On lattice-ordered algebras with infinitary operations

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Dedicated to Professor A. G. Kuroš on his 60-th birthday

§ 1. Introduction

In the papers [3], [4] L. FUCHS began the systematical investigation on partially ordered (universal) algebras with finitary operations. The purpose of this paper is to introduce a class of lattice-ordered algebras with infinitary operations which play an important rôle in the theory of algebras with infinitary operations.

Let C denote a closure system on a set A. An operation g on A is called C-admissible if every $X\in C$ is closed under it. In Proposition 2. 2 it is proved that every C-admissible ν-ary operation on A induces a ν-ary operation $\bar{g}$ on C such that $\bar{g}$ is *isotone* in its variables. Furthermore $\bar{g}$ is *contractive* in C, i.e. for every $X\in C$

$$\bar{g}(X, X, \dots, X, \dots)\subseteq X$$

holds. (The definitions of the fundamental notions are given in § 2.)

Let $\mathfrak{A}=\langle A; F\rangle$ be an *algebra* (with infinitary or finitary operations). We assume that the meet of all subalgebras of $\mathfrak{A}$ is not void. It is known that the set $S(\mathfrak{A})$ of all subalgebras of $\mathfrak{A}$ forms a closure system on A.

Let $\langle V; G\rangle$ denote an algebra with the following properties:

(i) V is a lattice under a partial order $\leqq$;

(ii) every fundamental operation $g(\in G)$ is *isotone* in its variables;

(iii) every $g(\in G)$ is contractive in V.

By an *algebra-lattice* $\mathfrak{B}=\langle V; G; \leqq\rangle$ we mean an algebra $\langle V; G\rangle$ with the properties (i), (ii), (iii). $\mathfrak{B}=\langle V; G; \leqq\rangle$ is called a *complete algebra-lattice*, if V is a *complete* lattice under the partial order $\leqq$. Making use of the Dedekind—MacNeille completion method we prove that every algebra-lattice is isomorphic to some subalgebra-lattice of a complete algebra-lattice. (See Theorem 5.1.)

In Theorem 2. 3 it is proved that every closure system C on a set A forms a complete algebra-lattice under the set inclusion and under the operations $\bar{g}_0, \bar{g}_1, \dots, \bar{g}_n, \dots$ induced by the C-admissible operations $g_0, g_1, \dots, g_n, \dots$ on A. This theorem implies that the set $S(\mathfrak{A})$ of all subalgebras of an algebra $\mathfrak{A}$ forms a complete algebra-lattice under suitable operations. (See Corollary 2. 4.)

Conversely, for every complete algebra-lattice $\mathfrak{B}=\langle V; G; \leqq\rangle$ one can construct an algebra $\mathfrak{A}=\mathfrak{A}(\mathfrak{B})$ uniquely determined by $\mathfrak{B}$ such that the complete algebra-lattice $S(\mathfrak{A})$ of all subalgebras of $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$. (See Theorem 3. 2.) The proof of this representation theorem is based on Lemma 3.1, in which we show that $S(\mathfrak{A})$ coincides with the set of all closed intervals $[0, a]$ $(a \in V)$.

A subalgebra $\mathfrak{W}$ of a complete algebra-lattice $\mathfrak{V} = \langle V; G; \leqq \rangle$ is called a *complete subalgebra-ideal of* $\mathfrak{V}$, if $\mathfrak{W}$ is a *complete ideal* of the complete lattice $\langle V; \leqq \rangle$. In Lemma 3.1 it is proved that the set $I(\mathfrak{V})$ of all complete subalgebra-ideals of $\mathfrak{V}$ coincides with the set of all closed intervals $[0, a]$ $(a \in V)$ too. Thus one can show that $\mathfrak{V}$ and $I(\mathfrak{V})$ are isomorphic complete algebra-lattices by the mapping

$$a \to [0, a] \qquad (a \in V).$$

(See Theorem 3. 3.)

For *algebraic* closure systems on a set and for *finitary algebras* we can sharpen the mentioned results. E.g. the set of all subalgebras of a finitary algebra forms a *compactly generated algebra-lattice* and conversely, for a compactly generated algebra-lattice $\mathfrak{V}$ there exists a finitary algebra, whose subalgebras form a compactly generated algebra-lattice isomorphic to $\mathfrak{V}$. (See Corollary 4. 3 and Theorem 4. 5. In these results the known lattice-theoretical theorems of BIRKHOFF and FRINK [1] are extended for algebra-lattices.)

In § 6 we define the *complete grupoid-lattice* which is a special complete algebra-lattice. Example 6. 2 (6. 4) shows that the set of all subgroups of a group (the set of all subrings of a ring) forms a complete grupoid-lattice under suitable operations. Examples 6. 8 and 6. 9 are applications of Corollary 2. 4 and Theorem 3. 2.

§ 2. On algebras and complete algebra-lattices

Let A be a non-void set and let ν be an ordinal number. A *ν-ary operation on* A is a function $f(x_0, x_1, \ldots, \ldots, x_\eta, \ldots)$ $(\eta < \nu)$ which associates with every sequence $a_0, a_1, \ldots, a_\eta, \ldots$ $(\in A; \eta < \nu)$ an element $f(a_0, a_1, \ldots, a_\eta, \ldots)(\eta < \nu)$ of A. If f is a (ν-ary) operation on the set A, then we say that A is *closed under the operation f*. In the case $\nu = n < \omega_0$, a n-ary operation is called *finitary*. A *0-ary (nullary) operation* picks out a certain distinguished element in A. The ν-ary operations, where $\nu \geqq \omega_0$ are called *infinitary*.

Let τ be an ordinal number. A sequence $\mathfrak{A} = \langle A; f_0, f_1, \ldots, f_\xi, \ldots \rangle$ $(\xi < \tau)$ in which A is a non-void set and f_ξ is a ν_ξ-ary operation on A is called an *algebra of type* $\Lambda = \langle \nu_0, \nu_1, \ldots, \nu_\xi, \ldots \rangle$ $(\xi < \tau)$. τ is called the *order of* Λ. In the algebra $\mathfrak{A} = \langle A; f_0, f_1, \ldots, f_\xi, \ldots \rangle$ A denotes the *underlying set* and $f_0, f_1, \ldots, f_\xi, \ldots$ are the *fundamental operations on A*. Two algebras of the same type are called *similar*. (We use mostly the notions and the terminology of J. SLOMINSKY [9]).

Let $\mathfrak{A} = \langle A; f_0, f_1, \ldots, f_\xi, \ldots \rangle$ be an algebra of type $\Lambda = \langle \nu_0, \nu_1, \ldots, \nu_\xi, \ldots \rangle$ $(\xi < \tau)$ of order τ and B a non-void subset of A. If the fundamental operations $f_0, f_1, \ldots, f_\xi, \ldots$ $(\xi < \tau)$ are also operations on B, that is if B is closed under each of the operations $f_0, f_1, \ldots, f_\xi, \ldots$ $(\xi < \tau)$, then the sequence $\mathfrak{B} = \langle B; f_0, f_1, \ldots, f_\xi, \ldots \rangle$ $(\xi < \tau)$ is also an algebra of type Λ. $\mathfrak{B}$ is called a *subalgebra of algebra* $\mathfrak{A}$. We also write $\langle A; F \rangle$ for $\mathfrak{A} = \langle A; f_0, f_1, \ldots, f_\xi, \ldots \rangle$ and $\langle B; F \rangle$ for the subalgebra $\langle B; f_0, f_1, \ldots, f_\xi, \ldots \rangle$ of $\mathfrak{A}$.

Let $S(\mathfrak{A})$ denote the set of all subalgebras $\mathfrak{B}_\gamma = \langle B_\gamma; F \rangle$ $(\gamma \in \Gamma)$ of an algebra $\mathfrak{A} = \langle A; F \rangle$. If $\mathfrak{B}_\delta = \langle B_\delta; F \rangle$ $(\delta \in \Delta \subseteqq \Gamma)$ is a family of subalgebras of $\mathfrak{A}$, then the set-theoretical intersection $\bigcap_{\delta \in \Delta} \mathfrak{B}_\delta = \langle \bigcap_{\delta \in \Delta} B_\delta; F \rangle$ of the subalgebras $\mathfrak{B}_\delta$ is either the void set or a subalgebra of $\mathfrak{A}$. *In this paper we assume that the intersection of all subalgebras of an algebra* $\mathfrak{A}$ *is always a subalgebra of* $\mathfrak{A}$. Thus the set $S(\mathfrak{A})$ of all subalgebras of an algebra $\mathfrak{A}$ forms a complete lattice under the set-theoretical inclusion.

If X is a non-void subset of the underlying set A of an algebra $\mathfrak{A}=\langle A; F\rangle$ and $\mathfrak{B}_\delta(\delta\in\Delta)$ is the family of all subalgebras of $\mathfrak{A}$ which contain X, then the intersection $\bigcap_{\delta\in\Delta}\mathfrak{B}_\delta$ is the *subalgebra of* $\mathfrak{A}$ *generated by* X. This subalgebra will be denoted by $\{X\}$.

Let A be an arbitrary set and $B(A)$ the set of all its subsets. A subset C of $B(A)$ is said to be a *closure system* if C is closed under intersections, i.e. for any subsystem $D\subseteq C$, we have $\cap D\in C$.

Note that $A\in C$, by the definition of the intersection of a void family of subsets. Since a closure system C admits arbitrary intersections, it is easy to prove that *C forms a complete lattice* (with respect to the set-theoretical inclusion). However, it need not be a sublattice of $B(A)$. (See P. M. Cohn [2] Ch. II. 1.)

Let $\mathfrak{A}=\langle A; F\rangle$ be an algebra. The complete lattice $S(\mathfrak{A})$ of all subalgebras of $\mathfrak{A}$ is e.g. a closure system of A.

