Rees factor lattices

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To Professor A. G. Kuroš on his 60 th birthday

1. Introduction

It is well-known that every lattice forms, with respect to the meet (or join) operation, a commutative idempotent semigroup (i.e. a semilattice). Accordingly, the concepts of the theory of semigroups may be applied to lattices, but many of these yield only trivial results.

In the theory of semigroups, the following construction is very important. Let S be a semigroup and I an ideal of S. Define an equivalence relation Θ on S as follows: let $a \equiv b(\Theta)$ $(a, b \in S)$ mean that either $a = b$ or else both a and b belong to I. Then Θ is, strictly speaking, a congruence on S, called the *Rees congruence of S modulo I*. The factor semigroup of S modulo Θ is called the *Rees factor semigroup of S modulo I* and denoted by S/I. Clearly, I is the zero element of S/I.

In this note the described construction will be applied to lattices. As the results of Section 4 show, the concept of Rees factor lattices seems to be useful.

We use the notations of the books [1] and [3]. For any sets A, B, the set of elements of A which are not in B will be denoted by $A - B$.

2. Definition of the Rees factor lattice

Let $\mathsf{L} = (L, \cap, \cup)$ be a lattice with respect to the meet operation $\cap$ and the join operation $\cup$. Let, further, I be an ideal of L. Then I is, a fortiori, an ideal of the semilattice $(L, \cap)$ and the Rees factor semigroup $(M, \wedge) = (L, \cap)/I$ is a semilattice again (where M denotes the set of equivalence classes of the Rees congruence of $(L, \cap)$ modulo I). For convenience, any element $\{a\}$ $(a \in L)$ of M (i.e., any one-element class $\{a\}$) will be identified with the element a of L.

Define, as usual, $a \leqq b$ to mean in M that $a \wedge b = a$. Then I will be the least element of M and, for any elements $a, b \neq I$, $a \leqq b$ holds in M if and only if $a \leqq b$ in L. Hence, the semilattice $(M, \wedge)$ with this ordering may be described as the result the ideal I of L collapsing into a single least element, while the other elements of L remain invariant.

Let a and b be any elements of $L - I$. Then, by a well-known property of lattice ideals, any upper bound of the set $\{a, b\}$ lies also outside of I. Hence, in M we have

$$\sup_M \{a, b\} = a \cup b \quad \text{for} \quad a, b \neq I.$$

Moreover,

$$\sup_M \{a, I\} = a \quad \text{for any} \quad a \in M.$$

Hence, any two-element subset of M has (not only an infimum, but) a supremum, too. Thus the set M forms a lattice with respect to the operations $\wedge$, $\vee$ defined by

$$a \wedge b = \begin{cases} a \cap b, & \text{if} \quad a, b \neq I \text{ in } M \text{ and } a \cap b \notin I, \\ I & \text{otherwise} \end{cases}$$

and

$$a \vee b = \begin{cases} a \cup b, & \text{if} \quad a, b \neq I \text{ in } M, \\ a, & \text{if} \quad b = I \text{ in } M, \\ b, & \text{if} \quad a = I \text{ in } M. \end{cases}$$

The lattice $\mathsf{M} = (M, \wedge, \vee)$ will be called the *Rees factor lattice of* L *modulo* I and denoted by L/I.

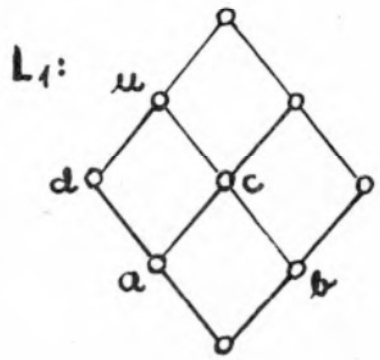

Fig. 1

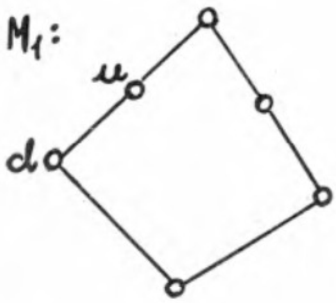

Fig. 2

Remark 1. The natural homomorphism of $(L, \cap)$ onto $(M, \wedge)$ is, in general, no join-homomorphism of L onto M. For example, the Rees factor lattice of the lattice L_1, represented by Fig. 1, modulo the principal ideal $(c]$ is the lattice M_1 in Fig. 2 and for the natural homomorphism $\varphi\colon L_1 \to M_1$ we have

$$\varphi(b \cup d) = \varphi(u) = u$$

but

$$\varphi(b) \vee \varphi(d) = I \vee d = d.$$

Remark 2. By Remark 1, a Rees factor lattice of a distributive (or modular) lattice is not necessarily distributive (modular).

