta(a, a'') = \theta(b, b'')$.*

Proof. We will consider $a \ll a''$ and $b \ll b''$ with $a \neq b$ because in the other cases the proof is identical. By Proposition 2.5 (i), $a'' \neq b''$ and, by the previous lemma, $a'' \nleq b$ and $b'' \nleq a$. Since $b \ll b''$ using the properties in the above remark we will have that the following conditions hold simultaneously:

(1) $a \wedge b = a \wedge b''$ or $a \vee b = a \vee b''$.

(2) $a'' \wedge b = a'' \wedge b''$ or $a'' \vee b = a'' \vee b''$.

We must consider the following four cases:

Case 1. $a \wedge b = a \wedge b''$ and $a'' \wedge b = a'' \wedge b''$.

The second equality implies $a'' \vee b \ll a'' \vee b''$ and since $(a'' \vee b)' = (a'' \vee b'')'$, the Proposition 2.5 (ii) implies either $a'' \vee b = (a'' \vee b'')''$ or $(a'' \vee b)'' = a'' \vee b''$. But then $a \ll a'' \leq a'' \vee b \ll a'' \vee b''$ and this contradicts Lemma 2.7.

Case 2. $a \vee b = a \vee b''$ and $a'' \wedge b = a'' \wedge b''$.

The second equality implies $a \wedge b = a \wedge b''$. Hence $b = b''$, a contradiction.

Case 3. $a \vee b = a \vee b''$ and $a'' \vee b = a'' \vee b''$.

The first equality, by the previous remark, implies $a \wedge b \ll a \wedge b''$. Since $(a \wedge b)'' = a'' \wedge b'' = (a \wedge b'')''$, applying Proposition 2.5 (ii), we have either $a \wedge b'' = (a \wedge b)''$ or $a \wedge b = (a \wedge b'')''$. Then $a \wedge b \ll a \wedge b'' \leq a \ll a''$, a contradiction.

Case 4. $a \wedge b = a \wedge b''$ and $a'' \vee b = a'' \vee b''$.

In this case $(b, b'') \in \theta(a, a'')$ so $\theta(b, b'') \leq \theta(a, a'')$.

Analogously we can conclude that $\theta(a, a'') \leq \theta(b, b'')$ and so $\theta(a, a'') = \theta(b, b'')$. □

Lemma 2.9. *Let $\mathcal{L} \in \boldsymbol{SDM} \backslash \boldsymbol{M}$ be a finitely subdirectly irreducible algebra. Then, for each $a \in \mathcal{L}$ such that $a \neq a''$, $\theta(a, a'') = \phi$.*

Proof. Suppose $(x, y) \in \phi$ and $x \neq y$. If $x = y''$ then $(x, y) \in \theta(y, y'')$. By Lemma 2.8, $(x, y) \in \theta(a, a'')$, so $\phi \leq \theta(a, a'')$. □

Theorem 2.10. *Let $\mathcal{L} \in \boldsymbol{SDM} \backslash \boldsymbol{M}$ be a finitely subdirectly irreducible algebra. Then $\mathcal{L}$ is a subdirectly irreducible algebra and ϕ is the minimum congruence in* $\mathrm{Con}(\mathcal{L}) \backslash \{\Delta\}$.

Proof. Let $\alpha \in \mathrm{Con}(\mathcal{L}) \backslash \{\Delta\}$. Consider $(x, y) \in \alpha$ such that $x \neq y$. We have $\theta(x, y) \leq \alpha$. Since $\mathcal{L}$ is not a De Morgan algebra, there exists $a \in \mathcal{L}$ such that $a \neq a''$. By the previous lemma $\theta(a, a'') = \phi$. As $\mathcal{L}$ is a finitely subdirectly irreducible algebra, there exists $(c, d) \in \theta(a, a'') \cap \theta(x, y)$ such that $c \neq d$. Then $c = d''$ or $c'' = d$. Suppose, without loss of generality, that $c'' = d$. Then $(c, c'') \in \theta(x, y)$ and so $\theta(c, c'') \leq \theta(x, y) \leq \alpha$. By Lemma 2.9, $\phi \leq \alpha$. □

Corollary 2.11. *Let $\mathcal{L} \in \boldsymbol{SDM}$. $\mathcal{L}$ is a subdirectly irreducible algebra if and only if $\mathcal{L}$ is a subdirectly irreducible De Morgan algebra or ϕ is the minimum element of* $\mathrm{Con}(\mathcal{L}) \backslash \{\Delta\}$.

We can now state a corollary which extends a result obtained for $K_{1,1}$ algebras by R. Beazer (see [2], Corollary 7).