A *closure operator* on a partially ordered set $\langle P; \leqq\rangle$ is a mapping Φ of P into itself with the properties:

(2. 1) $$\text{if}\quad x\leqq y,\quad \text{then}\quad \Phi(x)\leqq\Phi(y),$$

(2. 2) $$x\leqq\Phi(x),$$

(2. 3) $$\Phi(\Phi(x))=\Phi(x)$$

for all $x, y\in P$.

For a closure operator Φ the element $\Phi(x)$ is called the *Φ-closure of x*; if an element x coincides with its Φ-closure, it is said to be *Φ-closed.*

The following theorem is proved in Cohn [2].

Theorem 2. 1. (See [2] Ch. II. Theorem 1. 1). *Every closure system C on a set A defines a closure operator Φ on the complete lattice $B(A)$ of all subsets of A by the rule*

$$\Phi(X)=\bigcap_{X\subseteq Y\in C} Y \qquad (X\in B(A)).$$

Conversely, every closure operator Φ on $B(A)$ defines a closure system C on A by

$$C=\textit{the set of all}\ \ X\in B(A)\ \ \textit{with}\ \ \Phi(X)=X,$$

and the correspondence $C\leftrightarrow\Phi$ between closure systems and closure operators thus defined is bijective.

Let us consider a closure system C of a set A. We call an operation g on A *C-admissible* if every set $X\in C$ is closed under the operation g.

Let g be an arbitrary ν-ary operation on the set A which is C-admissible with respect to the closure system C on A. For a sequence $A_0, A_1, \ldots, A_\eta, \ldots\in C(\eta<\nu)$ the set of all elements $g(a_0, a_1, \ldots, a_\eta, \ldots)$ where a_η is an arbitrary element of A_η $(\eta<\nu)$ is denoted by $g(A_0, A_1, \ldots, A_\eta, \ldots)$. Let Φ denote the closure operator on $B(A)$ corresponding to C, determined by Theorem 2.1. We define the operation $\bar{g}$ on the set C by the equality

$$(2.4) \qquad \bar{g}(A_0, A_1, \ldots, A_\eta, \ldots) = \Phi(g(A_0, A_1, \ldots, A_\eta, \ldots)) \qquad (\eta < \nu).$$

We call $\bar{g}$ the operation on C induced by g. (Cf. L. Fuchs [4]).

Let $S = S(\mathfrak{A})$ denote the closure system of all subalgebras of an algebra $\mathfrak{A} = \langle A; F\rangle$. In view of Theorem 2.1 the closure operator Φ determined by $S = S(\mathfrak{A})$ has the form

$$\Phi(X) = \bigcap_{X \subseteqq Y \in S(\mathfrak{A})} Y = \{X\} \qquad (X \in S(\mathfrak{A})).$$

Let g be an arbitrary S-admissible ν-ary operation on A. For a sequence $\mathfrak{B}_0, \mathfrak{B}_1, \ldots, \ldots, \mathfrak{B}_\eta, \ldots (\eta < \nu)$ of subalgebras of $\mathfrak{A}$ the operation $\bar{g}$ on $S(\mathfrak{A})$ induced by g is defined by

$$(2.4') \qquad \bar{g}(\mathfrak{B}_0, \mathfrak{B}_1, \ldots, \mathfrak{B}_\eta, \ldots) = \{g(\mathfrak{B}_0, \mathfrak{B}_1, \ldots, \mathfrak{B}_\eta, \ldots)\} \qquad (\eta < \nu).$$

Let $\langle P; \leqq\rangle$ be a partially ordered set. A ν-ary operation f on P is *isotone in its variables* $x_0, x_1, \ldots, x_\eta, \ldots (\eta < \nu)$ if the inequalities $a_\eta \leqq b_\eta$ $(a_\eta, b_\eta \in P; 0 \leqq \eta < \nu)$ imply the inequality

$$f(a_0, a_1, \ldots, a_\eta, \ldots) \leqq f(b_0, b_1, \ldots, b_\eta \ \ldots) \qquad (\eta < \nu)$$

for each sequence $a_0, a_1, \ldots, a_\eta, \ldots \in P$.

A μ-ary operation g on P is called *contractive* in P if

$$(2.5) \qquad g(\overset{0}{\breve{a}}, \overset{1}{\breve{a}}, \ldots, \overset{\xi}{\breve{a}}, \ldots) \leqq a \qquad (\xi < \mu)$$

holds for every element a of P.

Proposition 2.2. *Let C be a closure system on a set A and g a C-admissible ν-ary operation on A. Then the ν-ary operation $\bar{g}$ on C induced by g has the following properties:*

(a) *$\bar{g}$ is isotone in its variables,*

(b) *$\bar{g}$ is contractive in C.*

Proof. Because of (2.1) and (2.4) the operation g has evidently the property (a). As g is a C-admissible operation on A, for every $B \in C$

$$\bar{g}(\overset{0}{\breve{B}}, \overset{1}{\breve{B}}, \ldots, \overset{\eta}{\breve{B}}, \ldots) = \Phi(g(\overset{0}{\breve{B}}, \overset{1}{\breve{B}}, \ldots, \overset{\eta}{\breve{B}}, \ldots)) \subseteqq \Phi(B) = B.$$

holds, where Φ denotes the closure operator on $B(A)$ corresponding to C. Thus $\bar{g}$ is contractive in C.

We can now define a class of partially ordered algebraic systems which plays an important rôle in this paper.

Let $\langle V; f_0, f_1, \ldots, f_\xi, \ldots\rangle$ be an algebra of type $\Lambda = \langle \nu_0, \nu_1, \ldots, \nu_\xi, \ldots\rangle$ $(\xi < \tau)$ of order τ with the following properties:

(i) in V a partial order $\leqq$ is defined such that V forms a lattice under it. (The lattice operations are denoted by $\wedge$ and $\vee$);

(ii) every fundamental operation $f_0, f_1, \ldots, f_\xi, \ldots (\xi < \tau)$ is isotone in its variables;

(iii) every $f_0, f_1, \ldots, f_\xi, \ldots (\xi < \tau)$ is contractive in V.

By an *algebra-lattice* $\mathfrak{B} = \langle V; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee \rangle$ we mean an algebra $\langle V; f_0, f_1, \ldots, f_\xi, \ldots \rangle$ of type $\Lambda = \langle \nu_0, \nu_1, \ldots, \nu_\xi, \ldots \rangle$ $(\xi < \tau)$ of order τ, satisfying the properties (i), (ii), (iii). (An algebra-lattice is a special lattice-ordered algebra with infinitary operations; for the definition of lattice-ordered algebras with finitary operations see in FUCHS [3] p. 117.)

If we assume instead of (i) the condition

(i′) in V a partial order $\leqq$ is defined such that V forms a complete lattice under it. (The lattice operations are denoted by $\wedge$ and $\vee$), then the algebra-lattice $\mathfrak{B} = \langle V; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee \rangle$ is called *complete*.

Naturally a complete algebra-lattice $\mathfrak{B} = \langle V; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee \rangle$ has a *least element* $0 = \bigwedge_{x \in V} x$ and a *greatest element* $e = \bigvee_{x \in V} x$.

With the help of Proposition 2. 2 we can prove

Theorem 2. 3. *Let C be a closure system on a set A, σ an ordinal number and $G = \langle g_0, g_1, \ldots, g_\eta, \ldots \rangle$ $(\eta < \sigma)$ a sequence of C-admissible operations on A. If $\bar{g}_\eta$ is the operation on C induced by g_η $(\eta < \sigma)$, then C forms a complete algebra-lattice under the operations $\bar{g}_0, \bar{g}_1, \ldots, \bar{g}_\eta, \ldots (\eta < \sigma)$ and under the set inclusion $\subseteqq$.*

PROOF. If $g_0, g_1, \ldots, g_\eta, \ldots (\eta < \sigma)$ are μ_0-ary, μ_1-ary, ..., μ_η-ary, ...$(\eta < \sigma)$ operations on A, then in view of Proposition 2. 2 $\bar{g}_0, \bar{g}_1, \ldots, \bar{g}_\eta, \ldots (\eta < \sigma)$ are μ_0-ary, μ_1-ary, ..., μ_η-ary, ... operations on C. Thus $\mathfrak{C} = \langle C; \bar{g}_0, \bar{g}_1, \ldots, \bar{g}_\eta, \ldots \rangle$ $(\eta < \sigma)$ is an algebra of type $\langle \mu_0, \mu_1, \ldots, \mu_\eta, \ldots \rangle$ of order σ. We have to show that the algebra $\mathfrak{C}$ satisfies the condition (i′), (ii), (iii). Evidently $\mathfrak{C}$ is a complete lattice under the set-theoretical inclusion. As every operation $\bar{g}_0, \bar{g}_1, \ldots, \bar{g}_\eta, \ldots$ $(\eta < \sigma)$ has the properties (a) and (b), therefore conditions (ii) and (iii) are fulfilled.

