Extra loops I.

By FERENC FENYVES (Debrecen)

To professor A. G. Kuroš on his 60 th birthday

§ 1. Introduction

The object of this paper is to define a special class of loops, the class of extra loops, and to examine some of its properties. A loop $(G, \cdot)$ is said to be an extra loop if and only if the identity $(xy \cdot z)x = x(y \cdot zx)$ holds for all $x, y, z \in G$.

The main results of the present part of our paper are the following:

The three identities $(xy \cdot z)x = x(y \cdot zx)$, $yx \cdot zx = (y \cdot xz)x$, $xy \cdot xz = x(yx \cdot z)$ are equivalent for any loop (Theorem 1). Every extra loop is a Moufang loop (Theorem 3). If a loop is isotopic to an extra loop, then they are isomorphic (Theorem 4). A commutative extra loop is an abelian group (Theorem 5). And so, we shall get that the class of all Moufang loops properly contains the class of all extra loops.

In the second part [2] we give all possible nontrivial identities of Bol—Moufang-type and then we establish their connections to each other when we consider loops.

§ 2. Preliminaries

In this section we summarize definitions, notation and known or elementary results in loops theory which will be needed later. For the concepts of loop, inverse property loop and their basic properties see, for example [1].

If $(G, \cdot)$ is a loop, then we shall denote by 1 the unit element of $(G, \cdot)$ and, if $x \in G$, then the permutations (one-to-one mappings of G onto G) R_x and L_x are defined by $yR_x = yx$ and $yL_x = xy$ for all y in G.

If $(G, \cdot)$ is an inverse property loop, then the inverse mapping J of $(G, \cdot)$ is defined by $xJ = x^{-1}$ for every $x \in G$. Moreover we have: $J^2 = I$ (it is the identity mapping of G), $JL_xJ = R_{x^{-1}} = R_x^{-1}$, $JR_xJ = L_{x^{-1}} = L_x^{-1}$.

By the inner mapping group $\mathscr{I}(G)$ of the loop $(G, \cdot)$ we mean the group generated by the mappings $R_{x,y} \overset{\text{def}}{=} R_x R_y R_{xy}^{-1}$, $L_{x,y} \overset{\text{def}}{=} L_x L_y L_{yx}^{-1}$, $T_x \overset{\text{def}}{=} R_x L_x^{-1}$ for all $x, y \in G$.

An ordered triple $\langle U, V, W \rangle$ of permutations of the set G, is called an autotopism of the loop $(G, \cdot)$ if and only if $xU \cdot yV = (xy)W$ for all $x, y \in G$. Note, that the set of all autotopisms of $(G, \cdot)$ forms a group under componantwise multiplication:

$$\langle U_1, V_1, W_1 \rangle \langle U_2, V_2, W_2 \rangle = \langle U_1 U_2, V_1 V_2, W_1 W_2 \rangle.$$

The unit element of this group is $\langle I, I, I \rangle$, and $\langle U, V, W \rangle^{-1} = \langle U^{-1}, V^{-1}, W^{-1} \rangle$.

A permutation P of a set G is said to be a pseudo-automorphism of the loop $(G, \cdot)$ if and only if there exists an element c of G, called a companion of P, such that $\langle P, PR_c, PR_c\rangle$ is an autotopism of $(G, \cdot)$.

An element m of a loop $(G, \cdot)$ is called a Moufang element of $(G, \cdot)$ if and only if $\langle L_m, R_m, L_mR_m\rangle$ is an autotopism of $(G, \cdot)$.

A loop $(G, \cdot)$ is said to be a Moufang loop, if every $x \in G$ is Moufang element of $(G, \cdot)$.

The following results are well-known: (See [1], Chapter VII, Lemma 2.1 and Theorem 2. 3)

Lemma A. *If $\langle U, V, W\rangle$ is an autotopism of the inverse property loop $(G, \cdot)$, then*

(i) $\langle W, JVJ, U\rangle$ *and* $\langle JUJ, W, V\rangle$ *are autotopisms of* $(G, \cdot)$,

(ii) $u=1U, v=1V,$ *and* $w=1W$ *are Moufang elements of* $(G, \cdot)$.