Remark 3. Let L be a lattice and D a dual ideal of L. Then, by the dual of the above procedure, we obtain again a lattice called *the Rees factor lattice of* L *modulo the dual ideal* D and denoted likewise by L/D. If the ideal I and the dual ideal D of the lattice L have no common element, then D is a dual ideal of L/I and I is an ideal of L/D, too. Hence, both of the symbols $(\mathsf{L}/I)/D$ and $(\mathsf{L}/D)/I$ have then a meaning and, clearly, these two lattices are isomorphic. Therefore, instead of these symbols we write $\mathsf{L}/(I, D)$ and call it the *Rees double factor lattice of* L *modulo* I *and* D.

3. Some properties of Rees factor lattices

We complete our Remark 1 by the following

Theorem 1. *Let* $\mathsf{M}=(M, \wedge, \vee)$ *be the Rees factor lattice of* $\mathsf{L}=(L, \cap, \cup)$ *modulo the ideal* I. *The natural homomorphism* φ *of* $(L, \cap)$ *onto* $(M, \wedge)$ *is a homomorphism of* L *onto* M *if and only if* $a \geqq b$ *holds in* L *for each pair* $a \in L-I$ *and* $b \in I$.

PROOF. Let a, b be any elements of L. If $a, b \in L-I$, then $a \cup b \in L-I$, too, whence $\varphi(a \cup b)=a \cup b=a \vee b=\varphi(a) \vee \varphi(b)$. If $a, b \in I$, then $a \cup b \in I$, too, whence $\varphi(a \cup b)=I=I \vee I=\varphi(a) \vee \varphi(b)$. Finally, if $a \in L-I$ and $b \in I$, then $a \cup b \in L-I$, too, whence

$$\varphi(a \cup b)=a \cup b$$

and

$$\varphi(a) \vee \varphi(b)=a \vee I=a.$$

Hence, $\varphi(a \cup b)=\varphi(a) \vee \varphi(b)$ for any $a, b \in L$ if and only if $a \cup b=a$ for each pair $a \in L-I$, $b \in I$ in L. Thus Theorem 1 is proved.

In contrast to Remark 2 we prove

Theorem 2. *The Rees factor lattice modulo a prime ideal of a distributive (modular, semimodular) lattice is likewise a distributive (modular, semimodular) lattice.*

PROOF. Let L be a distributive lattice and P a prime ideal of L. It suffices to show (see [3], p. 80, Corollary 1) that

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{1}$$

holds for any elements a, b, c of L/P.

If $a=P$, then (1) is trivial. If $b=P$ or $c=P$, then (1) holds by the absorption identity of the join. Finally, suppose that none of the elements a, b, c is equal to P. Then, P being a prime ideal in L, $b \cap c \notin P$ and thus $b \wedge c=b \cap c$. A fortiori, $a \cup (b \cap c) \in P$. Hence, in this case we have

$$a \vee (b \wedge c) = a \cup (b \cap c) = (a \cup b) \cap (a \cup c) = (a \vee b) \wedge (a \vee c),$$

indeed.

Next, let L be a modular lattice and P a prime ideal of L. We have to show that (1) holds for the elements a, b, c of L/P whenever $a \leqq c$. But, it is not hard to see that the preceding consideration can be applied also in this case.

Finally, let L be a semimodular lattice and P a prime ideal of L. Let, further, a, b and x be any elements of L/P such that

$$a \wedge b \prec x \prec a \quad \text{and} \quad a \| b.$$

Then $a \neq P$ and $b \neq P$ (in L/P) whence $a \cap b \notin P$ (in L) and $a \wedge b=a \cap b$. It follows that the interval $[a \cap b, b]$ of L belongs entirely to $L-P$. But, by the semimodularity of L, there exists an element t such that $a \cap b \prec t \leqq b$ and $(x \cup t) \cap a=x$. Since $x, t, x \cup t$ and a all belong to $L-P$, we get

$$(x \vee t) \wedge a=(x \cup t) \cap a=x,$$

completing the proof of the semimodularity of L/P.