Since the set $S(\mathfrak{A})$ of all subalgebras of an algebra $\mathfrak{A} = \langle A; F \rangle$ is a closure system on A, Theorem 2. 3 implies the following

Corollary 2. 4. *Let $\mathfrak{A} = \langle A; F \rangle$ be an algebra and $S = S(\mathfrak{A})$ the closure system of all subalgebras of $\mathfrak{A}$. If σ is an ordinal number and $G = \langle g_0, g_1, \ldots, g_\eta, \ldots \rangle$ $(\eta < \sigma)$ a sequence of S-admissible operations on A, then S forms a complete algebra-lattice under the operations $\bar{g}_0, \bar{g}_1, \ldots, \bar{g}_\eta, \ldots$ and under the set inclusion, where $\bar{g}_\eta$ is the operation on S induced by $g_\eta (\eta < \sigma)$.*

Remark 1. It is possible that two S-admissible operations f, g of an algebra $\mathfrak{A}$ are different but the operations $\bar{f}, \bar{g}$ on the set $S(\mathfrak{A})$ induced by f and g are identical. E.g. let R be an associative ring and let

$$f(a, b) = a + b, \qquad g(a, b) = a + b - ab \qquad (a, b \in R)$$

be two admissible binary operations on R. If A and B are two arbitrary subrings of R, then in view of (2. 4′)

$$\bar{f}(A, B) = \{a + b;\, a \in A,\, b \in B\} \quad \text{and} \quad \bar{g}(A, B) = \{a + b - ab;\, a \in A,\, b \in B\}$$

are two operations on the set $S(R)$ of all subrings of R. The elements $f(a, b)$

and $g(a, b)$ of R are generally not equal, but $\bar{f}(A, B)$ and $\bar{g}(A, B)$ both are the subring of R generated by A and B.

Remark 2. Evidently not all the identities of the algebra $\mathfrak{A}=\langle A; f_0, f_1, \ldots, \ldots, f_\xi, \ldots\rangle$ $(\xi<\tau)$ retain their validity in the algebra $\langle S(\mathfrak{A}); \bar{f}_0, \bar{f}_1, \ldots, \bar{f}_\xi, \ldots\rangle$ where $\bar{f}_\xi(\xi<\tau)$ is the operation on $S(\mathfrak{A})$ induced by f_ξ. (See Example 6. 3).

§ 3. A representation theorem on complete algebra-lattices

Let $\mathfrak{B}=\langle V; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee\rangle$ $(\xi<\tau)$ be a complete algebra-lattice. $\mathfrak{W}=\langle W; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee\rangle$ $(\xi<\tau)$ is a *complete subalgebra-lattice* of $\mathfrak{B}$, if $\mathfrak{W}$ is a subalgebra and a complete sublattice of $\mathfrak{B}$.

We call a (non-void) subset A of a complete lattice $\mathfrak{L}=\langle L; \wedge, \vee\rangle$ a *complete ideal of* $\mathfrak{L}$, if the following two conditions hold:

(1) if $a_\lambda(a_\lambda\in A; \lambda\in\Lambda)$ denotes an arbitrary system of elements of A, then $\bigvee_{\lambda\in\Lambda} a_\lambda\in A$;

(2) if $a\in A$ and $x\in L$, then $a\wedge x\in A$.

Evidently every complete ideal of a complete lattice $\mathfrak{L}$ is a complete sublattice of $\mathfrak{L}$.

A complete subalgebra-lattice $\mathfrak{W}=\langle W; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee\rangle$ of a complete algebra-lattice $\mathfrak{B}=\langle V; f_0, f_1, \ldots, f_\eta, \ldots; \wedge, \vee\rangle$ is called a *complete subalgebra-ideal* of $\mathfrak{B}$, if $\langle W; \wedge, \vee\rangle$ is a complete ideal of the complete lattice $\langle V; \wedge, \vee\rangle$.

In Corollary 2. 4 we give a method how one can obtain complete algebra-lattices from a given algebra. Now we shall define to every complete algebra-lattice $\mathfrak{B}$ a uniquely determined algebra which plays an important rôle in the representation of $\mathfrak{B}$.

Let $\mathfrak{B}=\langle V; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee\rangle$ be an arbitrary complete algebra-lattice. We define for every element a of V the following unary operation g_a on V:

$$g_a(x)=\begin{cases} a, & \text{if} \quad a\vee x=x, \\ 0, & \text{if} \quad a\vee x\neq x \end{cases} \qquad (x\in V). \tag{3. 1}$$

Let $g_a, g_b, \ldots(a, b, \ldots\in V)$ be a sequence of all unary operations defined by (3.1). We call the algebra $\mathfrak{A}=\mathfrak{A}(\mathfrak{B})=\langle V; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee; g_a, g_b, \ldots\rangle$ the *algebra belonging to the complete algebra-lattice* $\mathfrak{B}$. (If α is the cardinality of V and $\alpha\geqq\aleph_0$, then we consider the meet $\wedge$ and join $\vee$ as infinitary operations on V.)

Lemma 3. 1. *Let* $\mathfrak{B}=\langle V; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee\rangle$ *be a complete algebra-lattice. For a non-void subset* W *of* V *the following conditions are equivalent:*

(a) $\mathfrak{W}=\langle W; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee\rangle$ *is a complete subalgebra-ideal of* $\mathfrak{B}$;

(b) $0\in W$ *and* $\mathfrak{W}=\langle W; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee\rangle$ *is a convex* [1]) *complete subalgebra-lattice of* $\mathfrak{B}$;

(c) W *has the form* $W=[0, w]$, *where* $w=\bigvee_{x\in W} x$;

(d) $\mathfrak{B}=\langle W; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee; g_a, g_b, \ldots\rangle$ *is a subalgebra of the algebra* $\mathfrak{A}=\langle V; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee; g_a, g_b, \ldots\rangle$ *belonging to* $\mathfrak{B}$.

[1]) A subset of a partially ordered set is called *convex* if it contains the whole interval $[a, b]$ whenever it contains the endpoints a, b.

PROOF. The implication (a)$\Rightarrow$(b) follows immediately from the property (2).

(b)$\Rightarrow$(c). As W is closed under the operation $\vee$, the element $w=\bigvee_{x\in W} x$ belongs to W. Thus $0, w\in W$ and the convexity of W imply (c).

(c)$\Rightarrow$(d). Let us assume that $W=[0, w]$ $\left(w=\bigvee_{x\in W} x\right)$ holds. We have to prove that W is closed under every fundamental operation $f_0, f_1, \dots, f_\xi, \dots, \wedge, \vee$, $g_a, g_b, \dots$ $(a, b, \dots \in V)$ on V. Evidently $W=[0, w]$ is closed under the operations $\wedge$ and $\vee$. By the conditions (ii) and (iii) the operations $f_0, f_1, \dots, f_\xi, \dots$ are isotone in their variables and they are contractive in V, therefore $[0, w]$ $(w\in V)$ is closed under these operations too. Let us consider an arbitrary unary operation g_k $(k\in V)$. If for the element y of the closed interval $[0, w]$ the relation $k\vee y=y$ holds, then $0\leqq k\leqq y\leqq w$ and so because of (3. 1)

$$g_k(y)=k\in[0, w].$$

If z is an element of $[0, w]$ with the property $k\vee z\neq z$, then in view of (3. 1)

$$g_k(z)=0\in[0, w]$$

holds. Thus the interval $W=[0, w]$ is closed under each operation $g_k\in V$.

(d)$\Rightarrow$(a). Let $\mathfrak{B}=\langle W; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee; g_a, g_b, \dots\rangle$ be a subalgebra of the algebra $\mathfrak{A}=\mathfrak{A}(\mathfrak{V})=\langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee; g_a, g_b, \dots\rangle$ belonging to $\mathfrak{V}$. As the underlying set W is closed under the operations $f_0, f_1, \dots, f_\xi, \dots, \wedge$ and $\vee$, therefore $\mathfrak{W}=\langle W; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee\rangle$ is a complete subalgebra-lattice of $\mathfrak{V}$. We have only to show that $\langle W; \wedge, \vee\rangle$ is a complete ideal of the complete lattice $\langle V; \wedge, \vee\rangle$. By the definition of $\mathfrak{B}$ the property (1) holds evidently. If $a\in W$ and $x\in V$, then for the element $b=a\wedge x$

$$g_b(a)=b=a\wedge x \qquad (b\vee a=a)$$

holds. As W is closed under the unary operation g_b, therefore $a\wedge x=b=g_b(a)\in W$. Thus the property (2) holds too.

With the help of Lemma 3.1 we can prove our main result. First we must define some fundamental notions.

Let $\mathfrak{A}=\langle A; f_0, f_1, \dots, f_\xi, \dots\rangle$ and $\mathfrak{B}=\langle B; f_0', f_1', \dots, f_\xi' \dots\rangle$ be two similar algebras of type $\Delta=\langle \nu_0, \nu_1, \dots, \nu_\xi, \dots\rangle$ $(\xi<\tau)$ of order τ. Let us assume that φ is a one-to-one mapping of A onto B such that

$$\varphi\big(f_\xi(a_0, a_1, \dots, a_\eta, \dots)\big)=f_\xi'(\varphi(a_0), \varphi(a_1), \dots, \varphi(a_\eta), \dots) \quad (\xi<\tau; \eta<\nu_\xi)$$

holds for each pair of fundamental operations f_ξ, f_ξ' and for all sequences $a_0, a_1, \dots, \dots, a_\eta, \dots (\in A; \eta<\nu_\xi)$, then we say that the algebras $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic by the isomorphism* φ.

We call the (complete) algebra-lattices $\mathfrak{V}=\langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee\rangle$ and $\mathfrak{W}=\langle W; f_0', f_1', \dots, f_\xi', \dots; \wedge', \vee'\rangle$ *isomorphic*, if the similar algebras $\langle V; f_0, f_1, \dots, f_\xi, \dots\rangle$ and $\langle W; f_0', f_1', \dots, f_\xi', \dots\rangle$ furthermore the (complete) lattices $\langle V; \wedge, \vee\rangle$ and $\langle W; \wedge', \vee'\rangle$ are isomorphic by the same isomorphism φ.

