On compact objects in categories

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To Prof. A. G. Kuroš on his 60th birthday

Introduction

In the paper [6] we have introduced the concept of M-compact objects in a category. The M-compact objects correspond to the 'in a narrow sense linearly compact' modules introduced by LEPTIN [2] in the category of modules. M-compact objects take an important part in the characterisation of M-semi-simple objects (cf. [6]).

It is the purpose of this paper to derive some results concerning M-compact objects. In § 1 we establish assertions used in the subsequent sections. Some of them are of some interest, and complete the results of [6] concerning the inverse system belonging to an object.

In his paper [4] SULIŃSKI has defined the concept of M-representable ideals, and developed results concerning it. The aim of § 2 is to show a close relation between M-representable ideals and M-compact objects. It will turn out that in an M-compact object any ideal which contains the M-radical and which is an M-compact object, is M-representable and conversely, in an M-compact object any M-representable ideal is an M-compact object. Further, we obtain that any M-representable ideal of an M-compact M-semi simple object is a direct factor of this object.

In § 3 we introduce a closure operation on the subobjects of an object in a rather natural way and we prove that for M-compact objects this closure operation is topological. Thus M-compact objects can be considered as objects equipped with a topology. Making use of this closure operation we define dense subobjects of an object and also M-compactifications of an object. It will be proved that any M-semi-simple object has an M-compactification and can be embedded as a dense subobject in a direct product of M-objects.

§ 1. Preliminaries

Let C be a category. The objects and maps of C will be denoted by small Latin and small Greek letters, respectively. By definition C satisfies the following conditions

(C_1) *If* $\alpha: a \to b$ *and* $\beta: b \to c$ *are maps, then there is a uniquely defined map* $\alpha\beta: a \to c$ *which is called the product of the maps* α *and* β;

(C$_2$) *If* $\alpha: a \to b$, $\beta: b \to c$, $\gamma: c \to d$ *are maps, then* $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ *holds;*

(C$_3$) *For each object* $a \in C$ *there is a map* $\varepsilon_a: a \to a$, *called the identity map of* a *such that for any* $\alpha: b \to a$ *and* $\beta: a \to c$ *we have* $\alpha\varepsilon_a = \alpha$, $\varepsilon_a\beta = \beta$.

For the definitions of familiar concepts, such as monomorphism, epimorphism, equivalence, kernel, image, direct product, inverse limit, etc. we refer to the books [1], [3] and to the papers [4], [6], respectively. In this paper *an epimorphism will always mean a normal epimorphism, and it will be supposed that the product of two normal epimorphisms is again a normal one*. The subobject determined by the object a and the monomorphism α will be denoted by (a, α). If the map $\xi: a \to b$ is an equivalence then we shall write $a \sim b(\xi, \xi^{-1})$ or only $a \sim b$, if there is no fear of ambiguity. The zero maps and identity maps will be denoted by ω and ε respectively, and the zero objects by 0.

We say that a diagram consisting of rows and columns is *exact*, if its rows and columns are exact.

As it was done in [4] and [6], we shall suppose that the category C satisfies the following additional requirements:

(C$_4$) *C possesses zero objects;*

(C$_5$) *Every map has a kernel;*

(C$_6$) *Every map has an image;*

(C$_7$) *An image of an ideal by an epimorphism is always an ideal;*

(C$_8$) *Every family of objects has a direct product and a free product;*

(C$_9$) *The class of all subobjects of any object is a set;*

(C$_{10}$) *For each object $a \in C$ the set of all ideals of a is a complete lattice;*

(C$_{11}$) *Every inverse system has an inverse limit.*

We shall need the analogous statements of the Noetherian Isomorphism Theorems.

First Isomorphism Theorem. *Let* $(k, \varkappa) \subseteq (d, \delta)$ *be two ideals of an object* a, *and let*

$$0 \to k \overset{\varkappa}{\to} a \overset{\alpha}{\to} b \to 0$$

be an exact sequence. Denote the image of (d, δ) *by the epimorphism* α, *by* (m, χ). *Then there are such maps* γ, β *that*

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & k & \to & d & \to & m & \to & 0 \\
 & & & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\chi} & & \\
0 & \to & k & \overset{\varkappa}{\to} & a & \overset{\alpha}{\to} & b & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & c_1 & \to & c_2 & \to & 0 \\
 & & & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \\
 & & & & 0 & & 0 & &
\end{array}$$

is an exact commutative diagram.

For the proof we refer to [6] Theorem 2.1.

Second Isomorphism Theorem. *Let* $(k, \varkappa), (d_1, \delta_1)$ *and* (d_2, δ_2) *be ideals of an object* $a \in C$ *such that*

$$(k, \varkappa) = (d_1, \delta_1) \cap (d_2, \delta_2),$$

$$(a, \varepsilon) = (d_1, \delta_1) \cup (d_2, \delta_2)$$

are valid. If

$$0 \to k \to d_1 \to b_1 \to 0$$

and

$$0 \to d_2 \to a \to b_2 \to 0$$

are exact sequences, then the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & k & \to & d_1 & \to & b_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & d_2 & \to & a & \to & b_2 & \to & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

is exact and commutative.