Lemma B. *If $\langle U, V, W\rangle$ is an autotopism of the inverse property loop $(G, \cdot)$ and, if $1U=1$, then U is a pseudo-automorphism of $(G, \cdot)$ with companion $v=1V$.*

Lemma C. *Let $(G, \cdot)$ be a Moufang loop. Then a necessary and sufficient condition that every loop isotopic to $(G, \cdot)$ be isomorphic to $(G, \cdot)$ is that every element of G* be *a companion of some pseudo-automorphism of $(G, \cdot)$.*

§ 3. Definition and fundamental properties of extra loops

We begin with the

Theorem 1. *For a loop $(G, \cdot)$ the following identities are equivalent:*

$$(xy\cdot z)x = x(y\cdot zx), \tag{e_1}$$

$$yx\cdot xz = (y\cdot xz)x, \tag{e_2}$$

$$xy\cdot xz = x(yx\cdot z). \tag{e_3}$$

Moreover, if the loop $(G, \cdot)$ satisfies any one of these identities, then $(G, \cdot)$ has the inverse property.

Proof. Suppose that for the loop $(G, \cdot)$ (e_1) holds. For each $x \in G$ define x^{-1} by $x^{-1}x=1$. Then from (e_1), writing $y=x^{-1}$ and $z=1$ we get $xx^{-1}\cdot x = x\cdot x^{-1}x$, and hence $xx^{-1}=1$. Thus, also $(x^{-1})^{-1}=x$ holds. For arbitrary given elements x, y of G, with the help of the solutions u, v of the equations $x^{-1}u=y$ and $vx^{-1}=y$, and using (e_1), we get

$$yx\cdot x^{-1} = (x^{-1}u\cdot x)x^{-1} = x^{-1}(u\cdot xx^{-1}) = x^{-1}u = y,$$

that is

$$yx\cdot x^{-1} = y; \tag{1}$$

and

$$x^{-1}\cdot xy = x^{-1}(x\cdot vx^{-1}) = (x^{-1}x\cdot v)x^{-1} = vx^{-1} = y,$$

that is

$$x^{-1}\cdot xy = y \tag{2}$$

respectively.

By (1) and (2) the loop $(G, \cdot)$ with (e_1) has the inverse property.

On the other hand, the validity of (e_1) implies that, for all $x \in G$, $\langle L_x, R_x^{-1}, L_x R_x^{-1}\rangle$ is an autotopism of $(G, \cdot)$. So, by Lemma A, $\langle JL_xJ, L_xR_x^{-1}, R_x^{-1}\rangle = \langle R_x^{-1}, L_xR_x^{-1}, R_x^{-1}\rangle$ also is an autotopism of $(G, \cdot)$. That is for all x, y, z of G, $yx^{-1}\cdot(xz\cdot x^{-1}) = yz\cdot x^{-1}$. Taking $z = x^{-1}z$, we obtain $yx^{-1}\cdot zx^{-1} = (y\cdot x^{-1}z)x^{-1}$, and replacing here x^{-1} by x, we get

$$yx\cdot zx = (y\cdot xz)x.$$

Thus (e_1) implies (e_2).

Now we assume that the loop $(G, \cdot)$ satisfies (e_2). Define x^{-1} by $x^{-1}x = 1$ for all $x \in G$. Then from (e_2) with $y = x^{-1}$, $zx = (x^{-1}\cdot xz)x$ or

$$z = x^{-1}\cdot xz, \tag{3}$$

so $(G, \cdot)$ has the left inverse property. From (3) by $z = x^{-1}$, $x^{-1} = x^{-1}\cdot xx^{-1}$ and this implies $1 = xx^{-1}$, that is $(x^{-1})^{-1} = x$. Moreover in (e_2) we write $(xz)^{-1}$ instead of y and using (3), we obtain $[(xz)^{-1}\cdot x]zx = x = z^{-1}(zx)$ therefore $(xz)^{-1}\cdot x = z^{-1}$ and so

$$x = (xz)z^{-1}.$$

Hence $(G, \cdot)$ has the inverse property. Thus every step in the proof of $(e_1)\Rightarrow(e_2)$ is reversible. This means that (e_2) implies (e_1).