Theorem 3. 2. *Let* $\mathfrak{V}=\langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee\rangle$ $(\xi<\tau)$ *be a complete algebra-lattice and let* $\mathfrak{A}=\mathfrak{A}(\mathfrak{V})=\langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee; g_a, g_b, \dots\rangle=$ $=\langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee; g_a, g_b, \dots\rangle$ *be the algebra belonging to* $\mathfrak{V}$. *Let*

$\langle S=S(\mathfrak{A});\ \cap, \cup\rangle$ denote the complete lattice of all subalgebras of $\mathfrak{A}$. Then $\mathfrak{B}$ is isomorphic to the complete algebra-lattice $\mathfrak{S}=\langle S; \bar{f}_0, \bar{f}_1, \ldots, \bar{f}_\xi \ldots;\ \cap, \cup\rangle$, where $\bar{f}_\xi$ is the operation on $S=S(\mathfrak{A})$ induced by f_ξ $(\xi<\tau)$.

PROOF. From the equivalence of the conditions (c) and (d) in Lemma 3. 1 we obtain that the set $S=S(\mathfrak{A})$ of all subalgebras of the algebra $\mathfrak{A}=\langle V; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee; g_a, g_b, \ldots\rangle$ is exactly the set of all closed intervals $[0, a]$ $(a\in V)$. If $f_0, f_1, \ldots, \ldots, f_\xi, \ldots(\xi<\tau)$ are ν_0-ary, ν_1-ary, ..., ν_ξ-ary, ... operations on V, then the induced operations $\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_\xi, \ldots$ $(\xi<\tau)$ are ν_0-ary, ν_1-ary, ..., ν_ξ-ary, ... operations on $S(\mathfrak{A})$. Namely, for an arbitrary sequence $[0, a_0], [0, a_1], \ldots, [0, a_\eta], \ldots$ $(a_0, a_1, \ldots, \ldots, a_\eta, \ldots \in V;\ \eta<\nu_\xi)$ of subalgebras of $\mathfrak{A}$

$$(3.\ 2)\qquad \bar{f}_\xi([0, a_0], [0, a_1], \ldots, [0, a_\eta], \ldots)=\{f_\xi([0, a_0], [0, a_1], \ldots, [0, a_\eta], \ldots)\}$$
$$(\eta<\nu_\xi)$$

holds, according to (2. 4′). In view of Corollary 2. 4 $\mathfrak{S}=\langle S; \bar{f}_0, \bar{f}_1, \ldots, \bar{f}_\xi, \ldots; \cap, \cup\rangle$ is a complete algebra-lattice.

Now we will show that the given complete algebra-lattice $\mathfrak{B}=\langle V; f_0, f_1, \ldots, \ldots, f_\xi, \ldots; \wedge, \vee\rangle$ $(\xi<\tau)$ is isomorphic to $\mathfrak{S}$. First we remark that the algebras $\langle V; f_0, f_1, \ldots, f_\xi, \ldots\rangle$ and $\langle S; \bar{f}_0, \bar{f}_1, \ldots, \bar{f}_\xi \ldots\rangle$ both are of the same type $\Delta= =\langle \nu_0, \nu_1, \ldots, \nu_\xi, \ldots\rangle$ $(\xi<\tau)$ of order τ. It is enough to prove that the mapping

$$(3.\ 3)\qquad \varphi: a\to[0, a] \qquad (a\in V;\ [0, a]\in S(\mathfrak{A}))$$

of V onto $S(\mathfrak{A})$ is an isomorphism between the algebras $\langle V; f_0, f_1, \ldots, f_\xi, \ldots\rangle$, $\langle S(\mathfrak{A}); \bar{f}_0, \bar{f}_1, \ldots, \bar{f}_\xi, \ldots\rangle$ and between the complete lattices $\langle V; \wedge, \vee\rangle$, $\langle S(\mathfrak{A}); \cap, \cup\rangle$. Naturally φ is a one-to-one mapping of V onto $S(\mathfrak{A})$. Thus we have to show only the following properties of homomorphism:

(3. 4) for every sequence $a_0, a_1, \ldots, a_\eta \ldots$ $(\in V;\ \eta<\nu_\xi)$

$$f_\xi(a_0, a_1, \ldots, a_\eta, \ldots)\to[0, f_\xi(a_0, a_1, \ldots, a_\eta, \ldots)]=\bar{f}_\xi([0, a_0], [0, a_1], \ldots, [0, a_\eta], \ldots);$$

(3. 5) for every system of elements $a_\lambda(\in V;\ \lambda\in\Lambda)$ the relations

$$\bigwedge_{\lambda\in\Lambda} a_\lambda \to [0, \bigwedge_{\lambda\in\Lambda} a_\lambda] = \bigcap_{\lambda\in\Lambda} [0, a_\lambda] \quad\text{and}\quad \bigvee_{\lambda\in\Lambda} a_\lambda \to [0, \bigvee_{\lambda\in\Lambda} a_\lambda] = \bigcup_{\lambda\in\Lambda} [0, a_\lambda]$$

hold.

Because of the definition of the operation $\bar{f}_\xi$

$$(3.\ 6)\qquad f_\xi(a_0, a_1, \ldots, a_\eta, \ldots)\in\bar{f}_\xi([0, a_0], [0, a_1], \ldots, [0, a_\eta], \ldots) \qquad (\eta<\nu_\xi).$$

As $\bar{f}_\xi([0, a_0], [0, a_1], \ldots, [0, a_\eta], \ldots)$ is a subalgebra of $\mathfrak{A}$, therefore it must have the form

$$\bar{f}_\xi([0, a_0], [0, a_1], \ldots, [0, a_\eta], \ldots)=[0, b] \qquad (b\in V).$$

Thus (3. 6) implies $[0, f_\xi(a_0, a_1, \ldots, a_\eta, \ldots)\subseteqq\bar{f}_\xi([0, a_0], [0, a_1], \ldots, [0, a_\eta], \ldots)$. Conversely, according to (3. 2) the subalgebra $\bar{f}_\xi([0, a_0], [0, a_1], \ldots, [0, a_\eta], \ldots)$ is generated by all the elements $f_\xi(x_0, x_1, \ldots, x_\eta, \ldots)(x_0\in[0, a_0], x_1\in[0, a_1], \ldots, \ldots, x_\eta\in[0, a_\eta], \ldots(\eta<\nu_\xi)$. Since f_ξ is isotone in its variables, we have

$$0\leqq f_\xi(x_0, x_1, \ldots, x_\eta, \ldots)\leqq f_\xi(a_0, a_1, \ldots, a_\eta, \ldots) \quad (0\leqq x_\eta\leqq a_\eta;\ \eta<\nu_\xi),$$

whence $\bar{f}_\xi([0, a_0], [0, a_1], \ldots, [0, a_\eta], \ldots) \subseteq [0, f_\xi(a_0, a_1, \ldots, a_\eta, \ldots)]$ $(\eta < \nu_\xi)$ follows. Thus (3. 4) is proved.

Let us show that the relation (3. 5) is true. Namely, if x is an element of the subalgebra $[0, \bigwedge_{\lambda \in \Lambda} a_\lambda]$ then $0 \leqq x \leqq \bigwedge_{\lambda \in \Lambda} a_\lambda \leqq a_\lambda$ $(\lambda \in \Lambda)$ holds, whence $x \in \bigcap_{\lambda \in \Lambda} [0, a_\lambda]$. On the other hand, if the element y is contained in every subalgebra $[0, a_\lambda]$ $(\lambda \in \Lambda)$, then $y \leqq \bigwedge_{\lambda \in \Lambda} a_\lambda$. Hence $y \in [0, \bigwedge_{\lambda \in \Lambda} a_\lambda]$.

Because $\bigcup_{\lambda \in \Lambda} [0, a_\lambda]$ denotes the subalgebra of $\mathfrak{A}$ generated by the subalgebras $[0, a_\lambda]$ $(\lambda \in \Lambda)$, therefore $\bigvee_{\lambda \in \Lambda} a_\lambda \in \bigcup_{\lambda \in \Lambda} [0, a_\lambda]$. On the other hand the subalgebra $\bigcup_{\lambda \in \Lambda} [0, a_\lambda]$ of $\mathfrak{A}$ has the form $\bigcup_{\lambda \in \Lambda} [0, a_\lambda] = [0, u]$ with a suitable element $u \in V$, whence $[0, \bigvee_{\lambda \in \Lambda} a_\lambda] \subseteq \bigcup_{\lambda \in \Lambda} [0, a_\lambda]$. Conversely, every subalgebra $[0, a_\lambda]$ $(\lambda \in \Lambda)$ of $\mathfrak{A}$ is contained in the subalgebra $[0, \bigvee_{\lambda \in \Lambda} a_\lambda]$, therefore $\bigcup_{\lambda \in \Lambda} [0, a_\lambda] \subseteq [0, \bigvee_{\lambda \in \Lambda} a_\lambda]$ must hold.