The proof can be found in [4].

Combining the assertions of Theorem 2, 3 in [6] and Theorem 2, 5 of [4] we obtain

Proposition 1. *Let*

$$0 \to k \overset{\varkappa}{\to} a \to b \to 0$$

and

$$0 \to d_i \overset{\delta_i}{\to} a \to b_i \to 0 \qquad (i = 1, 2)$$

be exact sequences. If

$$(k, \varkappa) = (d_1, \delta_1) \cap (d_2, \delta_2)$$

and

$$(a, \varepsilon) = (d_1, \delta_1) \cup (d_2, \delta_2)$$

hold, then b *is a direct product of* b_1 *and* b_2: $b = b_1 \times b_2(\alpha_1, \alpha_2; \beta_1, \beta_2)$ *moreover,* (b_i, β_i) $(i = 1, 2)$ *is the image of* (d_i, δ_i) *by* α *and*

$$\operatorname{Ker} \alpha_1 = (b_2, \beta_2), \quad \operatorname{Ker} \alpha_2 = (b_1, \beta_1)$$

hold.

We can prove also the converse statement of this proposition.

Proposition 2. *Let*

$$0 \to k \overset{\varkappa}{\to} a \overset{\alpha}{\to} b_1 \times b_2(\pi_i; \varrho_i) \qquad (i = 1, 2)$$

be an exact sequence, and denote the complete counterimage of (b_i, ϱ_i) *by* (d_i, δ_i) $(i = 1, 2)$. *Then*

$$(k, \varkappa) = (d_1, \delta_1) \cap (d_2, \delta_2)$$

and

$$(a, \varepsilon) = (d_1, \delta_1) \cup (d_2, \delta_2)$$

hold.

PROOF. The image of $(d_1, \delta_1) \cap (d_2, \delta_2)$ by α is $(b_1, \varrho_1) \cap (b_2, \varrho_2) = (0, \omega)$, so its complete counterimage by α is $\operatorname{Ker} \alpha = (k, \varkappa)$.

Denote the union of (d_1, δ_1) and (d_2, δ_2) by (l, λ). Now

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & l & \longrightarrow & b_0 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow \varrho_0 & & \\
0 & \to & k & \to & a & \overset{\alpha}{\to} & b_1 \times b_2 & \to & 0
\end{array}
$$

is an exact commutative diagram such that $(b_0, \varrho_0) = (b_1, \varrho_1) \cup (b_2, \varrho_2)$ holds Since by the Second Isomorphism Theorem

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & b_2 & \longrightarrow & b_2 & \to & 0 \\
 & & \downarrow & & \downarrow \varrho_2 & & \downarrow \varepsilon & & \\
0 & \to & b_1 & \overset{\varrho_1}{\to} & b_1 \times b_2 & \overset{\pi_2}{\to} & b_2 & \to & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}
$$

is also an exact commutative diagram, so the complete counterimage of (b_2, ε) by π_2 is just (b_0, ϱ_0). Since π_2 is an epimorphism, we get

$$(b_0, \varrho_0) = b_1 \times b_2 \, (\pi_i; \varrho).$$

Thus the assertion is proved.

We recall the definition of M-radical given in [6]. For this reason, consider a class M of objects belonging to C such that

(i) *If* $a \in M$ *and* $a \sim b$ *then* $b \in M$;

(ii) *If* $a, b \in M$ *and* $\alpha \colon a \to b$ *is an epimorphism, then either* α *is a zero map, or* $a \sim b(\alpha, \alpha_0^{-1})$.

The objects belonging to M will be called *M-objects*. We say that the ideal (d, δ) is an *M-ideal* of $a \in C$, if there is an exact sequence $0 \to d \overset{\delta}{\to} a \to b \to 0$ such that $b \in M$. The set M_a consisting of all M-ideals $(\neq (a, \varepsilon))$ of an object a is called the *structure M-space* of a

The *M-radical* of an object $a \in C$ is the intersection of all M-ideals of a. If a has no M-ideals, then its M-radical is (a, ε), and a is called an *M-radical object*.

A subset Δ of M_a is called an *independent M-system, of a*, if for any M-ideals $(d_1, \delta_1), \ldots, (d_n, \delta_n) \in M_a$ and exact sequences

$$0 \to d_i \overset{\delta_i}{\to} a \to b_i \to 0 \qquad (i = 1, \ldots, n)$$

the sequence

$$0 \to \bigcap_{i=1}^{n} (d_i, \delta_i) \to a \to b_1 \times \ldots \times b_n \to 0$$

is also exact. Every object has *maximal independent M-systems*, and the M-radical of an object is just the intersection of the M-ideals of any independent M-system of a (cf. [6]).

