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Linear and quadratic predictability for homogeneous bilinear time series of Hermite degree two

By J. MÁTH (Debrecen)

Abstract. The linear and quadratic predictors are considered for bilinear realizable Hermite degree-2 processes. We give a sufficient condition for the equivalence of the two predictors based on the bispectrum of the noise of the best linear predictor. This gives an example, where the two predictors are equivalent but the process is not Gaussian.

1. Introduction

It is a well known fact that the best least squares predictor with respect to the past of stochastic process is the conditional expectation. A method has been given by Masani and Wiener (1959) for finding the best predictor for stationary processes. It has been shown that under certain circumstances the Hilbert space spanned by all the polynomials of the past is the same as the Hilbert space generated by the random variables with second moments, measurable with respect to the σ- algebra generated by the past.

In this paper we are considering cases when the linear predictor is as good as the linear and quadratic ones together and the process is not necessarily Gaussian. The assumption of the best linear predictability is concerned with the bispectrum of the innovation series originating from the best linear predictor. We focus on bilinear realizable Hermite degree-2 processes with separable kernel. That is an example of the situation when although the process is non-Gaussian, the linear predictor is the best among all possible nonlinear ones. We are giving a necessary and sufficient condition of the linear predictability in a simplest but nontrivial case.

2. Linear and quadratic predictor

Suppose there is given a zero mean time series Y_t stationary up to the third order with finite fourth moments.

The construction of the linear predictor $\hat{Y}_L(t+1) = \sum_{k=0}^{\infty} a_k Y_{t-k}$ is well known (see PRIESTLEY (1981)), and based on the spectrum S_Y of the process. One needs only the Szegő assumption, i.e.,

$$\int_0^1 \log S_Y(z) d\lambda > -\infty \tag{1}$$

to be fulfilled. Let

$$e_t = Y_t - \hat{Y}_L(t),$$

be the innovation process. Note that under the assumption (1), Y_t has a moving average representation

$$Y_t = \sum_{k=0}^{\infty} d_k e_{t-k}.$$

The spectrum S_Y is denoted by

$$S_Y(z) = \sum_{k=-\infty}^{\infty} c_Y(k) z^{-k},$$

where $z = e^{i2\pi\lambda}$, $\lambda \in [0,1]$, $c_Y(k) = \mathsf{E} Y_0 Y_k$.

The quadratic predictor of one lag is of the form

$$\hat{Y}_Q(t+1) = \sum_{k=0}^{\infty} a_k Y_{t-k} + \sum_{j,k=0}^{\infty} a_{jk} Y_{t-j} Y_{t-k}, \tag{2}$$

and the coefficients $a_k, a_{j,k}$ are chosen such that the mean square error

$$\mathsf{E} \left| Y_{t+1} - \hat{Y}_Q(t+1) \right|^2,$$

is minimum.

It is well known that if the process Y_t is Gaussian then the conditional expectation of Y_{t+1} with respect to $Y_t, Y_{t-1}, Y_{t-2}, \ldots$ is linear, i.e., $\hat{Y}_L(t)$ is the best predictor. However if the process Y_t is non-Gaussian then it can happen that the variance of the error process according to the quadratic predictor, i.e., $Y_{t+1} - \hat{Y}_Q(t+1)$ is smaller than the variance of the linear innovation process e_t. Recently it was shown by TERDIK and SUBBA RAO (1989) that the variance of the best linear predictor of a bilinear process driven by Gaussian white noise u_t is greater than the variance of the noise process u_t.

Our question is whether the contribution of the quadratic term in (2) is significant or not, i.e., whether the linear predictor $\hat{Y}_L(t+1)$ is the same (in the mean square sense) as the quadratic one $\hat{Y}_Q(t+1)$.

The main tool we base our analysis on is the bispectrum of the process Y_t

$$B_Y(z_1, z_2) = \sum_{k,j=-\infty}^{\infty} c_{YY}(k,j) z_1^{-k} z_2^{-j},$$

where $z_1 = e^{i2\pi\lambda_1}$, $z_2 = e^{i2\pi\lambda_2}$, $\lambda_1, \lambda_2 \in [0,1]$, and $c_{YY}(k,j) = \mathsf{E} Y_0 Y_k Y_j$.