In the rest of this section we deal with Rees factor lattices of relatively complemented lattices. First we prove the following generalization of a well-known proposition (see, e.g., [3], p. 157, Exercise 11):

Theorem 3. *Every Rees factor lattice of a relatively complemented modular lattice is semimodular.*

Proof. Let $\mathsf{L}=(L, \cap, \cup)$ be a relatively complemented modular lattice and I any ideal of L. Let, further, a, b and x be any elements of the Rees factor lattice $\mathsf{M}=(M, \wedge, \vee)=\mathsf{L}/I$ such that

$$(2) \qquad a \wedge b < x \prec a \quad \text{and} \quad a \| b.$$

We have to find an element t such that

$$(3) \qquad a \wedge b \prec t \leqq b \quad \text{and} \quad (x \vee t) \wedge a = x.$$

We distinguish two cases according as $a \cap b < x$ holds in L or not.

Case 1: $a \cap b < x$.

By (2), the elements a, b, x and, a fortiori, $x \cup b$ and $(x \cup b) \cap a (\geqq x)$ belong to $L-I$. Hence, in this case we get, using also the modularity of L,

$$(x \vee b) \wedge a = (x \cup b) \cap a = x \cup (b \cap a) = x,$$

i.e. that (3) is satisfied by $t=b$.

Case 2: $a \cap b < x$ does not hold.

By (2), $a \wedge b \neq a \cap b$ in this case. Hence, by the definition of the meet operation in L/I, $a \cap b \in I$ (and $a \wedge b = I$).

Let r denote a relative complement (in L) of $a \cap b$ with respect to the elements $u = x \cap (a \cap b)$ and b. Then $r \notin I$, for in the contrary case $a \cap b \in I$ and $r \in I$ would imply that also $b = (a \cap b) \cup r \in I$, in contradiction to our assumption (2) by which $b \notin I$. But $r \notin I$ and $a \wedge b = I$ imply that $r \neq a \wedge b$ and thus

$$a \wedge b \prec r \leqq b$$

in M. Moreover, by the modularity of L, by the inequality $r \leqq b$ and by the definition of r,

$$(x \vee r) \wedge a = (x \cup r) \cap a = x \cup (r \cap a) = x \cup (r \cap b \cap a) = x \cup u = x,$$

i.e., (3) is satisfied by $t=r$. Thus Theorem 3 is proved.

It is quite natural to ask, whether the Rees factor lattices of a relatively complemented semimodular lattice also are semimodular or not. The answer to this problem is not yet known. We prove, however, the following

Theorem 4. *Let* L *be a relatively complemented lattice satisfying the lower covering condition. Then any Rees factor lattice of* L *also satisfies this condition.*

Corollary. *Every Rees factor lattice of a relatively complemented semimodular lattice of finite length is itself semimodular.*

Proof. Let I be any ideal of L and $\mathsf{M}=\mathsf{L}/I$. Define $a \prec b$ $(a, b \in L)$ [and $a \dashv b$ $(a, b \in M)$] to mean that a is covered by b in L [in M, respectively].

Let a, b be any elements of M such that $a \wedge b \dashv b$ (and, consequently, $b \neq I$). We have to prove that $a \dashv a \vee b$. This is evident if $a = I$.

In what follows we assume that $a, b \neq I$ and we show that then $a \wedge b \dashv b$ implies $a \cap b \prec b$, too. Suppose the contrary, i.e. that there exists an element u such that $a \cap b < u < b$ in L. Then $u \in I$, because $u \notin I$ would imply $a \wedge b < u < b$ in M, in contradiction to our assumption $a \wedge b \dashv b$. Let r be a relative complement of u in L with respect to the pair of elements $a \cap b$, b. Then $a \cap b < r < b$.

Suppose $r \in I$. Then, by $u \in I$, also $b = u \cup r \in I$ in L and this is in contradiction to our assumption that $b \neq I$ in M.

Suppose $r \notin I$. Then $a \wedge b < r < b$ in M and this contradicts again the assumption $a \wedge b \dashv b$.

By the preceding paragraphs, $a \cap b \prec b$ indeed. Hence $a \prec a \cup b$ in L, by the lower covering condition. But, by $a, b \neq I$ we get $a \vee b = a \cup b$, and consequently, $a \dashv a \vee b$ in M. This completes the proof of Theorem 4.