Any (not necessarily maximal) independent M-system Δ induces an inverse system Ω_Δ as follows (cf. [6]). Consider the set $\mathbf{F}_\Delta$ — a so-called *filter* — consisting of all finite intersections (c_i, γ_i) $(i \in I)$ of M-ideals belonging to Δ. We regard I as a partially ordered set by the definition: $i \geqq j$ if and only if $(c_i, \gamma_i) \leqq (c_j, \gamma_j)$. For each $(c_i, \gamma_i) \in \mathbf{F}_\Delta$ let us consider an exact sequence

$$0 \to c_i \to a \xrightarrow{\alpha_i} b_i \to 0.$$

Using the First Isomorphism Theorem we get that for any ideals $(c_i, \gamma_i) \leqq (c_j, \gamma_j)$ there is an epimorphism $\pi^i_j \colon b_i \to b_j$ such that

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & c_i & \to & c_j & \to & m^i_j & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & c_i & \to & a & \xrightarrow{\beta_i} & b_i & \to & 0 \\
 & & & & \downarrow{\scriptstyle \beta_j} & & \downarrow{\scriptstyle \pi^i_j} & & \\
 & & 0 & \to & b_j & \xrightarrow{\varepsilon} & b_j & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

is an exact commutative diagram, further we have $\beta_i \pi^i_j \pi^j_k = \beta_i \pi^i_k$ for any $i \geqq j \geqq k \in I$. Since β_i is an epimorphism, there follows $\pi^i_j \pi^j_k = \pi^i_j$. Thus $\Omega_\Delta = [b_i, \pi^i_j]$ forms an inverse system, and there is a uniquely defined map, the so called *canonical map* of a belonging to Δ, $\beta \colon a \to \varprojlim \Omega_\Delta$ such that $\beta \pi_i = \beta_i$ $(i \in I)$ holds where π_i means the projection $\pi_i \colon \varprojlim \Omega_\Delta \to a_i$.

It is obvious that if $\Delta = \{(d_K, \delta_K) | k \in K\}$ is an independent M-system, then the elements $k \in K$ are just the minimal elements of the index set I of $\mathbf{F}_\Delta$.

Theorem 1. *If*

$$0 \to d_k \to a \to a_k \to 0 \qquad (k \in K)$$

are exact sequences, then $\varprojlim \Omega_\Delta$ *is a direct product of the objects* a_k $(k \in K)$.

PROOF. According to the construction of $\Omega_\Delta = [b_i, \pi^i_j]$ any b_i is determined by the exact sequence

$$0 \to (c_i, \gamma_i) = \bigcap_{\text{finite}} (d_k, \delta_k) \to a \to b_i \to 0.$$

Since Δ is an independent M-system, b_i is a direct product $b_i = \prod_{\text{finite}} a_k(\tau^{(i)}_k, \vartheta^{(i)}_k)$ of the objects a_k such that

$$0 \to d_k \to a \to a_k \to 0$$

is an exact sequence for all $k \in K$ occurring in the direct decomposition of b_i. Thus by definition for the projections $\pi_i\colon \varprojlim \Omega_\Delta \to b_i$ the relation $\pi_i \pi_j^i = \pi_j$ is valid for all $i \geqq j \in I$, and for any system of maps $\chi_i\colon h \to b_i$ satisfying $\chi_i \pi_j^i = \chi_j$, there is a unique map $\chi\colon h \to \varprojlim \Omega_\Delta$ such that $\chi \pi_i = \chi_i$ is valid for each $i \in I$.

We have to prove that $\varprojlim \Omega_\Delta$ is a direct product of the objects a_k $(k \in K)$. Obviously we have *)

$$\varrho_k^{(i)}\colon \varprojlim \Omega_\Delta \to a_k$$

for $\varrho_k^{(i)} = \pi_i \tau_k^{(i)}$ $(k \in K \subseteq I)$ and $\varrho_k^{(j)} = \varrho_k^{(i)}$ for $i \geqq j$. Let h be an object and $\alpha_k\colon h \to a_k$ $(k \in K)$ a system of maps. Now we can complete this system of maps by setting $\chi_i^{(k)} = \alpha_k \vartheta_k^{(i)}\colon h \to b_i$. Clearly $\chi_i^{(k)} \pi_j^i = \chi_j^{(k)}$ is valid for all $i \geqq j \in I$. Thus by the definition of the inverse limit there exists a unique map χ such that $\chi \pi_i = \chi_i^{(k)}$. So we get

$$\chi \varrho_k^{(i)} = \chi \pi_i \tau_k^{(i)} = \chi_i^{(k)} \tau_k^{(i)} = \alpha_k \vartheta_k^{(i)} \tau_k^{(i)} = \alpha_k,$$

therefore $\varprojlim \Omega_\Delta$ is indeed a direct product of the objects a_k $(k \in K)$.

Again, let $\Delta = \{(d_k, \delta_k) | k \in K\}$ be an independent M-system of the object a.

Theorem 2. *If α denotes the canonical map $\alpha\colon a \to \varprojlim \Omega_\Delta$ then* Ker $\alpha = \bigcap_{k=K} (d_k, \delta_k)$ *holds.*

PROOF. Denote $\bigcap_{k=K} (d_k, \delta_k)$ by (l, λ). By Theorem 1 there is an equivalence

$$\xi\colon \varprojlim \Omega_\Delta \to \prod_{k \in K} a_k(\tau_k, \vartheta_k)$$

such that

$$0 \to d_k \to a \to a_k \to 0$$

is an exact sequence for each $k \in K$. This implies $\operatorname{Ker} \alpha = \operatorname{Ker} \beta$ $(\beta = \alpha \xi)$.