B_Y exists for all $\lambda_1, \lambda_2 \in [0,1]$ if

$$\sum_{k,l=-\infty}^{\infty} |c_{YY}(k,l)| < \infty.$$

The following symmetry properties are fulfilled for the third order moments c_{YY}

$$\begin{aligned} c_{YY}(k,l) &= c_{YY}(l,k) = c_{YY}(-k, l-k) \\ &= c_{YY}(l-k, -k) = c_{YY}(-l, k-l) = c_{YY}(k-l, -l). \end{aligned} \tag{3}$$

From the definition of B_Y and from (3) one can prove the following properties

$$B_Y(z_1, z_2) = \overline{B_Y(z_1^{-1}, z_2^{-1})},$$

$$\begin{aligned} B_Y(z_1, z_2) &= B_Y(z_2, z_1) = B_Y(z_1, z_1^{-1} z_2^{-1}) \\ &= B_Y(z_1^{-1} z_2^{-1}, z_1) = B_Y(z_2, z_1^{-1} z_2^{-1}) = B_Y(z_1^{-1} z_2^{-1}, z_2). \end{aligned}$$

Let L be the backshifting operator, i.e., $\mathsf{L} Y_t = Y_{t-1}$ and let P and Q be two polynomials with roots outside of the unit circle. The operator $P(\mathsf{L})/Q(\mathsf{L})$ defines a linear filter on Y_t. It is known that the spectrum of the process $\tilde{Y}_t = [P(\mathsf{L})/Q(\mathsf{L}) Y_t]$ is given by

$$S_{\tilde{Y}}(z) = \left| \frac{P(z)}{Q(z)} \right|^2 S_Y(z),$$

moreover the bispectrum is

$$B_{\tilde{Y}}(z_1, z_2) = \frac{P(z_1) P(z_2) P(z_1^{-1} z_2^{-1})}{Q(z_1) Q(z_2) Q(z_1^{-1} z_2^{-1})} B_Y(z_1, z_2).$$

We use Masani and Wiener's (1959) definition of the *spectrum of a distribution*, which is often used in the measure theory.

Now we are in the position to prove the following

Theorem 1. *Let the time series Y_t be stationary in third order with spectral density S_Y. Moreover suppose that the fourth order moments of Y_t exist, and let S_Y fulfil the Szegő condition (1) and let all finite-dimensional distributions of Y_t have positive spectrum. Then the necessary and sufficient condition for the equivalence of the linear $\hat{Y}_L(t)$ and the quadratic $\hat{Y}_Q(t)$ predictor is that the bispectrum $B_e(z_1, z_2)$ of the innovation process e_t has the form*

$$B_e(z_1, z_2) = f(z_1) + f(z_2) + f(z_1^{-1} z_2^{-1}), \tag{4}$$

where

$$f(z) = \sum_{k=0}^{\infty} c_k z^k,$$

and

$$z = e^{i2\pi\lambda},\ z_1 = e^{i2\pi\lambda_1},\ z_2 = e^{i2\pi\lambda_2}.$$

The proof is given in Terdik and Máth (1993b). Here we note that the assumption (4) is automatically fulfilled when the bispectrum of the process Y_t is zero for all frequencies because the bispectrum of the linearly filtered process e_t is given as a product of the bispectrum of the process Y_t and the filter. This implies that the bispectrum of the innovation process e_t is also zero. Therefore it may happen that although the linearity test fails, the best predictor is linear.

3. Bilinear realizable processes with Hermite degree two

The so called bilinear realizable model is given in the following way

$$\begin{aligned} X_t &= AX_{t-1} + DX_{t-1}\varepsilon_{t-1} + \mathbf{b}\varepsilon_t + \mathbf{f}, \\ Y_t &= \mathbf{c}' X_t. \end{aligned} \tag{5}$$

We shall consider a simplified case of this model, the so called homogeneous bilinear model with Hermite degree 2.