The corollary follows immediately from the theorem (see [3], p. 143, Corollary).

4. Applications

In this section we discuss two applications of Rees factor lattices.

Let L be a lattice with least element o and greatest element i. Let, further, e be any element of L and denote by C_e the set of all complements of e. It is well-known that, for any element e of a distributive lattice, C_e has at most one element and, in case of a modular lattice, every set C_e is either empty or completely unordered (with respect to the ordering relation of L). If, however, L is only semimodular, then C_e can contain subchains of more than one element. By a *maximal (minimal) complement* of e we mean a maximal (minimal) element of the partly ordered set C_e.

Theorem 5. *Let $\mathsf{L} = (L, \cap, \cup)$ be a modular lattice with least element o and greatest element i. Let, further, I be any principal ideal of L and $\mathsf{M} = (M, \wedge, \vee)$ the Rees factor lattice of L modulo I. Then, for every element e of M, the set C_e is either empty or has a maximal element.*

Remark. For lattices of finite length (and, more generally, for lattices satisfying the maximum condition) our assertion is, of course, trivial. The problem whether every non-empty C_e has a minimal element too, is open .

Proof. Clearly, it suffices to discuss only that case when e is no bound element of M and the set C_e is non-empty. Accordingly, consider an arbitrary inner element e of M and let e' be a complement of e in M. Then $e, e' \in L - I$ and thus $e \cup e' = e \vee e' = i$. Hence, by the isomorphism theorem of modular lattices, the mapping φ defined by

$$\varphi(y) = e \cap y \qquad (y \in [e', i])$$

is an isomorphism of the sublattice $[e', i]$ of L onto the sublattice $[e \cap e', e]$.

Let u denote the greatest element of the principal ideal I. Then $y \in [e', i]$ is a complement of e in M if and only if $e \cap y \leqq u$ in L.

Summarizing, we obtain that an element y of $[e', i]$ is a maximal complement of e if (and only if) the corresponding y is maximal in the subset $[e \cap e', e] \cap (u]$

of L. But $e \cap e' \leqq u$ in L (because $e \wedge e' = I$ in M, by assumption) and this implies that

$$[e \cap e', e] \cap (u] = [e \cap e', e \cap u].$$

Hence, the element $y = \varphi^{-1}(e \cap u)$ is maximal in C_e.

We conclude this note by constructing a complemented semimodular lattice in which no inner element has either a maximal or a minimal complement.

Let S be any infinite set and $\mathsf{P}(S)$ the subset lattice of S. The family F of all finite subsets of S is an ideal in $\mathsf{P}(S)$. Similarly, the family C of all subsets X of S with $S - X \in \mathsf{F}$ is a dual ideal in $\mathsf{P}(S)$. Clearly, F and C have no common element. Thus, according to Remark 3, we can construct the Rees double factor lattice

$$\mathsf{R}(S) = \mathsf{P}(S)/(\mathsf{F}, \mathsf{C})$$

which we shall call the *reduced subset lattice* of S. We prove

Theorem 6. *The reduced subset lattice of any infinite set is semimodular, dually semimodular and complemented, but none of its inner element has a maximal or a minimal complement.*

Corollary. *The reduced subset lattice of an infinite set is not modular.*

Proof. Since $\mathsf{R}(S)$ is evidently self-dual and complemented, we need only to show that it is semimodular, but none of its inner elements has a maximal complement.

Let A be any inner element of $\mathsf{R}(S)$ and A' a complement of A in this lattice. Then any set of the form

$$A'' = A' \cup \{e_1, e_2, \ldots, e_n\} \qquad (e_i \in A, \ n \text{ finite})$$

is again a complement of A in $\mathsf{R}(S)$. This shows that A has no maximal complement in $\mathsf{R}(S)$, implying also the corollary.

In order to prove that $\mathsf{R}(S)$ is semimodular, consider the elements A, B, X of this lattice such that

$$(4) \qquad A \wedge B \prec X < A \quad \text{and} \quad A \| B$$

in $\mathsf{R}(S)$ ($\wedge$ and $\vee$ mean the meet and the join in $\mathsf{R}(S)$, respectively); by these assumptions, A, B, X and $S-A$, $S-B$, $S-X$ all are infinite subsets of S. We have to find an element T in $\mathsf{R}(S)$ that satisfies the conditions

$$(5) \qquad A \wedge B \prec T \leqq B \quad \text{and} \quad (X \vee T) \wedge A = A.$$

We distinguish the following three cases:

1. $A \cap B \in \mathsf{R}(S)$ and $B - A \notin \mathsf{R}(S)$.
2. $A \cap B \in \mathsf{R}(S)$ and $B - A \in \mathsf{R}(S)$.
3. $A \cap B \notin \mathsf{R}(S)$, i.e. $A \cap B$ is a finite set.