If $\beta_k = \beta \pi_k$, then we have $\lambda \beta_k = \omega$ for each $k \in K$. Thus ω is the uniquely defined map $\omega\colon l \to \prod_{k \in K} a_k (\tau_k, \vartheta_k)$ and this implies $\lambda \beta = \omega$.

Let $\gamma\colon c \to a$ be a map satisfying $\gamma \beta = \omega$. Now $\gamma \beta_k = \gamma \beta \pi_k = \omega$ holds for each $k \in K$, therefore there are maps $\gamma_k\colon c \to d_k$ $(k \in K)$ such that $\gamma_k \delta_k = \gamma$. Because of $(l, \lambda) = \bigcap_{k \in K} (d_k, \delta_k)$ there exists a map $\gamma'\colon c \to l$ such that $\gamma' \lambda = \gamma$ holds. Thus the assertion $\operatorname{Ker} \beta = (l, \lambda)$ is proved.

As an immediate consequence of Theorem 2 we obtain

Proposition 3. *If Δ is a maximal independent M-system of a and α is the canonical map $\alpha\colon a \to \varprojlim \Omega_\Delta$, then* Ker α *is just the M-radical of a.*

In [6] we have introduced the concept of M-compactness. Let us consider a maximal independent M-system Δ_0 of an object a. The object a is said to be M-*compact*, if the canonical map $\alpha\colon a \to \varprojlim \Omega_{\Delta 0}$ is an epimorphism.

* The maps having indices in brackets, essentially, do not depend on these indices.

Proposition 4. *If a is an M-compact object, then there is a one-to-one correspondence between the M-ideals of a and those of* $\varprojlim \Omega_{\Delta_0}$.

PROOF. Applying the First Isomorphism Theorem, we have the exact commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & k & \to & d & \longrightarrow & m & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & k & \xrightarrow{\varkappa} & a & \to & \varprojlim \Omega_{\Delta_0} & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \to & c_1 & \longrightarrow & c_2 & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

where $(k, \varkappa)$ means the M-radical of a, further (d, δ) and (m, χ) are M-ideals of a and $\varprojlim \Omega_{\Delta_0}$ respectively. Thus $(d, \delta) \leftrightarrow (m, \chi)$ induces a one-to-one correspondence between the M-ideals of a and $\varprojlim \Omega_{\Delta_0}$.

Let M_a denote the structure M-space of an object a. M_a induces a filter $\mathbf{F}_M$, and there belongs an inverse system Ω_M to $\mathbf{F}_M$. In [6] we have proved

Proposition 5. *If Δ_0 denotes a maximal independent M-system of a then the inverse system Ω_{Δ_0} is a cofinal subsystem of Ω_M, and so $\varprojlim \Omega_{\Delta_0} \sim \varprojlim \Omega_M$ hlods.*

The class M satisfying the conditions (i) and (ii) is said to be a *modular class* of objects, if M satisfies also the following conditions (cf. [4], [6]):

(iii) If *(p, π) is an ideal of a and $p \in M$, then there is a uniquely defined ideal $(m, \chi) \in M_a$ such that $(p, \pi) \cap (m, \chi) = (0, \omega)$;*

(iv) *If (l, λ) is an ideal of a, and (q, ϑ) is an M-ideal of l, then $(q, \vartheta\lambda)$ is an ideal of a.*

A non-zero object a is called *simple* if its only ideals are $(0, \omega)$ and (a, ε). Throughout this paper we suppose that *M denotes a modular class of simple objects.*

Proposition 6. (see [4], Proposition 4. 7). *Let (l, λ) be an ideal of an object $a \in C$. If (d, δ) is an M-ideal of a, then $(d, \delta) \cap (l, \lambda)$ is either an M-ideal of l or it is equal to (l, ε). Conversely, any M-ideal (q, ϑ) of l can be represented as an intersection $(q, \vartheta\lambda) = (d, \delta) \cap (l, \lambda)$ where (d, δ) is an M-ideal of a. The correspondence $(q, \vartheta) \leftrightarrow (d, \delta)$ is one-to-one.*

As a consequence of Proposition 6 one can obtain

Proposition 7. (see [4] Theorem 4. 10) *If (l, λ) is an ideal of an object $a \in C$ and (c_a, γ_a) denotes the M-radical of a then $(c_l, \gamma_l\lambda) = (c_a, \gamma_a) \cap (l, \lambda)$ is valid where (c_l, γ_l) denotes the M-radical of l.*

§ 2. M-representable ideals

Following SULIŃSKI [4], we say that the ideal (l, λ) of an object $a \in C$ is M-*representable*, if (l, λ) can be represented as the intersection of all M-ideals containing it.

In this section we shall show a close connection between the M-compact objects and M-representable ideals. First we prove the following statement.