In this case the process Y_t can be given by the following state space equations

$$
\begin{aligned}
&\sum_{k=0}^{P1} a_k^{(1)} X_{t-k}^{(1)} = \varepsilon_t,\\
(6)\qquad &\sum_{k=0}^{P2} a_k^{(2)} X_{t-k}^{(2)} = \sum_{m=1,n=0}^{R,S} c_{m,m+n} X_{t-m-n}^{(1)} \varepsilon_{t-m} + \text{const.},\\
&Y_t = X_t^{(2)}.
\end{aligned}
$$

The Wiener-Ito integral representation of the stationary solution of this model is

$$
(7)\qquad t = \int_0^1 \int_0^1 e^{i2\pi(\omega_1+\omega_2)t} \frac{\gamma(z_1, z_1 z_2)}{\alpha_{22}(z_1 z_2)\alpha_{21}(z_1)}, W(d\omega_1, d\omega_2),
$$

where W denotes the stochastic spectral measure with respect to the Gaussian white noise series ε_t. The polynomials $\alpha_{21}(z), \alpha_{22}(z)$ and $\gamma(z,v)$ are given by

$$
\begin{aligned}
\alpha_{21}(z) &= \sum_{k=0}^{P_1} a_k^{(1)} z^{-k}; \quad a_0^{(1)} = 1,\\
\alpha_{22}(z) &= \sum_{k=0}^{P_2} a_k^{(2)} z^{-k}; \quad a_0^{(2)} = 1,\\
\gamma(z,v) &= \sum_{m=1,n=0}^{R,S} c_{m,m+n} z^{-n} v^{-m},\\
\gamma_0(v) &= \sum_{m=1}^{R} c_{m,m} v^{-m}.
\end{aligned}
$$

Let us denote the roots of α_{21} by $\alpha_1, \dots, \alpha_{P_1}$ and the roots of α_{22} by $\beta_1, \dots, \beta_{P_2}$ These roots are supposed to be inside the unit circle.

The process Y_t is called separable if the polynomial γ is the product of two polynomials of a single variable, i.e.,

$$
\gamma(z,v) = \gamma_0(v)\gamma_1(z).
$$

The spectrum and the bispectrum for bilinear realizable processes with Hermite degree-2 are explicitly given (see TERDIK and MEAUX (1991)), i.e.,

$$S(z_1) = \sigma^4 \left| \frac{\gamma_0(z_1)}{\alpha_{22}(z_1)} \right|^2 + \frac{\sigma^4}{|\alpha_{22}(z_1)|^2} \int_0^1 \left| \frac{\gamma(z, z_1)}{\alpha_{21}(z)} \right|^2 d\lambda \tag{8}$$

and

$$B(z_1, z_2) = \psi(z_1, z_2, z_1^{-1} z_2^{-1}) \tag{9}$$

where

$$\begin{aligned} \psi(z_1, z_2, z_3) = & \frac{6\sigma^6}{\alpha_{22}(z_1)\alpha_{22}(z_2)\alpha_{22}(z_3)} \left[\frac{\gamma_0(z_1)\gamma_0(z_2)\gamma_0(z_3)}{3} \right. \\ & \left. + \operatorname{sym} \left(\int_0^1 \frac{\gamma(z^{-1}, z_1)\gamma(z^{-1}z_1^{-1}, z_2)\gamma(z, z_3)}{|\alpha_{21}(z)|^2 \alpha_{21}(z^{-1}z_1^{-1})} d\lambda \right) \right]. \end{aligned} \tag{10}$$

Theorem 2. *If the homogeneous bilinear realizable Hermite degree-2 process (6) is separable and the roots of γ_0 are inside the unit circle, then the best linear predictor is the best quadratic one as well.*

PROOF. In this case the spectrum and the bispectrum have the following form

$$\begin{aligned} S_Y(z_1) &= \sigma^4 \left| \frac{\gamma_0(z_1)}{\alpha_{22}(z_1)} \right|^2 \left[1 + \int_0^1 \left| \frac{\gamma_1(z)}{\alpha_{21}(z)} \right|^2 d\lambda \right] = \sigma_e^2 \left| \frac{\gamma_0(z_1)}{\alpha_{22}(z_1)} \right|^2, \\ B(z_1, z_2) &= \psi(z_1, z_2, z_1^{-1} z_2^{-1}), \end{aligned} \tag{11}$$

with

$$\begin{aligned} \psi(z_1, z_2, z_3) = & \frac{2\sigma^6 \gamma_0(z_1)\gamma_0(z_2)\gamma_0(z_3)}{\alpha_{22}(z_1)\alpha_{22}(z_2)\alpha_{22}(z_3)} \\ & \times \left[1 + \operatorname{sym} \left(\int_0^1 \frac{\gamma_1(z^{-1})\gamma_1(z^{-1}z_1^{-1})\gamma_1(z)}{|\alpha_{21}(z)|^2 \alpha_{21}(z^{-1}z_1^{-1)}} d\lambda \right) \right], \end{aligned} \tag{12}$$