Assume again that $(G, \cdot)$ satisfies (e_2), then $(G, \cdot)$ is an inverse property loop. Taking inverses of both sides of (e_2), and replacing x^{-1}, y^{-1}, z^{-1} by x, z, y respectively we obtain the identity (e_3). Similarly, (e_3) implies (e_2). This completes the proof of Theorem 1.

Definition. A loop satisfying any one of the (equivalent) identities (e_1), (e_2), (e_3) is called an extra loop.

For example, the Cayley loop is an extra loop.

We need

Theorem 2. *A loop $(G, \cdot)$ is an extra loop if and only if, for all $x \in G$, $(G, \cdot)$ satisfyes any one of the following (equivalent) conditions:*

(a_1) $A_1(x) = \langle L_x, R_x^{-1}, L_xR_x^{-1}\rangle$ *is an autotopism of* $(G, \cdot)$,

(a_2) $A_2(x) = \langle R_x, L_x^{-1}R_x, R_x\rangle$ *is an autotopism of* $(G, \cdot)$,

(a_3) $A_3(x) = \langle R_x^{-1}L_x, L_x, L_x\rangle$ *is an autotopism of* $(G, \cdot)$.

Proof. Indeed, it is easy to see that (e_i) is equivalent to (a_i), $(i = 1, 2, 3)$, and so the proof is complete.

Let $(G, \cdot)$ be an extra loop, then using Theorem 2, $\langle R_x^{-1}L_x, L_x, L_x\rangle$ is an autotopism of $(G, \cdot)$ for each $x \in G$. But here $1R_x^{-1}L_x = 1$ and $1L_x = x$. So by Lemma A, x is a Moufang element of $(G, \cdot)$, and by Lemma B, $R_x^{-1}L_x$ is a pseudo-automorphism of $(G, \cdot)$ with companion x. Taking still into account Lemma C, we obtain the following two theorems:

Theorem 3. *Every extra loop $(G, \cdot)$ is a Moufang loop.*

Theorem 4. *If $(G, \cdot)$ is an extra loop, then every loop isotopic to $(G, \cdot)$ is isomorphic to $(G, \cdot)$.*

Theorem 5. *Let $(G, \cdot)$ be a commutative extra loop, then $(G, \cdot)$ is a commutative group.*

PROOF. Suppose $(G, \cdot)$ is a commutative extra loop. Then by the commutativity we obtain from (e_1) $(xy\cdot z)x=(y\cdot zx)x$ and this implies $z\cdot xy=zx\cdot y$ for all $x, y, z\in G$. This shows that $(G, \cdot)$ is a commutative group.

We remark that the class of all Moufang loops properly contains the class of all extra loops, because there are such commutative Moufang loops, which are not abelian groups (see [1], Chapter VIII).

Theorem 6. *Let $(G, \cdot)$ be an extra loop. Then T_x is a pseudo-automorphism with a companion x^{-1}, and $R_{x,y}$, $L_{x,y}$ are automorphisms of $(G, \cdot)$ for all $x, y\in G$.*

PROOF. Indeed, from $A_3(x^{-1})=\langle R_xL_x^{-1}, L_x^{-1}, L_x^{-1}\rangle=\langle T_x, L_x^{-1}, L_x^{-1}\rangle$ we obtain by $1T_x=1$, $1L_x^{-1}=x^{-1}$ that T_x is a pseudo-automorphism of $(G, \cdot)$ with companion x^{-1}, for each $x\in G$. Moreover, for all x, y in G, $A_2(x)A_2(y)A_2^{-1}(xy)=\langle R_{x,y}, V, R_{x,y}\rangle$ is an autotopism where $V=L_x^{-1}R_xL_y^{-1}R_yR_{xy}^{-1}L_{xy}$, $1R_{x,y}=1$ and $1V=1$ so, we get that $R_{x,y}$ is a pseudo-automorphism with companion 1, hence $R_{x,y}$ is an automorphism of $(G, \cdot)$. Similar considerations show that $L_{x,y}$ is also an automorphism. This completes the proof of Theorem 6.

References

[1] R. H. BRUCK, A survey of binary systems, *Berlin—Göttingen—Heidelberg*, 1958.
[2] F. FENYVES, Extra loops II., Publ. Math. Debrecen, **16** (1969) *(to appear)*.

(Received August 25, 1967.)