Proposition 8. *If* $\Delta=\{(d_i, \delta_i)|i\in I\}$ *is a not necessarily maximal independent M-system of an M-compact object a, and J, K are subsets of I such that* $J\cup K=I$ *and* $J\cap K=\varnothing$, *then*

$$\Big(\bigcap_{j\in J}(d_j, \delta_j)\Big)\cup\Big(\bigcap_{k\in K}(d_k, \delta_k)\Big) = (a, \varepsilon)$$

holds.

Proof. Consider exact sequences

$$0\to d_i\to a\to a_i\to 0 \qquad (i\in I)$$

and denote $\bigcap_{i\in I}(d_i, \delta_i)$, $\bigcap_{j\in J}(d_j, \delta_j)$ and $\bigcap_{k\in K}(d_k, \delta_k)$ by (l, λ), (l_J, λ_J) and (l_K, λ_K), respectively. Since Δ is an independent M-system, so by Theorem 1 the sequences

$$0\to l_J\to a\to \prod_{j\in J} a_j\to 0,$$

$$0\to l_K\to a\to \prod_{k\in K} a_k\to 0,$$

$$0\to l\to a\xrightarrow{\alpha} \prod_{j\in J} a_j\times \prod_{k\in K} a_k\to 0$$

are exact. Consider the complete counterimage of $\prod_{j\in I} a_j$ and $\prod_{k\in K} a_k$ by α, they are obviously (l_K, λ_K) and (l_J, λ_J), respectively. Thus by Proposition 2

$$(l_J, \lambda_J)\cup(l_K, \lambda_K)=(a, \varepsilon)$$

holds, and the assertion is proved.

Theorem 3. *Let M be a modular class of simple objects. If (l, λ) is an M-representable ideal of an M-compact object a, then the object l is also M-compact.*

Proof. Consider a maximal independent M-system $\Delta_1=\{(d_j, \delta_j)|i\in J\}$ of M-ideals of a containing the ideal (l, λ). By Zorn's Lemma there exists such an M-system.

We establish the relation $(l, \lambda)=\bigcap_{j\in J}(d_j, \delta_j)$. $(l, \lambda)\leqq\bigcap_{j\in J}(d_j, \delta_j)$ is obviously true. To prove $(l, \lambda)\geqq\bigcap_{j\in J}(d_j, \delta_j)$ let us consider the sequence

$$0\to \bigcap_{j\in J}(d_j, \delta_j)\xrightarrow{\varkappa} a\xrightarrow{\alpha} \varprojlim \Omega_{\Delta_1}\to 0,$$

where

$$\Delta_1=\{(d_j, \delta_j)|j\in J\}.$$

According to Theorem 2 this sequence is exact. Consider a map $\gamma\colon c\to a$ such that $\gamma\alpha=\omega$. Since $\operatorname{Ker}\alpha=\bigcap_{j\in J}(d_j, \delta_j)$ holds, there exists a map $\gamma'\colon c\to\operatorname{Ker}\alpha$ such that $\gamma'\varkappa=\gamma$.

By the First Isomorphism Theorem we have for any M-ideal (d_j, δ_j) $(j \in J)$ the following exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \bigcap_{j \in J} (d_j, \delta_j) & \rightarrow & d_j & \longrightarrow & m_j & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow \chi_j & & \\
0 & \rightarrow & \bigcap_{j \in J} (d_j, \delta_j) & \rightarrow & a & \xrightarrow{\alpha} & \varprojlim \Omega_{\Delta_1} & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & c_1 & \longrightarrow & c_2 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and it is easy to see that $\Delta' = \{(m_j, \chi_j) | j \in J\}$ forms a maximal independent M-system of $\varprojlim \Omega_{\Delta_1}$. If (d, δ) is an arbitrary M-ideal containing (l, λ), and (m, χ) is its image by α, then with respect Proposition 5 to (m, χ) there is an intersection $\bigcap_{\text{finite}} (m_j, \chi_j)$ such that $\bigcap_{\text{finite}} (m_j, \chi_j) \leqq (m, \chi)$. Hence for their complete counter-images $\bigcap_{\text{finite}} (d_j, \delta_j) \leqq (d, \delta)$ holds. Thus there exists a map $\gamma_j: c \rightarrow d$ such that $\gamma_j \delta_j = \gamma$, moreover there exists a map $\gamma_0: c \rightarrow \bigcap_{(d, \delta) \geqq (l, \lambda)} (d, \delta) = (l, \lambda)$ such that $\gamma_0 \lambda = \gamma$. Hence $(l, \lambda) = \operatorname{Ker} \alpha$ is proved.

Complete the M-system Δ_1 to an independent M-system Δ of a Thus we have $\Delta = \Delta_1 \cup \Delta_2$ where Δ_2 consists of certain M-ideals (d_k, δ_k) $(k \in K)$ not containing (l, λ). So $J \cap K = \emptyset$ is valid, further denote $J \cup K$ by I.

