On some properties of commutator subsemigroups

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Dedicated to Professor A. G. Kuroš on the occasion of his sixtieth birthday

A. SUSCHKEWITSCH was the first who introduced the graph representation of a transformation of degree n i.e. a mapping of a set of n elements into itself (see [8]). For a certain generalization see [9].

One can formulate A. Suschkewitsch's graph representation as follows: to every transformation of degree n there may correspond uniquely a directed graph having n labelled vertices in such a way that the vertices are labelled by the natural numbers $1, 2, \ldots, n$ and if the transformation maps i to j then the graph has a directed edge from i to j (see [8]).

The same graph representation was rediscovered by O. ORE, F. HARARY and others. (See e.g. [4], [6]).

Several properties of the graph representation have been investigated by the author and some of the results were contained in three of the author's papers (see [1], [2], [3]).

It is easy to see that a directed graph corresponds to a transformation if and only if each of its connected components contains a singele cyclically directed circuit and directed rooted trees. Such graphs with n vertices will be called *$F(n)$ graphs*. A transformation corresponding to a component of an $F(n)$ graph is conveniently called *generalized cycle*. Deleting the trees (apart from their roots) from an $F(n)$ graph one can obtain a special $F(k)$ graph ($k \leqq n$), containing circuits only. The transformation corresponding to the $F(k)$ graph is a permutation: it is called the *main-permutation* of the original transformation.

If α denotes an arbitrary transformation of degree n, then the *quasiinverse* will be defined so that the quasiinverse of α is its power α^s with the least exponent s whose main permutation is the inverse of the main permutation of α. Obviously if α is a permutation its quasiinverse is equal to its inverse. Therefore the notation (α^{-1}) of the quasiinverse of α will not be troublesome.

For an arbitrary abstract semigroup S of order n there exists a subsemigroup S' of the symmetric semigroup of degree $n+1$ such that S and S' are isomorphic. If the representation is given by the correspondence

$$a_i \leftrightarrow \begin{pmatrix} a_1 & a_2 & \ldots & a_n & a_{n+1} \\ a_1 a_i & a_2 a_i & \ldots & a_n a_i & a_i \end{pmatrix}$$

where $a_i \in S$, $i = 1, 2, \ldots, n$ it is called a *regular representation*. By means of the regular representation one can define the quasiinverse of an arbitrary element of an abstract semigroup in a similar way as it would be for a transformation. Let us consider the abstract semigroup S and its regular representation T; if $\alpha \in T$ and $a \in S$ then $\alpha \leftrightarrow a$ implies $\alpha^{-1} \leftrightarrow a^{-1}$ and the *commutator subsemigroup* of S will be defined as the subsemigroup which is generated by the *commutators* $aba^{-1}b^{-1}$ where $a, b \in S$. If A_n denotes the alternating group of degree n, S_n denotes the symmetric group of degree n, and the symmetric semigroup of degree n i.e. the semigroup of all transformations of degree n will be denoted by F_n, then for the commutator subsemigroup K_n of F_n the equality

$$K_n = (F_n \setminus S_n) \cup A_n$$

holds, where $\setminus$ denotes the set-theoretical difference. The proof which we omit here has been published in [2].

Since K_n plays a similar role in F_n as A_n in S_n, K_n will be called *alternating semigroup* of degree n.

It is almost trivial, that $\alpha K_n \alpha^{-1} = K_n$ $(\alpha \in F_n)$ holds, i.e. K_n is a *normal subsemigroup* of F_n. Further K_n is a maximal normal subsemigroup, since $\{K_n, \alpha\} = F_n$ if $\alpha \notin K_n$ holds.

When a group coincides with its commutator subgroup it is said to be a *perfect group*. It is well-known that A_n is perfect. A semigroup will be called *perfect* if it coincides with its commutator subsemigroup. To exhibit the strong analogy between A_n and K_n we shall prove that K_n is perfect.

Since A_n is perfect and all the transformations of the form $\begin{pmatrix} 1 & 2 \dots i-1 & i & i+1 \dots n \\ 1 & 2 \dots i-1 & j & i+1 \dots n \end{pmatrix}$ $i \neq j$, $i, j = 1, 2, \dots, n$ (they are called *singular transformations*) are idempotent elements and so commutators, there remains to prove that A_n and the set T_n of all singular transformations generate K_n i.e.

$$K_n = \{T_n, A_n\}$$

holds. It is obvious, that an arbitrary element of K_n whose main permutation is even can be represented as the product of the elements of A_n and T_n.

The proof will be completed when it is pointed out that for any $\alpha \in K_n$ whose main permutation is odd $\alpha \in \{T_n, A_n\}$ holds. This is true since if k is odd then the permutation $\varrho = (1 \;\; 2 \dots k)$ $(k \leqq n)$ is contained in A_n and the singular transformation $\sigma = \begin{pmatrix} 1 & 2 \dots k-1 & k \\ 1 & 2 \quad k-1 & 1 \end{pmatrix}$ is an element of T_n. Obviously $\varrho\sigma$ is a transformation whose main permutation is odd, since $\varrho\sigma = \begin{pmatrix} 1 & 2 \dots k-1 & k \\ 2 & 3 \quad 1 & 1 \end{pmatrix}$. By iteration the proof can be easily extended to an arbitrary transformation (which is not a permutation) and whose main permutation is odd.

N. Ito [5] and O. Ore [8] proved: if $n \geqq 5$ all the elements of A_n are commutators. The author has the conjecture, that a similar theorem holds for the alternating semigroups i.e. all the elements of K_n $(n \geqq 5)$ are commutators.

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