Thus (3. 5_2) is proved and *the proof of Theorem* 3. 2 *is finished.*

Let us consider a complete algebra-lattice $\mathfrak{B} = \langle V; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee \rangle$. It follows from Lemma 3. 1 that the set $I(\mathfrak{B})$ of all complete subalgebra-ideals of $\mathfrak{B}$ coincides with the set of all closed intervals $[0, a]$ $(a \in V)$. Hence $I(\mathfrak{B})$ is a closure system on V such that the operations $f_0, f_1, \ldots, f_\xi, \ldots$ are $I(\mathfrak{B})$-admissible. If f_ξ is a ν_ξ-ary operation and $[0, a_0], [0, a_1], \ldots, [0, a_\eta], \ldots$ $(\eta < \nu_\xi;\ a_0, a_1, \ldots, a_\eta, \ldots \in V)$ is an arbitrary sequence of complete subalgebra-ideals of $\mathfrak{B}$, then because of (2. 4) and the properties of f_ξ the operation $\bar{f}_\xi$ on $I(\mathfrak{B})$ induced by f_ξ satisfies the relation

$$(3.\ 7) \qquad \bar{f}_\xi([0, a_0], (0, a_1], \ldots, [0, a_\eta], \ldots) = [0, f_\xi(a_0, a_1, \ldots, a_\eta, \ldots)] \quad (\eta < \nu_\xi).$$

In view of Theorem 2. 3 $\mathfrak{T} = \langle I(\mathfrak{B}); f_0, f_1, \ldots, f_\xi, \ldots;\ \cap,\ \cup \rangle$ is a complete algebra-lattice. Lemma 3. 1 and Theorem 3. 2 imply that $\mathfrak{B}$ is isomorphic to $\mathfrak{T}$ by the mapping

$$(3.\ 8) \qquad a \to [0, a] \qquad (a \in V).$$

So we have obtained:

Theorem 3. 3. *Let* $\mathfrak{B} = \langle V; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee \rangle$ $(\xi < \tau)$ *be a complete algebra-lattice and* $I(\mathfrak{B})$ *the set of all complete subalgebra-ideals of* $\mathfrak{B}$. *By the mapping* (3. 8) $\mathfrak{B}$ *is isomorphic to the complete algebra-lattice* $\mathfrak{T} = \langle I(\mathfrak{B}); \bar{f}_0, \bar{f}_1, \ldots, \bar{f}_\xi, \ldots$ $\cap,\ \cup \rangle$, *where* $\bar{f}_\xi$ *is the operation on* $I(\mathfrak{B})$ *induced by* f_ξ $(\xi < \tau)$.

We mention the following special case of Theorem 3. 3.

Corollary 3. 4. *Let L be a complete lattice. The set of all complete ideals of L forms under the set theoretical inclusion a complete lattice such that is isomorphic to L by the mapping* (3. 8).

§ 4. On finitary algebras and compactly generated algebra-lattices

An algebra $\mathfrak{A} = \langle A; f_0, f_1, \ldots, f_\xi, \ldots \rangle$ of type $\Delta = \langle n_0, n_1, \ldots, \ldots, n_\xi, \ldots \rangle$ $(\xi < \tau)$ of order τ is called *finitary*, if $n_0, n_1, n_\xi, \ldots$ are non-negative integers. For finitary algebras we can sharpen Theorems 2. 3 and 3. 2 and Corollary 2. 4.

An element a of a complete lattice $\langle L; \wedge, \vee \rangle$ is called *compact* if the following condition is satisfied: if $a \leqq \bigvee_{\gamma \in \Gamma} x_\gamma (x_\gamma \in L)$, then there exists a finite subset $\Gamma'(\subseteq \Gamma)$ such that $a \leqq \bigvee_{\gamma \in \Gamma'} x_\gamma$.

By a *compactly generated algebra-lattice* $\mathfrak{B} = \langle V; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee \rangle$ we mean a finitary algebra $\langle V; f_0, f_1, \ldots, f_\xi, \ldots \rangle$ of type $\Delta = \langle n_0, n_1, \ldots, n_\xi \ldots \rangle$ $(\xi < \tau)$ of order τ satisfying the properties (i′), (ii), (iii) and

(iv) every element b of the complete lattice $\langle V; \wedge, \vee \rangle$ can be written as $b = \bigvee_\mu a_\mu$, where all the a_μ are compact elements of V, i.e. $\langle V; \wedge, \vee \rangle$ is a *compactly generated lattice*.

A closure system C on a set A is called *algebraic (or inductive)* if every chain in C has a supremum in C.

The following result is known. (See G. Grätzer [6] Chapter 0, § 6, Theorem 4.)

Theorem 4. 1. *Every algebraic closure system C on a set A forms a compactly generated lattice with respect to the set inclusion.*

Theorems 2. 3 and 4.1 imply

Theorem 4. 2. *Let C be an algebraic closure system on a set A, σ an ordinal number and $G = \langle g_0, g_1, \ldots, g_\eta, \ldots \rangle$ $(\eta < \sigma)$ a sequence of C-admissible finitary operations on A. If $\bar{g}_\eta$ is the operation on C induced by g_η $(\eta < \sigma)$, then C forms a compactly generated algebra-lattice under the operations $\bar{g}_0, \bar{g}_1, \ldots, \bar{g}_\eta, \ldots$ $(\eta < \sigma)$ and under the set inclusion.*

In P. M. Cohn [2] it is proved that the set $S(\mathfrak{A})$ of all subalgebras of a finitary algebra $\mathfrak{A}$ is an algebraic closure system. (See [2] Theorem 5. 2.) Birkhoff—Frink [1] proved directly that $S(\mathfrak{A})$ forms a compactly generated lattice under the set inclusion.[2])

These results, Theorem 4. 2 and Corollary 2. 4 imply

Corollary 4. 3. *Let $\mathfrak{A} = \langle A; F \rangle$ be a finitary algebra and $S = S(\mathfrak{A})$ the algebraic closure system of all subalgebras of $\mathfrak{A}$. If σ is an ordinal number and $G = \langle g_0, g_1, \ldots, \ldots, g_\eta, \ldots \rangle$ $(\eta < \sigma)$ is a sequence of S-admissible finitary operations on A, then $\langle S; \bar{g}_0, \bar{g}_1, \ldots, \bar{g}_\eta, \ldots; \cap, \cup \rangle$ is a compactly generated algebra-lattice, where $\bar{g}_\eta$ is the operation on S induced by g_η $(\eta < \sigma)$.*

Let K denote the set of all compact elements of a compactly generated lattice $\mathfrak{L} = \langle L; \wedge, \vee \rangle$. Evidently the zero element 0 of $\mathfrak{L}$ is always compact. As the union of two compact elements is a compact element, $\langle K; 0, \vee \rangle$ forms a semi-lattice with 0 under the nullary operation 0 and the binary operation $\vee$. We define for every element k of K the following unary operation g_k on K:

$$g_k(x) = \begin{cases} k, & \text{if} \quad k \vee x = x, \\ 0, & \text{if} \quad k \vee x \neq x \end{cases} \qquad (x \in K). \tag{4. 1}$$

Let $g_k, g_l, \ldots$ $(k, l \ldots \in K)$ be a sequence of all unary operations defined by (4. 1).

2) This theorem is generalized in G. Grätzer[5] for the subalgebras of an algebra (with infinitary operations).

We call the finitary algebra $\mathfrak{A}=\mathfrak{A}(K)=\langle K; 0, \vee, g_k, g_l, \ldots\rangle$ of type $\langle 0, 2, 1, 1, \ldots\rangle$ the *finitary algebra belonging to K*.

An *ideal* of a semilattice $\mathfrak{F}=\langle F; \vee\rangle$ is a non-void set A of F such that, for all $a, b \in F$

$$a\vee b \in A \quad \text{if and only if} \quad a, b\in A.$$

We prove the following analogon of Lemma 3.1.

Lemma 4. 4. (Cf. GRÄTZER—SCHMIDT [7]). *Let K denote the set of all compact elements of the compactly generated lattice $\mathfrak{L}=\langle L; \wedge, \vee\rangle$. For a non-void subset B of K the following conditions are equivalent:*

(a′) *B is an ideal of the semilattice $\langle K; \vee\rangle$;*

(c′) *B has the form $B=[0, b]\cap K=[0, b]_K$, where $b=\bigvee_{y\in B} y$;*

(d′) *$\mathfrak{B}=\langle B; 0, \vee, g_k, g_l, \ldots\rangle$ is a subalgebra of the finitary algebra $\mathfrak{A}=\mathfrak{A}(K)=\langle K; 0, \vee, g_k, g_l, \ldots\rangle$ belonging to K.*

PROOF. (a′)⇒(c′). If B is an ideal of $\langle K; \vee\rangle$, then for the element $b=\bigvee_{y\in B} y$

$$B\subseteqq [0, b]\cap K=[0, b]_K$$

holds. To show that $[0, b]_K=B$, we must prove that $x\in[0, b]_K$ implies $x\in B$. If $x\in[0, b]_K$, then x is a compact element with the property

$$x\leqq b=\bigvee_{y\in B} y.$$

Hence, $x\leqq \bigvee_{y\in B'} y$ for some finite subset B' of B. Set $B'=\langle y_1, y_2, \ldots, y_n\rangle$ and $b'=y_1\vee y_2\vee\ldots\vee y_n$. Then $x\vee b'=b'\in B$ and the ideal property of B imply $x\in B$, which was to be proved.

(c′)⇒(d′). Let $B=[0, b]\cap K=[0, b]_K$ with $b=\bigvee_{y\in B} y$. Evidently $0\in[0, b]_K$ and $[0, b]_K$ is closed under the binary operation $\vee$. Now let us consider an arbitrary unary operation g_k $(k\in K)$. (4. 1) implies for every element y of $[0, b]_K=B$ that $g_k(y)\in[0, b]_K$ holds. Thus $\langle B; 0, \vee, g_k, g_l, \ldots\rangle$ is a subalgebra of $\langle K; 0, \vee, g_k, g_l, \ldots\rangle$, indeed.

(d′)⇒(a′). Now let $\mathfrak{B}=\langle B; 0, \vee, g_k, g_l, \ldots\rangle$ be a subalgebra of $\mathfrak{A}=\mathfrak{A}(K)=\langle K; 0, \vee, g_k, g_l, \ldots\rangle$. If $x, y\in B$, then evidently $x\vee y\in B$. Conversely, let us assume that for the elements $k, l\in K$

$$k\vee l\in B$$

holds. We have to show that $k, l\in B$. Since B is closed under the opertaions g_k and g_l $(k, l\in K)$, it follows from (4.1)

$$g_k(k\vee l)=k\in B \quad\text{and}\quad g_l(k\vee l)=l\in B.$$

Thus B is an ideal of the semilattice $\langle K; \vee\rangle$.