To prove the theorem we need

Proposition 9. *Consider the ideals* $(q_k, \vartheta_k \lambda) = (d_k, \delta_k) \cap (l, \lambda)$ $(k \in K)$. *The system* $\Phi = \{(q_k, \vartheta_k) | k \in K\}$ *is an independent M-system of the object l. If Δ is a maximal independent M-system, then so is Φ.*

By Proposition 8 for any finite intersection $(f, \varphi) = \bigcap_{\text{finite}} (d_k, \delta_k)$ we have $(f, \varphi) \cup (l, \lambda) = (q, \varepsilon)$, and so, using the Second Isomorphism Theorem we obtain the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & (f, \varphi) \cap (l, \lambda) & \rightarrow & l & \rightarrow & b' & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & f & \longrightarrow & a & \rightarrow & \Pi a_k & \rightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Thus Φ is an independent M-system of l.

Suppose Φ is not maximal, i.e. there exists an ideal (q, ϑ) of l, such that $\Phi' = \Phi \cup \{(q, \vartheta)\}$ is also independent. By Proposition 6

$$(q, \vartheta \lambda) = (d, \delta) \cap (l, \lambda)$$

holds for an M-ideal (d, δ) of a. Consider any finite intersection $\bigcap_{\text{finite}} (d_k, \delta_k)$. By Proposition 5 there are finite many ideals of Δ such that

$$\bigcap_{\text{finite}} (d_i, \delta_i) \leqq \bigcap_{\text{finite}} (d_k, \delta_k) \cap (d, \delta) = (n, \nu)$$

holds. From Proposition 8 there follows

$$\Big(\bigcap_{\text{finite}} (d_i, \delta_i)\Big) \cup (l, \lambda) = (a, \varepsilon),$$

hence

$$(n, \nu) \cup (l, \lambda) = (a, \varepsilon)$$

is valid too.

The intersection $(n, \nu) \cap (l, \lambda) = (g, \psi)$ is obviously a finite intersection of ideals of Φ' and (q, ϑ) is one of the components of this intersection. Since Φ' is independent, there is an exact sequence

$$0 \to f \to l \to a' \times \prod_{\text{finite}} a_k \to 0$$

such that

$$0 \to q \to l \to a' \to 0$$

is an exact sequence. Using the Second Isomorphism Theorem we obtain the following exact commutative diagrams:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & g & \to & l & \to & a' \times \prod_{\text{finite}} a_k & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & n & \to & a & \longrightarrow & b & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

and

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & q & \to & l & \to & a' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & d & \to & a & \to & a'' & \to & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

Hence $\Delta_2 \cup \{d, \delta)\}$ is an independent M-system such that (d, δ) does not contain (l, λ). This contradicts the choice of Δ_2, and thus the proposition results proved.

Now we continue the proof of Theorem 3. Assume that Δ is a maximal independent M- system of a. In view of Theorem 1 and Proposition 9 we obtain

$$\varprojlim \Omega_\Phi \sim \prod_{k \in K} a_k \sim \varprojlim \Omega_{\Delta_2}. \tag{1}$$

Let (l_0, λ_0) denote the M-radical of a, and denote $\bigcap_{k \in K} (d_k, \delta_k)$ by (l', λ'). Now $(l_0, \lambda_0) = (l, \lambda) \cap (l', \lambda')$ is valid, and Proposition 8 implies

$$(a, \varepsilon) = (l, \lambda) \cup (l', \lambda').$$

Taking into account the M-compactness of a, from Theorems 1 and 2 we infer by the Second Isomorphism Theorem that

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & l_0 & \to & l & \to & b_0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & l' & \to & a & \to & \prod\limits_{k\in K} a_k & \to & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

is an exact commutative diagram. Hence by (1) $b_0 \sim \varprojlim \Omega_\Phi$ is valid. Thus l is M-compact, and Theorem 3 is proved.

Corollary. *If a is an M-compact, M-semi-simple object, then every M-reperesentable ideal of a is a direct factor of a.*

If (l, λ) is an M-representable ideal of a, then by Theorem 3 the object l is M-compact. Hence we have the exact commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & 0 & \to & l & \longrightarrow & \prod a_k & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & 0 & \to & a & \to & \prod\limits_{j\in J} a_j \times \prod\limits_{k\in K} a_k & \to & 0 \end{array}$$

and so (l, λ) is a direct factor of a.

The statement of this corollary is analogous to well-known facts in the category of modules (cf [5]).

Now we show by a simple example that an M-compact object a in which every M-representable ideal is a direct factor, need not be M-semi simple. For this sake, let a be a direct product of an M-radical object $l \neq 0$ and an object $p \in M$: $a = l \times p$ $(\pi_1, \pi_2; \varrho_1, \varrho_2)$. It is obvious that $(l, \varrho_1) \neq (0, \omega)$ is the M-radical of a, and every M-representable ideal (namely only (l, ϱ_1)) is a direct factor of a.

The converse statement of Theorem 3 is also true.