where σ_e^2 is the variance of the residual series of the best linear predictor. Assuming that the roots of γ_0 are inside the unit circle the residual series has the form

$$e_t = \frac{\alpha_{22}(\mathsf{L})}{\gamma_0(\mathsf{L})} Y_t,$$

where L is the backward shift operator and the bispectrum of the residual series is also simple, morover

$$\psi(z_1, z_2) = 2\sigma^6(f(z_1) + f(z_2) + f(z_1^{-1}z_2^{-1})),$$

where

$$f(z_1) = 1/3 + \int_0^1 \frac{|\gamma_1(z)|^2\,\gamma_1(z^{-1}z_1^{-1})}{|\alpha_{21}(z)|^2\,\alpha_{21}(z^{-1}z_1^{-1})}d\lambda.$$

As $\psi(z_1, z_2)$ satisfies the necessary and sufficient condition of Theorem 1, the proof is completed.

Now, we assume that the best linear and the best quadratic predictor are the same and we try to find a necessary condition.

From (8) it is easy to infer that the error of the best linear predictor has the form

$$e_t = \frac{\alpha_{22}(\mathsf{L})}{h(\mathsf{L})}Y_t, \tag{13}$$

and the degree of h is just S.

Using (13) we can prove the following

Lemma 1. *If the best linear and quadratic predictor are the same then*

$$\frac{h(z_1)h(z_2)h(z_1^{-1}z_2^{-1})}{\alpha_{22}(z_1)\alpha_{22}(z_2)\alpha_{22}(z_1^{-1}z_2^{-1})}$$

will be a divisor of the bispectrum of Y_t.

Let

$$B_1(z_1, z_2) = \alpha_{22}(z_1)\alpha_{22}(z_2)\alpha_{22}(z_1^{-1}z_2^{-1})B(z_1, z_2). \tag{14}$$

We assume that $\alpha_1, \dots, \alpha_{P1}$, are different. So we can write

$$I(z_1, z_2, z_3) = \int_0^1 \frac{\gamma(z^{-1}, z_1)\gamma(z^{-1}z_1^{-1}, z_2)\gamma(z, z_3)}{|\alpha_{21}(z)|^2\,\alpha_{21}(z^{-1}z_1^{-1})}d\lambda \tag{15}$$

$$= \sum_{k=1}^{P1} \frac{\gamma(\alpha_k^{-1}, z_1)\gamma(\alpha_k^{-1}z_1^{-1}, z_2)\gamma(\alpha_k, z_3)}{A_k\alpha_{21}(\alpha_k^{-1}z_1^{-1})}, \tag{16}$$

with some constants A_k. As

$$B_1(z_1, z_2) = 6\sigma^6\left[\frac{\gamma_0(z_1)\gamma_0(z_2)\gamma_0(z_3)}{3} + \operatorname{sym} I(z_1, z_2, z_1^{-1}z_2^{-1})\right] \tag{17}$$

the poles of B_1 are

$$z_1 = \alpha_k^{-1}\alpha_j^{-1}, z_2 = \alpha_k^{-1}\alpha_j^{-1}, z_1^{-1}z_2^{-1} = \alpha_k^{-1}\alpha_j^{-1}, \quad k, j = 1, \ldots P_1 \tag{18}$$

Using (16) we have

$$\begin{aligned} &\lim_{z_1 \to \alpha_k^{-2}} B_1((z_1, z_2, z_1^{-1}z_2^{-1})(1-\alpha_k^2 z_1) \\ &\qquad = \frac{\gamma(\alpha_k^{-1}, \alpha_k^{-2})\gamma(\alpha_k, z_2)\gamma(\alpha_k, \alpha_k^2 z_2^{-1})}{A_k^1}. \end{aligned} \tag{19}$$

From the Lemma and (19) we get

$$\gamma(\alpha_k^{-1}, \alpha_k^{-2})\gamma(\alpha_k, r_i)\gamma(\alpha_k, \alpha_k^2 r_i^{-1}) = 0. \tag{20}$$

where $r_i, i=1, \ldots S$ are the roots of h. Let us assume that $\gamma(\alpha_k^{-1}, \alpha_k^{-2}) \neq 0$. Then together with the consequence of the Lemma we have the necessary conditions

$$B(r_i, \cdot) = B(\cdot, r_i) = 0, \tag{21}$$

$$\gamma(\alpha_k, r_i)\gamma(\alpha_k, \alpha_k^2 r_i^{-1}) = 0, \quad k = 1, \ldots P_1, \quad i = 1, \ldots S. \tag{22}$$