For the compactly generated algebra-lattices, Theorem 3. 2 has the following analogon:

Theorem 4. 5. *Let $\mathfrak{B}=\langle V; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee\rangle$ be a compactly generated algebra-lattice and let K denote the set of all compact elements of the complete lattice $\langle V; \wedge, \vee\rangle$. The set $S(\mathfrak{A})$ of all subalgebras of the finitary algebra $\mathfrak{A}=\mathfrak{A}(K)$ belonging* *to K forms under suitable operations a compactly generated algebra-lattice isomorphic to* $\mathfrak{B}$.

PROOF. First we shall show that the mapping

$$a \to [0, a]_K = [0, a] \cap K \qquad (a \in V) \tag{4. 2}$$

is one-to-one. Let us assume that for the elements $a, b \in V$ the relation $[0, a]_K = [0, b]_K$ holds. Because of the condition (iv) one can write the element $a (\in V)$ in the form $a = \bigvee_{\lambda \in \Lambda} a_\lambda$, where $a_\lambda \in K$. This implies $a_\lambda \in [0, a] \cap K = [0, a]_K$ for every $\lambda \in \Lambda$, whence we obtain

$$a = \bigvee_{\lambda \in \Lambda} a_\lambda \leqq \bigvee_{x \in [0, a]_K} x \leqq a,$$

that is $a = \bigvee_{x \in [0, a]_K} x$. Similarly $b = \bigvee_{y \in [0, b]_K} y$. Thus the assumption $[0, a]_K = [0, b]_K$ implies $a = b$.

In view of Lemma 4. 4 we obtained that the set $S(\mathfrak{A})$ of all subalgebras of the finitary algebra $\mathfrak{A} = \langle K; 0, \vee, g_k, g_l, \dots \rangle$ belonging to K is exactly the set of all intersections $[0, a]_K = [0, a] \cap K$ $(a \in V)$. It is known that $S(\mathfrak{A})$ forms a compactly generated lattice under the set-theoretical inclusion. Now we have to define the „suitable" operations on $S(\mathfrak{A})$. If the operations [3] $f_0, f_1, \dots, f_\xi, \dots, (\xi < \tau)$ on V are n_0-ary, n_1-ary, ..., n_ξ-ary, ..., then we can define the following n_0-ary, n_1-ary, ..., ..., n_ξ-ary, ... operations $f'_0, f'_1, \dots, f'_\xi, \dots$ $(\xi < \tau)$ on $S(\mathfrak{A})$. For an arbitrary sequence $[0, a_1]_K, [0, a_2]_K, \dots, [0, a_{n_\xi}]_K$ $(a_1, a_2, \dots, a_{n_\xi} \in V)$ of subalgebras of $\mathfrak{A}$ let

$$f'_\xi([0, a_1]_K, [0, a_2]_K, \dots, [0, a_{n_\xi}]_K) = [0, f_\xi(a_1, a_2, \dots, a_{n_\xi})]_K. \tag{4. 3}$$

It is obvious that the operation f'_ξ on $S(\mathfrak{A})$ defined by (4. 3) is isotone in its variables and it is contractive in $S(\mathfrak{A})$.

Denoting by $\cap$ and $\cup$ the lattice-operations in $S(\mathfrak{A})$ we obtained that $\mathfrak{S} = \langle S(\mathfrak{A}); f'_0, f'_1, \dots, f'_\xi, \dots; \cap, \cup \rangle$ is a compactly generated algebra-lattice, furthermore $\langle V; f_0, f_1, \dots, f_\xi \dots \rangle$ and $\langle S(\mathfrak{A}); f'_0, f'_1, \dots, f'_\xi, \dots \rangle$ $(\xi < \tau)$ are finitary algebras of the same type $\langle n_0, n_1, \dots, n_\xi, \dots \rangle$ of order τ. We shall show that the mapping (4. 2) is an isomorphism of the given compactly generated algebra-lattice $\mathfrak{B}$ onto $\mathfrak{S} = \langle S(\mathfrak{A}); f'_0, f'_1, \dots, f'_\xi, \dots; \cap, \cup \rangle$. As we have seen (4. 2) is a one-to-one mapping.

For the operation f_ξ the property of homomorphism $f_\xi(a_1, a_2, \dots, a_{n_\xi}) \to [0, f_\xi(a_1, a_2, \dots, a_{n_\xi})]_K = f'_\xi([0, a_1]_K, [0, a_2]_K, \dots, [0, a_{n_\xi}]_K)$ $(a_1, a_2, \dots, a_{n_\xi} \in V)$ follows from (4. 3).

To complete the proof, we have to show the relations

$$\bigwedge_{\lambda \in \Lambda} a_\lambda \to [0, \bigwedge_{\lambda \in \Lambda} a_\lambda]_K = \bigcap_{\lambda \in \Lambda} [0, a_\lambda]_K \quad \text{and} \quad \bigvee_{\lambda \in \Lambda} a_\lambda \to [0, \bigvee_{\lambda \in \Lambda} a_\lambda]_K = \bigcup_{\lambda \in \Lambda} [0, a_\lambda]_K \tag{4. 4}$$

for an arbitrary system $a_\lambda (\lambda \in \Lambda)$ of elements of V. The proof of (4. 4) is analogous to the proof of (3. 5).

It follows from Lemma 4. 4 and the proof of Theorem 4. 5 the following analogon of Theorem 3. 3.

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[3]) Since the operations $f_0, f_1, \dots, f_\xi, \dots$ $(\xi < \tau)$ generally are not $S(\mathfrak{A})$-admissible, we cannot use Corollary 4.3.

Theorem 4. 6. *Let* $\mathfrak{B}=\langle V; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee\rangle$ $(\xi<\tau)$ *be a compactly generated algebra-lattice and let* $\langle K; \vee\rangle$ *denote the semilattice of all compact elements of* $\langle V; \wedge, \vee\rangle$. *If* $I(K)$ *denotes the set of all ideals of* $\langle K; \vee\rangle$, *then by the mapping* (4. 2) $\mathfrak{B}$ *is isomorphic to the compactly generated algebra-lattice* $\langle I(K); f_0', f_1', \ldots, \ldots, f_\xi', \ldots; \cap, \cup\rangle$, *where the operation* f_ξ' $(\xi<\tau)$ *is defined by* (4. 3).

This theorem is an extension of the mentioned result of GRÄTZER and SCHMIDT [7] for the compactly generated algebra-lattices.

§ 5. Embedding of algebra-lattices into complete algebra-lattices

We shall prove the following

Theorem 5. 1. *Every algebra-lattice is isomorphic to some subalgebra-lattice of a suitable complete algebra-lattice.*

This theorem is an analogon of the known result of MACNEILLE [8] about the embedding of lattices into complete lattices. (See e.g. G. SZÁSZ [10] Corollary of Theorem 27).

WARD [11] proved the following result.

Proposition 5. 2. *Let* Φ *be a closure operator of a complete lattice* L. *The set* Z_Φ *of the* Φ*-closed elements of* L *forms a complete lattice under the operations*

$$\inf_{Z_\Phi} R=\inf_L R,\ \sup_{Z_\Phi} R=\Phi(\sup_L R)$$

for an arbitrary subset R *of* Z_Φ. (See e.g. G. SZÁSZ [10] Theorem 25.).

For a subset X of a partially ordered set $\langle P; \leqq\rangle$ an *upper (lower) bound* of X in P is an element $u\in P$ $(v\in P)$ such that $x\leqq u$ $(x\geqq v)$ for every $x\in X$. The set of all upper (lower) bounds of X will be denoted by $U(X)(L(X))$. The set $L(U(X))=D(X)$ is called the *Dedekind cut* determined by X. It is easy to show that the following relations hold:

(5. 1) $X\subseteq Y\ (\subseteq P)$ implies $U(X)\supseteq U(Y)$ and $L(X)\supseteq L(Y)$,

(5. 2) $X\subseteq L(U(X))$ and $X\subseteq U(L(X))$,

(5. 3) $U(L(U(X)))=U(X)$ and $L(U(L(X)))=L(X)$.

One gets from (5. 1), (5. 2) and (5. 3):

(5. 4) if $X\subseteq Y(\subseteq P)$, then $D(X)=L(U(X))\subseteq L(U(Y))=D(Y)$,

(5. 5) $X\subseteq D(X)=L(U(X))$ $\quad (X\subseteq P)$,

(5. 6) $D(D(X))=L[U(L(U(X)))]=L(U(X))=D(X)$ $\quad (X\subseteq P)$.

As the set $B(P)$ of all subsets of the partially ordered set P forms a complete lattice under the set-theoretical inclusion, we have verified the following known result:

Proposition 5. 3. *Let $B(P)$ denote the complete lattice of all subsets of the partially ordered set P. The mapping*

$$X \to L(U(X)) = D(X) \qquad (X \in B(P))$$

of $B(P)$ into itself is a closure operator on $B(P)$.

Now we begin the proof of Theorem 5. 1. Let $\mathfrak{V} = \langle V; f_0, f_1, \ldots, f_\xi, \ldots; \wedge, \vee \rangle$ be an arbitrary algebra-lattice. If $B(\mathfrak{V})$ denotes the complete lattice of all subsets of the lattice $\langle V; \wedge, \vee \rangle$, then in view of Proposition 5. 3 the mapping

$$X \to L(U(X)) = D(X) \qquad (X \in B(\mathfrak{V}))$$

of $B(\mathfrak{V})$ into itself is a closure operator on $B(\mathfrak{V})$. Using Proposition 5. 2 for the closure operator D on the complete lattice $B(\mathfrak{V})$ we get that the set Z_D of all D-closed elements of $B(\mathfrak{V})$ is a complete lattice under the operations of $\inf_{Z_D}$ and $\sup_{Z_D}$, defined in Proposition 5. 2. We shall show that Z_D forms under suitable operations a complete algebra-lattice, in which $\mathfrak{V}$ can be embedded isomorphically.