Theorem 4. *Let M be a modular class of simple objects. If l and a are M-compact objects and (l, λ) is an ideal of a containing the M-radical of a, then (l, λ) is an M-representable ideal of a, moreover, (l, λ) is the intersection $\bigcap\limits_{j\in J} (d_j, \delta_j)$ of M-ideals which form an independent M-system $\Delta_J = \{(d_j, \delta_j) | j \in J\}$ of a.*

Proof. Consider the set of all M-ideals of a containing (l, λ), and choose a maximal independent M-system $\Delta_K = \{(d_k, \delta_k) | k \in K\}$ from these M-ideals. Complete Δ_K to a maximal independent M-system Δ of a. Obviously $\Delta = \Delta_K \cup \Delta_J$ holds where $\Delta_J = \{(d_j, \delta_j) | j \in J\}$ denotes an independent M-system consisting of M-ideals not containing (l, λ).

Let us denote the M-radical of a by (c, γ). Taking into account Proposition 3 and Theorem 1, we obtain the exact sequence

$$0 \to c \to a \to \prod_{i\in I} a_i \to 0 \qquad (I = K \cup J)$$

where a_i $(i \in I)$ are M-objects such that

$$0 \to d_i \xrightarrow{\delta_i} a \to a_i \to 0$$

is an exact sequence, meanwhile (d_i, δ_i) ranges over the maximal independent M-system Δ.

Consider the set $\Phi = \{(q_k, \vartheta_k) | k \in K\}$ such that $(q_k, \vartheta_k \lambda) = (d_k, \delta_k) \cap (l, \lambda)$. According to Proposition 9 Φ is a maximal independent M-system of l.

Since l is an M-compact object so by Theorem 1 and Proposition 3 we get the exact sequence

$$0 \to c \xrightarrow{\gamma_1} l \to \prod_{k \in K} a_k \to 0$$

and here, according to Proposition 7 (c, γ_1) is the M-radical of l. The First Isomorphism Theorem yields the following exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & c & \to & l & \to & \prod_{k \in K} a_k & \to & 0 \\
 & & & & \downarrow & & \downarrow \varrho & & \\
0 & \to & c & \to & a & \xrightarrow{\alpha} & \prod_{i \in I} a_i & \to & 0 \\
 & & & & \downarrow \varphi & & \downarrow \pi & & \\
 & & & 0 \to & b & \to & \prod_{j \in J} a_j & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Hence $\operatorname{Ker} \varphi = (l, \lambda)$ holds.

According to Proposition 8 there follows from the Second Isomorphism Theorem that

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & c & \longrightarrow & \bigcap_{j \in J} (d_j, \delta_j) & \to & b' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \bigcap_{k \in K} (d_k, \delta_k) & \longrightarrow & a & \longrightarrow & \prod_{k \in K} a_k & \to & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

is an exact commutative diagram. Thus the image of $\bigcap_{j \in J} (d_j, \delta_j)$ by $\alpha\pi$ is $(0, \omega)$. Hence $\bigcap_{j \in J} (d_j, \delta_j) \leqq (l, \lambda)$ is valid. Thus $(l, \lambda) = \bigcap_{j \in J} (d_j, \delta_j)$ holds and (l, λ) is an M-representable ideal of a.

§ 3. M-closure operation

We can introduce a closure operation in the set of all subobjects of an object $a \in C$ in a natural way.

Again, let M denote a modular class of simple objects, and let (l, λ) be a subobject of the object a. Denote the intersection of all M-ideals of a containing (l, λ) by $(\bar{l}, \bar{\lambda})$. If the set of such M-ideals is empty, then we put $(\bar{l}, \bar{\lambda}) = (a, \varepsilon)$.

The ideal $(\bar{l}, \bar{\lambda})$ will be called the *M-closure* of the subobject (l, λ).

In other words: consider the structure M-space of a, and denote the set of all intersections of M-ideals by $\mathbf{C}$. $\mathbf{C}$ is a closure-system, and the closure operation belonging to $\mathbf{C}$ is just the M-closure operation. The ideals with $(l, \lambda) = (\bar{l}, \bar{\lambda})$ (i.e. the M-representable ideals) will be called *M-closed ideals*.

Now the question arises: *does the M-closure operation define a topology on a?* (i.e.: Is the union of finitly many M-closed ideals again closed?) Concerning this problem we establish the following result.

Theorem 5. *If a is an M-compact object, then the M-closure operation defines a topology on a.*

PROOF. It is sufficient to prove that the union of any two M-closed ideals (l_1, λ_1), (l_2, λ_2) is again M-closed.

Consider the set of all M-ideals containing both of (l_1, λ_1) and (l_2, λ_2), and choose a maximal independent M-system $\Delta_G = \{(d_g, \delta_g) | (g \in G\}$ of such ideals. Complete Δ_G to maximal independent M-systems $\Delta_J = \{(d_j, \delta_j) | j \in J\}$ and $\Delta_K = \{(d_k, \delta_k) | k \in K\}$ of M-ideals containing (l_1, λ_1) and (l_2, λ_2), respectively.

It is obvious that $J \cap K = G$ is valid, moreover $\{(d_i, \delta_i) | i \in (J \cup K) \setminus G\}$ forms an independent M-system of a.