Moreover using the notation

$$\gamma(z, v) = \sum_{n=0}^{S} \gamma_n(v) z^{-n},$$

where

$$\gamma_n(v) = \sum_{m=1}^{R} c_{m,m+n} v^{-m},$$

we can say that if

$$\gamma(\alpha_{k_j}, r_i) = 0, \quad j = 1, \ldots, S+1, \tag{23}$$

holds then r_i is the root of $\gamma_n,\ n = 0, \ldots S$, and in this case

$$\gamma(z, v) = (1 - r_i v^{-1}) \sum_{n=0}^{S} \gamma_n'(v) z^{-n}. \tag{24}$$

If (23) holds for all the roots of h and these roots are different that means γ is separable.

4. An example

To give the general form of the matrices according to (5) is not easy so we shall do it in a particular case which will be considered later, parallel with (5). The particular model is the following

$$\begin{aligned} X_t^{(1)} + a_1^{(1)} X_{t-1}^{(1)} + a_2^{(1)} X_{t-2}^{(1)} &= \varepsilon_t, \\ X_t^{(2)} + a_1^{(2)} X_{t-1}^{(2)} + a_2^{(2)} X_{t-2}^{(2)} &= X_{t-1}^{(1)}\varepsilon_{t-1} + g_1 X_{t-2}^{(1)}\varepsilon_{t-1} + cX_{t-2}^{(1)}\varepsilon_{t-2} \\ &\quad + cg_2 X_{t-3}^{(1)}\varepsilon_{t-2} - \sigma^2(1+c^2), \\ Y_t &= X_t^{(2)}. \end{aligned} \tag{25}$$

Here $P_1 = 2, P_2 = 2, R = 2, S = 1$,

$$\begin{aligned} \alpha_{21}(z) &= (1-\alpha_1 z^{-1})(1-\alpha_2 z^{-1}), \\ \alpha_{22}(z) &= (1-\beta_1 z^{-1})(1-\beta_2 z^{-1}), \\ \gamma(z,v) &= v^{-1}(1+g_1 z^{-1} + cv^{-1} + cg_2 z^{-1}v^{-1}), \\ \gamma_0(v) &= v^{-1}(1+cv^{-1}). \end{aligned}$$

In the case of separability $g_1 = g_2$. From $S = 1$ it follows that h can be written as

$$h(z) = K(1 - rz^{-1}). \tag{26}$$

Using the model (5) and writing

$$\begin{aligned} C_1 &= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \\ A_1 &= \operatorname{diag}(\alpha_1, \alpha_1, \alpha_2, \alpha_2), \\ B_1 &= \frac{1}{(\alpha_1 - \alpha_2)} \begin{pmatrix} \alpha_1 & 1 & 0 & 0 \\ 0 & 0 & \alpha_1 & 1 \\ -\alpha_2 & -1 & 0 & 0 \\ 0 & 0 & -\alpha_2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ g_1 \\ c \\ cg_2 \end{pmatrix}, \\ C_2 &= (1,1), \\ A_2 &= \operatorname{diag}(\beta_1, \beta_2), \\ B_2 &= \frac{1}{\beta_1 - \beta_2} \begin{pmatrix} \beta_1 & 1 \\ -\beta_2 & -1 \end{pmatrix}, \end{aligned}$$

the matrixes according to (5) are the following

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 \\ B_2C_1 & 0 \end{pmatrix},$$
$$\mathbf{b} = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \qquad \mathbf{c} = (0, C_2).$$

Particularly, from (25) (21) and (22) we have the following simple formulae

$$B(r,\cdot) = B(\cdot, r) = 0, \tag{27}$$
$$\gamma(\alpha_1, r)\gamma(\alpha_1, \alpha_1^2 r^{-1}) = 0, \tag{28}$$
$$\gamma(\alpha_2, r)\gamma(\alpha_2, \alpha_2^2 r^{-1}) = 0. \tag{29}$$