Let us consider the fundamental operation f_ξ of V. If f_ξ is a ν_ξ-ary operation, then we consider an arbitrary sequence $A_0, A_1, \ldots, A_\eta, \ldots \in Z_D$ $(\eta < \nu_\xi)$. Let $f_\xi(A_0, A_1, \ldots, A_\eta, \ldots)$ denote the set of all elements $f_\xi(a_0, a_1, \ldots, a_\eta, \ldots)$ $(a_0 \in A_0, a_1 \in A_1, \ldots, a_\eta \in A_\eta, \ldots)$. We define by

$$\text{(5. 7)} \qquad f_\xi^*(A_0, A_1, \ldots, A_\eta, \ldots) = L(U(f_\xi(A_0, A_1, \ldots, A_\eta, \ldots))) = = D(f_\xi(A_0, A_1, \ldots, A_\eta, \ldots)) \quad (\eta < \nu_\xi)$$

a ν_ξ-ary operation [4]) on Z_D. We are going to show that $\langle Z_D; f_0^*, f_1^*, \ldots, f_\xi^*, \ldots; \inf_{Z_D}, \sup_{Z_D} \rangle$ is a complete algebra-lattice.

If $A_0, A_1, \ldots, A_\eta, \ldots \in Z_D$ and $B_0, B_1, \ldots, B_\eta, \ldots \in Z_D$ are two sequences with the property $A_\eta \subseteq B_\eta$ $(0 \leqq \eta < \nu_\xi)$, then

$$f_\xi(A_0, A_1, \ldots, A_\eta, \ldots) \subseteq f_\xi(B_0, B_1, \ldots, B_\eta, \ldots) \qquad (\eta < \nu_\xi).$$

This implies by (5. 4) and (5. 7)

$$f_\xi^*(A_0, A_1, \ldots, A_\eta, \ldots) \subseteq f_\xi^*(B_0, B_1, \ldots, B_\eta, \ldots) \qquad (\eta < \nu_\xi).$$

So f_ξ^* is isotone in its variables, i.e. property (ii) is fulfilled for the operation f_ξ^*.

Now let us consider an arbitrary D-closed set $A(\in Z_D)$ and let $u(\in V)$ be an upper bound of A. If $f_\xi(a_0, a_1, \ldots, a_\eta \ldots)$ $(a_0, a_1, \ldots, a_\eta, \ldots \in A)$ denotes an arbitrary element of the set $f_\xi(\overset{0}{\breve{A}}, \overset{1}{\breve{A}}, \ldots, \overset{\eta}{\breve{A}}, \ldots)$ $(\eta < \nu_\xi)$, then because of the conditions (ii) and (iii) given for f_ξ

$$f_\xi(a_0, a_1, \ldots, a_\eta, \ldots) \leqq f_\xi(\overset{0}{\breve{u}}, \overset{1}{\breve{u}}, \ldots, \overset{\eta}{\breve{u}}, \ldots) \leqq u \qquad (\eta < \nu_\xi)$$

holds. This means that every upper bound $u(\in V)$ of A is an upper bound of the set $f_\xi(\overset{0}{\breve{A}}, \overset{1}{\breve{A}}, \ldots, \overset{\eta}{\breve{A}}, \ldots)$, that is

$$U(A) \subseteq U(f_\xi(\overset{0}{\breve{A}}, \overset{1}{\breve{A}}, \ldots, \overset{\eta}{\breve{A}}, \ldots)) \qquad (\eta < \nu_\xi).$$

---

[4]) In (5. 7) we extend the definition (8) of Fuchs [3] for infinitary operations.

This implies because of (5. 1), (5. 7) and $D(A)=L(U(A))=A$

$$f_\xi^*(\overset{0}{\breve{A}}, \overset{1}{\breve{A}}, \dots, \overset{\eta}{\breve{A}}, \dots) = D\left(f_\xi(\overset{0}{\breve{A}}, \overset{1}{\breve{A}}, \dots, \overset{\eta}{\breve{A}}, \dots)\right) =$$

$$= L\left(U\left(f_\xi(\overset{0}{\breve{A}}, \overset{1}{\breve{A}}, \dots, \overset{\eta}{\breve{A}}, \dots)\right)\right) \subseteqq L\left(U(A)\right) = D(A) = A,$$

that is the operation f_ξ^* is contractive in Z_D.

Thus $\langle Z_D; f_0^*, f_1^*, \dots, f_\xi^*, \dots; \inf_{Z_D}, \sup_{Z_D}\rangle$ is a complete algebra-lattice indeed.

Now we show that the mapping

$$x \to D(x) = L(x) \qquad (x \in V) \tag{5. 8}$$

is an isomorphism of the algebra-lattice $\mathfrak{V} = \langle V; f_0, f_1, \dots, f_\xi, \dots; \wedge, \vee\rangle$ into the complete algebra-lattice

$$\mathfrak{Z} = \langle Z_D; f_0^*, f_1^*, \dots, f_\xi^*, \dots; \inf_{Z_D}, \sup_{Z_D}\rangle.$$

Evidently (5. 8) is a one-to-one mapping. One can verify the laws of homomorphism

$$x \wedge y \to D(x \wedge y) = \inf_{Z_D}\left(D(x), D(y)\right) \qquad (x, y \in V),$$

$$x \vee y \to D(x \vee y) = \sup_{Z_D}\left(D(x), D(y)\right) \qquad (x, y \in V)$$

by the usual method. (See e.g. the proof of Theorem 27 in G. SZÁSZ [10]). We have only to prove the relation

$$\begin{aligned} f_\xi(x_0, x_1, \dots, \dots, x_\eta, \dots) &\to D\left(f_\xi(x_0, x_1, \dots, x_\eta, \dots)\right) = \\ &= f_\xi^*\left(D(x_0), D(x_1), \dots, D(x_\eta), \dots\right) \qquad (\eta < \nu_\xi) \end{aligned} \tag{5. 9}$$

for an arbitrary sequence $x_0, x_1, \dots, x_\eta, \dots \in V$. Because of $L(x) = D(x) \in Z_D$ and (5. 7) we have

$$\begin{aligned} &f_\xi^*\left(D(x_0), D(x_1), \dots, D(x_\eta), \dots\right) = f_\xi^*\left(L(x_0), L(x_1), \dots, L(x_\eta), \dots\right) = \\ &= D\left(f_\xi(L(x_0), L(x_1), \dots, L(x_\eta), \dots)\right) = L\left(U\left(f_\xi(L(x_0), L(x_1), \dots, L(x_\eta), \dots)\right)\right) (\eta < \nu_\xi). \end{aligned} \tag{5. 10}$$

As $f_\xi(x_0, x_1, \dots, x_\eta, \dots)$ is the greatest element of the set $f_\xi\left(L(x_0), L(x_1), \dots, L(x_\eta), \dots\right)$ therefore we have

$$U\left(f_\xi(L(x_0), L(x_1), \dots, L(x_\eta), \dots)\right) = U\left(f_\xi(x_0, x_1, \dots, x_\eta, \dots)\right) \qquad (\eta < \nu_\xi).$$

This and (5.10) imply

$$f_\xi(x_0, x_1, \dots, x_\eta, \dots) \to D\left(f_\xi(x_0, x_1, \dots, x_\eta, \dots)\right) = L\left(U\left(f_\xi(x_0, x_1, \dots, x_\eta, \dots)\right)\right) =$$

$$= L\left(U\left(f_\xi(L(x_0), L(x_1), \dots, L(x_\eta), \dots)\right)\right) = f_\xi^*\left(D(x_0), D(x_1), \dots, D(x_\eta), \dots\right)$$

that is (5. 9) is true.

The proof of Theorem 5.1 is finished.

Remark 3. By the embedding of the algebra-lattice $\mathfrak{V}$ into the complete algebra-lattice $\mathfrak{Z}$ the identities between operations of $\mathfrak{V}$ carry over only exceptionally to $\mathfrak{Z}$. In FUCHS [3] a sufficient condition is given for an identity to carry over to $\mathfrak{Z}$.

§ 6. Examples and applications

First we define a special complete algebra-lattice which plays an important rôle in the following examples.

Let L be a *partially ordered groupoid*, i.e. let in L a binary multiplication and a partial order $\leqq$ be defined with the property

(6. 1) $\quad a \leqq b (a, b \in L)$ implies $ac \leqq bc$ and $ca \leqq cb$ for all $c \in L$.

We assume that the following conditions hold in L:

(6. 2) $$a^2 \leqq a \quad (\text{for all } a \in L)$$

and L is a *complete lattice* with respect to the partial order $\leqq$, whose *least*, *greatest elements* are 0 and e, respectively, satisfying

(6. 3) $$0 \cdot e = e \cdot 0 = 0.$$

The partially ordered groupoid L with the mentioned properties is called a *complete groupoid-lattice*.

From the conditions (6. 1), (6. 3) it follows

(6. 4) $$0 \cdot a = a \cdot 0 = 0 \qquad (\text{for all } a \in L).$$

Evidently every complete groupoid-lattice L is a special complete algebra-lattice, as properties (i′), (ii) and (iii) are fulfilled in L.

Example 6. 1. Let $\mathfrak{L}_0 = \langle C_0; \cdot, 0 \rangle$ be a groupoid with 0. If A, B are two subgroupoids with zero of $\mathfrak{L}_0$, then in view of (2. 4′) the multiplication defined on C_0 induces the following binary multiplication

(6. 5) $$AB = \{ab;\, a \in A,\, b \in B\}$$

on the set L_1 of all subgroupoids with zero of $\mathfrak{L}_0$. It is easy to show that L_1 forms a complete groupoid-lattice under the multiplication (6. 5) and under the set inclusion. We denote by $\cap$ and $\cup$ the lattice operations in the complete lattice L_1. Evidently 0 is the least element and C_0 is the greatest one of L_1.