Since by Theorem 3 l_1 and l_2 are M-compact objects, taking into account Theorem 4 we get $(l_1, \lambda_1) = \bigcap_{j \in J} (d_j, \delta_j)$ and $(l_2, \lambda_2) = \bigcap_{k \in K} (d_k, \delta_k)$.

Consider the ideal

$$(m, \chi) = \begin{cases} \bigcap_{g \in G} (d_g, \delta_g) & \text{if} \quad G \neq \varnothing \\ (a, \varepsilon) & \text{if} \quad G = \varnothing \end{cases}$$

and the ideals (d_i, δ_i) $(i \in (J \cup K) \setminus G)$. To each of them there exists an M-ideal (m_i, χ_i) of m such that

$$(m_i, \chi_i) = (d_i, \delta_i) \cap (m, \chi)$$

holds. From Proposition 9 it follows that

$$\Phi = \{m_i, \chi_i) | i \in (J \cup K) \setminus G\}$$

is an independent M-system of m. By definition we have

$$(l_1, \lambda_1) = \bigcap_{j \in J} (d_j, \delta_j) = \bigcap_{j \in J \setminus G} (d_j, \delta_j) \cap (m, \chi) = \bigcap_{j \in J \setminus G} (m_j, \chi)$$

and

$$(l_2, \lambda_2) = \bigcap_{k \in K} (d_k, \delta_k) = \bigcap_{k \in K \setminus G} (d_k, \delta_k) \cap (m, \chi) = \bigcap_{k \in K \setminus G} (m_k, \chi_k).$$

Since $J\backslash G$ and $K\backslash G$ are disjoint sets, Proposition 8 implies

$$\Big(\bigcap_{j\in J\backslash G}(m_j,\chi_j)\Big)\cup\Big(\bigcap_{k\in K\backslash G}(m_k,\chi_k)\Big)=(m,\varepsilon).$$

Thus

$$(l_1,\lambda_1)\cup(l_2,\lambda_2)=\begin{cases}\bigcap_{g\in G}(d_g,\delta_g) & \text{if} \quad G\neq\varnothing\\ (a,\varepsilon) & \text{if} \quad G=\varnothing\end{cases}$$

is valid and the theorem is proved.

In [2] LEPTIN has proved (among other results) that the sum of two closed submodules of a linearly compact module is again closed, and so linearly compact. The same holds for so-called 'in a narrow sense linearly compact' modules, i.e. for modules which are inverse limits of modules satisfying the descending chain condition. As it has been remarked in [6], the concept of M-compactness is a generalization of this concept. So Theorem 5 can be regarded as a category-theoretical generalization of LEPTIN's result.

The subobject (l,λ) of an object $a\in C$ will be called *dense* in a, if its M-closure is (a,ε).

By virtue of this definition, it is clear that a dense subobject (l,λ) cannot be contained in any M-ideal of a. Moreover, according to Proposition 6 there is a one-to-one correspondence between the M-ideals of a dense ideal (l,λ) and those of a.

If $\bar{a}$ is an M-compact object such that (a,α) is a dense subobject of $\bar{a}$, then $\bar{a}$ will be called an *M-compactification* of a.

If $\bar{a}_1$ and $\bar{a}_2$ are M-compactifications of a, then by the remark made above and in view of the Second Isomorphism Theorem, it is clear that the inverse systems Ω_a, $\Omega_{\bar{a}_1}$ and $\Omega_{\bar{a}_2}$ induced by the structure M-spaces of a, $\bar{a}_1$ and $\bar{a}_2$, respectively, are equivalent. However, $\bar{a}_1$ and $\bar{a}_2$ need not be equivalent objects. For instance, consider an M-radical object $l\neq 0$ and a direct product $\bar{a}_1=\prod_{i=1}^{\infty}a_i(\pi_i,\varrho_i)$ of M-objects $a_1,a_2,\dots$. If a is the discrete direct product $\bigcup_{i=1}^{\infty}(a_i,\varrho_i)$, then both $\bar{a}_1$ and $\bar{a}_2=\bar{a}_1\times l$ are M-compactifications of a but $\bar{a}_1$ and $\bar{a}_2$ are not equivalent.

Problem. *Has any object an M-compactification*? If no, then give a necessary and sufficient condition for the existence of M-compactifications of an object.

Now we prove the existence of an M-compactification of M-semi-simple objects (they are objects whose M-radical is a zero object).

Theorem 6. *Every M-semi-simple object has an M-compactification, moreover, every M-semi-simple object is a dense subobject of a direct product of M-objects.*

PROOF. By Proposition 3 and Theorem 1 $a\in C$ can be embedded by a monomorphism α in a direct product of M-simple objects which is M-compact and M-semi-simple. If (d,δ) is the M-closure of (a,α), then according to Theorem 3 d is an M-compact object, moreover, by Corollary 4, 12 of SULIŃSKI [4] also d is M-semi-simple. Thus d is a direct product of M-objects. By Proposition 6 a can be embedded as a dense subobject in d.

Theorem 6 generalizes the statement of Satz 15 in LEPTIN [2]. This theorem asserts that any semi-simple ring can be embedded as a dense subring in a direct product of linearly compact (topologically) simple rings.

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