Corollary 2.12. *Let $\mathcal{L} \in \boldsymbol{SDM} \backslash \boldsymbol{M}$ be a finite subdirectly irreducible algebra. Then* $\mathrm{Con}(\mathcal{L})$ *is a boolean lattice with a new minimum element.*

Proof. $\mathrm{Con}(\mathcal{L}/\phi)$ is a boolean lattice (see [7], Theorem 3.3). So $[\phi, \nabla]$ is also a boolean lattice and the result follows. □

The precedent statements allow us to achieve alternative proofs of some known results; in particular, Theorem 5.5 in [8], Corollaries 3.2, 3.3 and Theorem 5.2 in [9]. As an example we present the following:

Proposition 2.12 ([9], Corollary 3.2). *The class of subdirectly irreducible demi-p-lattices is a universal class.*

Proof. It is known that this class is an elementary class (see [9], Theorem 3.1). Consider a subdirectly irreducible demi-p-lattice $\mathcal{L}$ and let $\mathcal{L}_1$ be a subalgebra of $\mathcal{L}$. We must show that $\mathcal{L}_1$ is a subdirectly irreducible algebra. If $\mathcal{L} \in \boldsymbol{M}$, then $\mathcal{L} = \widehat{C}_0$, the boolean algebra with two elements, and the statement is trivially verified.

If $\mathcal{L} \notin \boldsymbol{M}$ and $\mathcal{L}_1 \in \boldsymbol{M}$, all the elements of $\mathcal{L}_1$ satisfy $x = x''$, so $\mathcal{L}_1 \subseteq B(\mathcal{L}) = \{a \in DM(\mathcal{L}) : a \wedge a' = 0\}$. Suppose that there exists $c \in \mathcal{L}_1 \backslash \{0,1\}$. As $c \in B(\mathcal{L}) \backslash \{0,1\}$ it is known from Lemma 5.1 in [8] that $c \vee c' \neq 1$. But $c \vee c' \in \mathcal{L}_1$ so $c \vee c' = (c \vee c')'' = (c' \wedge c'')' = 0' = 1$, a contradiction. Then $\mathcal{L}_1 = \widehat{C}_0$.

If $\mathcal{L} \notin \boldsymbol{M}$ and $\mathcal{L}_1 \notin \boldsymbol{M}$, $\phi|_{\mathcal{L}_1} \neq \Delta|_{\mathcal{L}_1}$. Consider $\alpha \in \mathrm{Con}(\mathcal{L}_1)$ such that $\alpha \neq \Delta|_{\mathcal{L}_1}$. Since $\mathcal{L}$ has the extension congruence property ([7], Corollary 4.7) there exists $\beta \in \mathrm{Con}(\mathcal{L})$ such that $\beta|_{\mathcal{L}_1} = \alpha$ and then $\beta \neq \Delta$. By Theorem 2.10 $\beta \geq \phi$, so $\beta|_{\mathcal{L}_1} \geq \phi|_{\mathcal{L}_1}$ and consequently $\alpha \geq \phi|_{\mathcal{L}_1}$. Thus $\mathcal{L}_1$ is a subdirectly irreducible algebra. □

3. Coherent semi-De Morgan algebras

An algebra $\mathcal{L}$ is called coherent if each subalgebra containing a class of a congruence θ of $\mathcal{L}$ is a union of classes of θ. In [3], R. Beazer proved that the coherent algebras of $K_{1,1}$ are exactly the coherent De Morgan algebras. This result extends to $\boldsymbol{SDM}$. In the same paper the author showed that a pseudocomplemented lattice is coherent if and only if it is a boolean algebra. This remains true for the algebras of the variety of demi-p-lattices.

Theorem 3.1. *Let $\mathcal{L} \in \boldsymbol{SDM}$. Then $\mathcal{L}$ is coherent if and only if it is a coherent De Morgan algebra.*

Proof. Let $\mathcal{L} \in \boldsymbol{SDM}$ be a coherent algebra. We will prove that for each $x \in \mathcal{L}$, $x = x''$. Consider $S = \{0\} \cup 1/\phi$. Clearly S is a subalgebra of $\mathcal{L}$ and $0/\phi = \{0\}$.

For each $x \in \mathcal{L} \backslash \{0,1\}$ let denote by $S[x']$ the subalgebra of $\mathcal{L}$ generated by x'. It is easily verified that

$$S[x'] = \left\{0, x', x'', x' \wedge x'', x' \vee x'', (x' \wedge x'')', 1\right\}.$$

Since $0/\phi = \{0\}$ is contained in $S[x']$ it must contain $x''/\phi = x/\phi$, so $x \in S[x']$. If $x = x' \vee x''$ we have $x' = x'' \wedge x'$ and $x' \leq x''$. Thus $x' \vee x'' = x''$ and $x = x''$. For the other elements of $S[x']$ it is also true that $x = x''$ since they belong to $DM(\mathcal{L})$. So $\mathcal{L} \in \boldsymbol{M}$. □

Corollary 3.2. *Let $\mathcal{L}$ be a demi-p-lattice. $\mathcal{L}$ is coherent if and only if it is a boolean algebra.*

Proof. If $\mathcal{L}$ is a coherent demi-p-lattice then for each $x \in \mathcal{L}$, $x = x''$, so $x' \wedge x'' = x' \wedge x = 0$ and this implies that $\mathcal{L}$ is a boolean algebra. □

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