Example 6. 2. Let L_2 denote the set of all subgroups of a group G. It is known that L_2 forms a complete lattice under the set inclusion. Evidently the commutator-forming $[x, y] = xyx^{-1}y^{-1}$ $(x, y \in G)$ is an L_2-admissible binary operation on G. If H, K denote two arbitrary subgroups of G, then which respect to (2. 4′) the operation $[\cdot, \cdot]$ induces the following binary operation

(6. 6) $$[H, K] = \{[h, k];\, h \in H,\, k \in K\}$$

on the complete lattice L_2. It is easy to see that L_2 forms a complete groupoid-lattice under the „multiplication" (6. 6) and under the lattice operations. The least element of L_2 is the unit element of G and its greatest element is G. It is remarkable that in general $[x, y] = [y, x]$ $(x, y \in G)$ does not hold, but $[H, K] = [K, H]$ holds for all $H, K \in L_2$.

Example 6. 3. Let $\mathfrak{F}_0 = \langle F_0; \cdot, 0 \rangle$ be a semigroup with 0. Similarly to Example 6. 1 the set L_3 of all subsemigroups with zero of $\mathfrak{F}_0$ forms a complete groupoid-lattice. It is interesting that the associative multiplication on F_0 induces by (6. 5) an in general non-associative multiplication on L_3. Namely if A, B, C, are subsemigroups with zero of $\mathfrak{F}_0$, then for the elements $a_1, a_2 \in A$, $b_1 b_2 \in B$ and $c \in C$

$$d = (a_1 b_1 a_2 b_2)c \in (AB)C \text{ but in general } d \notin A(BC).$$

Example 6. 4. Analogously to Example 6.1 the set L_4 of all subrings of a (not necessarily associative) ring R forms a complete groupoid-lattice, whose least element is 0 and the greatest one is R.

In the foregoing examples we considered only one (binary) operation of the given algebraic system and the set of all algebraic subsystems formed always a complete groupoid-lattice. In an other paper we want to write about complete groupoid-lattices, which satisfy some conditional associative and distributive laws. Now we give some illustrating examples for the results of §§ 2, 3 and 4.

Example 6. 5. Let us consider again the set L_2 of all subgroups of a group G. If A, B are two subgroups of G, then the multiplication on G induces by (2. 4′) the multiplication $AB = \{ab;\, a \in A,\, b \in B\}$ in L_2, satisfying the relation $AB = A \cup B$. The inverse-forming $a^{-1} (a \in G)$ is an L_2-admissible unary operation in G, it induces by (2. 4′) the unary operation $A^{-1} = A (\in L_2)$ in L_2. (It is evident that not the same identities hold for the multiplication and inverse-forming on G as for the induced operations on L_2.) In view of Corollary 4. 3 L_2 forms a compactly generated algebra-lattice under the mentioned two induced operations and under the set inclusion.

Example 6. 6. Let $\mathfrak{L} = \langle L;\, \wedge, \vee \rangle$ be a complete lattice with the least element 0 and with the greatest element e. Let $S_0(\mathfrak{L})$ denote the set of all complete sublattices with zero of $\mathfrak{L}$. If $A_\gamma (\gamma \in \Gamma)$ is a system of elements of $S_0(\mathfrak{L})$ then in view of (2. 4′) the operations $\wedge$ and $\vee$ induce the following operations $\overline{\wedge}$ and $\overline{\vee}$ on $S_0(\mathfrak{L})$:

$$\overline{\bigwedge_{\gamma \in \Gamma}} A_\gamma = \{\bigwedge_{\gamma \in \Gamma} x_\gamma;\, x_\gamma \in A_\gamma (\gamma \in \Gamma)\},$$

$$\overline{\bigvee_{\gamma \in \Gamma}} A_\gamma = \{\bigvee_{\gamma \in \Gamma} x_\gamma;\, x_\gamma \in A_\gamma (\gamma \in \Gamma)\}.$$

It is easy to show that $\overline{\wedge}$ and $\overline{\vee}$ are commutative and associative (infinitary) operations on $S_0(\mathfrak{L})$. Naturally $S_0(\mathfrak{L})$ forms a complete lattice under the set inclusion. If we denote by $\cap$ and $\cup$ the operations of this complete lattice, then $\langle S_0(\mathfrak{L});\, \overline{\wedge}, \overline{\vee};\, \cap, \cup \rangle$ is a complete algebra-lattice.

Example 6. 7. Let us consider the following operations $a+b$, $-a$, ab, $a \circ b = a + b - ab$ of an associative ring R $(a, b \in R)$. If A, B are two subrings of R, then the given operations induce by (2. 4′) the following operations on the set L_5 of all subrings of R:

$$A + B = \{a + b;\, a \in A;\, b \in B\},$$

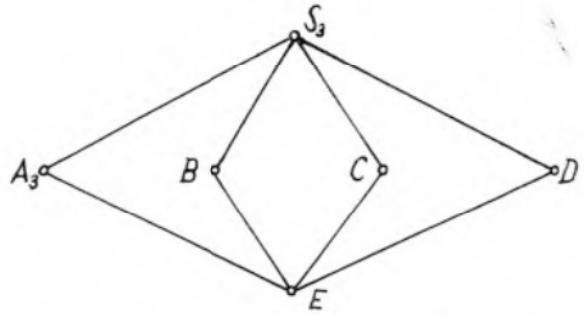
$$-A = \{-a;\, a \in A\},$$

$$AB = \{ab;\, a \in A;\, b \in B\},$$

$$A \circ B = \{a \circ b;\, a \in A;\, b \in B\}.$$

Naturally L_5 forms a complete lattice under the set inclusion. Let $\cap$ and $\cup$ denote the lattice operations. It is easy to show that the following relations hold [5]) $A+B=$ $=A\circ B=A\cup B$ and $-A=A$. Similarly to Example 6. 3 the multiplication AB may not be associative.

Example 6. 8. a) Let S_3 denote the symmetric permutation group of order 3. The subgroups of S_3 are: S_3; $A_3=(1), (123), (132)$; $B=(1), (12)$; $C=(1), (13)$; $D=(1), (23)$; $E=(1)$. These subgroups form under the set inclusion a complete lattice H, whose diagram is:



In view of Example 6. 2 H forms under the binary operation (6. 6) a commutative groupoid, whose Caley-table is:

(6. 8)

	E	B	C	D	A_3	S_3
E	E	E	E	E	E	E
B	E	E	A_3	A_3	A_3	A_3
C	E	A_3	E	A_3	A_3	A_3
D	E	A_3	A_3	E	A_3	A_3
A_3	E	A_3	A_3	A_3	E	A_3
S_3	E	A_3	A_3	A_3	A_3	A_3

Thus H is a complete groupoid-lattice and the operations on H are given by (6. 7) and (6. 8).

b) Conversely let $\mathfrak{V}=\langle V; \cdot; \wedge, \vee\rangle$ be a complete groupoid-lattice with the underlying set $V=0, a, b, c, d, e$. It is assumed that $\mathfrak{V}$ is isomorphic to the complete groupoid-lattice H and the one-to-one mapping is given by

$$0\to E,\ a\to A_3,\ b\to B,\ c\to C,\ d\to D,\ e\to S_3.$$

(The operations on V are defined according to (6. 7) and (6. 8).)

[5]) Cf. Remark 1.

In view of Theorem 3. 2 the algebra $\mathfrak{A}=\langle V; \cdot, \wedge, \vee, g_a, g_b, \ldots\rangle$ $(a, b, \ldots \in V)$ belonging to $\mathfrak{B}$ has the property that the complete algebra-lattice of all subalgebras of $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$.

(We remark that $\mathfrak{B}$ is naturally a compactly generated algebra-lattice and $\mathfrak{A}$ is a finitary algebra.)

Ex ample 6. 9. Let N_0 denote the multiplicative semigroup of the non-negative -gers. If we define on N_0 the following partial order $\leqq$:

$$a \leqq b \quad \text{if and only if } b|a \quad (a, b \in N_0),$$

then N_0 forms under the relation $\leqq$ a complete lattice. Its greatest (least) element is the number 1 (zero). For any subset M of N_0, sup M is the greatest commons divisor of the elements of M. Moreover for finite M, inf M is the least common multiple of the elements of M and for infinite M, it is 0.

It is easy to see that the relations (6.1), (6. 2) and (6. 3) hold in N_0, thus $\mathfrak{N}_0=\langle N_0; \cdot; \sup, \inf\rangle$ is a complete groupoid-lattice. According to (3.1) one can define for every number $a(\in N_0)$ the following unary operation g_a on N_0:

$$g_a(x)=\begin{cases} a, & \text{if } x\,|\,a, \\ 0, & \text{if } x \nmid a \end{cases} \qquad (x \in N_0).$$

Let us consider the algebra $\mathfrak{A}=\langle N_0; \cdot; \sup, \inf, g_a, g_b, \ldots\rangle$ $(a, b, \ldots \in N_0)$ belonging to $\mathfrak{N}_0$. From Lemma 3.1 it follows that every subalgebra of $\mathfrak{A}$ has the form $[0, a]$ $(a \in N_0)$, where the interval $[0, a]$ consists of the multiples of the number a.

In view of Theorem 3. 2 the set $S(\mathfrak{A})$ of all subalgebras of $\mathfrak{A}$ forms a complete groupoid-lattice isomorphic to $\mathfrak{N}_0$. The isomorphism is given by

$$a \to [0, a] \qquad (a \in N_0).$$

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