Let us consider the value r in this particular case. Using (8) we have

$$S_Y(z_1) = H_1 z_1^{-1} + H_0 + H_1 z_1,$$

where

$$H_0 = \sigma^4 \left[1 + c^2 + E_1(1 + c^2 + g_1^2 + c^2 g_2^2) + 2E_2(g1 + c^2 g_2)\right],$$
$$H_1 = \sigma^4 \left[c + E_1(c + c g_1 g_2) + E_2 c(g_1 + g_2)\right],$$

and

$$E_1 = \frac{1}{(1-\alpha_1\alpha_2)(\alpha_1 - \alpha_2)} \left[\frac{\alpha_1}{(1-\alpha_1^2)} - \frac{\alpha_2}{(1-\alpha_2^2)}\right],$$
$$E_2 = \frac{1}{(1-\alpha_1\alpha_2)(\alpha_1 - \alpha_2)} \left[\frac{\alpha_1^2}{(1-\alpha_1^2)} - \frac{\alpha_2^2}{(1-\alpha_2^2)}\right],$$

which gives

$$r = \frac{-\frac{H_0}{H_1} \pm \sqrt{\frac{H_0^2}{H_1^2} - 4}}{2}.$$

We must choose the solution which is inside the unit circle.

It is more difficult to calculate the bispektrum. Let

$$A1 = 1 + c z_1^{-1}, \quad A2 = 1 + c z_2^{-1}, \quad A3 = 1 + c z_1 z_2,$$
$$B1 = g_1 + c g_2 z_1^{-1}, \quad B2 = z_1(g_1 + c g_2 z_2^{-1}), \quad B3 = g_1 + c g_2 z_1 z_2.$$

With the notations

$$\begin{aligned} C_{-1} &= A1A2B3 \\ C_0 &= A1A2A3 + A1B2B3 + B1A2B3 \\ C_1 &= A1B2A3 + B1A2A3 + B1B2B3 \\ C_2 &= B1B2A3 \end{aligned}$$

$$IN_{-1} = \int_0^1 \frac{z^{-1}}{|\alpha_{21}(z)|^2 \alpha_{21}(z^{-1} z_1^{-1})} d\lambda$$

$$= \frac{(\alpha_1+\alpha_2)(1+z_1-\alpha_1^2 z_1 - \alpha 2^2 z_1)}{(1-\alpha_1^2)(1-\alpha_1^2 z_1)(1-\alpha_2^2)(1-\alpha_2^2 z_1)(1-\alpha_1\alpha_2)(1-\alpha_1\alpha_2 z_1)},$$

$$\begin{aligned} IN_k &= \int_0^1 \frac{z^k}{|\alpha_{21}(z)|^2 \alpha_{21}(z^{-1} z_1^{-1})} d\lambda \\ &= \frac{1}{(\alpha_1-\alpha_2)(1-\alpha_1\alpha_2)(1-\alpha_1\alpha_2 z_1)} \\ &\times \left[\frac{\alpha_1^{k+1}}{(1-\alpha_1^2)(1-\alpha_1^2 z_1)} - \frac{\alpha_2^{k+1}}{(1-\alpha_2^2)(1-\alpha_2^2 z_1)} \right], \quad k > 0, \end{aligned}$$

we get

$$\psi(z_1, z_2) = C_{-1} IN_{-1} + C_0 IN_0 + C_1 IN_1 + C_2 IN_2.$$

The bispectrum is the following:

$$B(z_1, z_2) = (1 + cz_1)(1 + cz_2)(1 + cz_1^{-1} z_2^{-1}) + \text{sym}\ \psi(z_1, z_2),$$

where

$$\begin{aligned} \text{sym}\,\psi(z_1, z_2) = (&\psi(z_1, z_2) + \psi(z_1, z_1^{-1} z_2^{-1}) + \psi(z_2, z_1^{-} 1 z_2^{-1}) \\ &\psi(z_2, z_1) + \psi(z_1^{-1} z_2^{-1}, z_1) + \psi(z_1^{-} 1 z_2^{-1}, z_2))/6. \end{aligned}$$

To find the explicit solutions of (27), (28) and (29) is too difficult. Therefore we solved them numerically on a rectangle of the parameter space. The parameters are $\alpha_1, \alpha_2, c, g_1, g_2$ and the rectangle is $(-1, 1) \times (-1, 1) \times (-1, 1) \times (-2, 2) \times (-2, 2)$. On this rectangle we found separable solutions. This suggests that in this particular case the separability is not only sufficient but also a necessary condition for the equality of the two predictors.

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J. MÁTH
DEPARTMENT OF PSYCHOLOGY
H–4010 DEBRECEN
HUNGARY

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