Case 1. First we show that $X \cup B \in \mathsf{R}(S)$ in this case. Let $H = S - (A \cup B) = (S - A) \cap (S - B)$. Using the well-known set-theoretical identity $B - A = B \cap (S - A)$ we get

$$H \cup (B - A) = (S - A) \cap ((S - B) \cup B) = S - A.$$

Since $S-A$ is an infinite set by assumption, this equality implies that either H or $B-A$ must be infinite. But $B-A$ is finite in the present case. Consequently, H is infinite and thus $A\cup B\in\mathsf{R}(S)$. A fortiori, $X\cup B\in\mathsf{R}(S)$.

Moreover, the assumption $A\cap B\in\mathsf{R}(S)$ implies that $A\wedge B=A\cap B$ and thus the first part of (4) can be written as

$$A\cap B\subset X\subset A. \tag{6}$$

Finally, by the inclusions $X\cup B\supseteqq X$ and $A\supseteqq X$, also $(X\cup B)\cap A$ belongs to $\mathsf{R}(S)$. Consequently, we get

$$(X\vee B)\wedge A=(X\cup B)\cap A=(X\cap A)\cup(B\cap A)=X.$$

Hence, (5) is satisfied by $T=B$.

Case 2. The inclusions in (6) hold also in this case. Moreover, the set $B-A=B-(A\cap B)$ is infinite. Hence, one can find a set Y such that

$$A\cap B\subset Y\subset B \tag{7}$$

and the set $B-Y$ is also infinite. For such Y, $X\cup Y\in\mathsf{R}(S)$. In fact,

$$S-(X\cup Y)=(S-X)\cap(S-Y)\supseteqq(S-A)\cap(B-Y)$$

and, by (7), $B-Y$ and A have no common element so that $B-Y\subseteqq S-A$. Consequently,

$$S-(X\cup Y)\supseteqq B-Y,$$

and thus the set $S-(X\cup Y)$ is infinite, i.e. $X\cup Y\in\mathsf{R}(S)$, as asserted.

Moreover, (7) also implies the inclusions $A\cap B\subseteqq A\cap Y\subseteqq A\cap B$, whence $A\cap Y=A\cap B$.

Finally, by $(X\cup Y)\cap A\supseteqq X$, also $(X\cup Y)\cap A$ belongs to $\mathsf{R}(S)$. Hence, in this case we have

$$(X\vee Y)\wedge A=(X\cup Y)\cap A=(X\cap A)\cup(Y\cap A)=X\cup(A\cap B)=X,$$

the last equality being true by (6). This result together with (7) shows that the conditions prescribed in (5) are satisfied by $T=Y$.

Case 3. Since $B=(B\cap A)\cup(B-A)$ for any sets A and B, the set $B-A$ must be infinite in this case. Choose an infinite subset Z of $B-A$ such that also $B-Z$ be infinite. Since $A\wedge B$ is now the least element of $\mathsf{R}(S)$, the first part of (5) is trivially satisfied by $T=Z$. Moreover, the set $Y=Z\cup(A\cap B)$ is infinite, it satisfies (7), and $B-Y$ is also infinite. Thus, by the same calculation as in Case 2, $X\cup Y=X\cup Z\cup(A\cap B)$ belongs to $\mathsf{R}(S)$. This implies, by $X\subseteqq X\cup Z\subseteqq X\cup Y$, that also $X\cup Z\in\mathsf{R}(S)$. Hence

$$(X\vee Z)\wedge A=(X\cup Z)\cap A=(X\cap A)\cup(Z\cap A)=X,$$

because $Z\cap A$ is empty by assumption. Thus we have obtained that (5) is satisfied now by $T=Z$ and this completes the proof of Theorem 6.

By the Corollary, $\mathsf{R}(S)$ is a semimodular and dually semimodular lattice which is however not modular. The first example for such lattices is due to *R.* Croisot [2], but this has a more complicated structure than our $\mathsf{R}(S)$.

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(Received November 1, 1967)