On the iteration of the divisor-function

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1. Let $d(n)$ denote the number of divisors of n, and let

$$d_r(n) = d_{r-1}(d(n)), \qquad r = 2, 3, \ldots; \qquad d_1(n) = d(n). \tag{1. 1}$$

Let further

$$D_r(x) = \sum_{n \leqq x} d_r(n). \tag{1. 2}$$

It is well-known, that

$$D_1(x) = (1+o(1))x \log x, \quad \text{as} \quad x \to \infty.$$

R. BELLMAN and H. SHAPIRO [1] called the attention to the investigation of $D_r(x)$ for $r \geqq 2$. It was conjectured by them, that

$$D_r(x) = c_r(1+o(1))x \log_r x \quad \text{as} \quad x \to \infty \tag{1. 3}$$

for every $r \geqq 1$, where $\log_r x$ denotes the r-fold iterated logarithmus of x, i.e.

$$\log_r x = \log(\log_{r-1} x), \quad \log_1 x = \log x; \qquad r = 2, 3, \ldots. \tag{1. 4}$$

It was remarked in a footnote that P. ERDŐS proved the relation (1. 3) for $r = 2$.

The aim of this paper is to prove the relation (1. 3) for $r = 3$.

The proof of (1. 3) seems to be very difficult for $r \geqq 4$.

Let

$$\bar{D}_r(x) = \sum_{p \leqq x} d_r(p-1), \tag{1. 5}$$

where in the sum p runs over the prime numbers.

We shall prove that

$$\bar{D}_r(x) = c'_r(1+o(1)) \frac{x}{\log x} \log_r x \tag{1. 6}$$

in the case $r = 2$.

We remark that the validity of (1. 6) for $r = 1$ (which is a more difficult problem) was proved by YU. V. LINNIK [2].

It seems to be difficult to verify the relation (1. 6) for $r \geqq 3$. We formulate now our assertions.

Theorem 1.

$$D_2(x) = \bigl(1+o(1)\bigr)c_1 x \log_2 x, \tag{1.7}$$

$$\bar{D}_2(x) = \bigl(1+o(1)\bigr)c_2 \frac{x}{\log x} \log_2 x, \tag{1.8}$$

as x tends to infinity, where c_1, c_2 denote suitable positive constants.

Theorem 2.

$$D_3(x) = \bigl(1+o(1)\bigr)c_3\, x \log_3 x \tag{1.9}$$

*as x tends to infinity, where c_3 denotes a suitable positive constant.**

For the proof of our assertions we need a theorem due to ERDŐS [3] which we state as

Lemma 1. *Denote $P_k(x)$ the number of square-free integers $\leqq x$ having exactly k prime factors. Then*

$$P_k(x) = \bigl(1+o(1)\bigr)\frac{6}{\pi^2}\frac{x}{\log x}\frac{(\log_2 x)^{k-1}}{(k-1)!} \tag{1.10}$$

uniformly for every k in the interval $J_x(c)$ defined by

$$\log_2 x - c\,(\log_2 x)^{1/2} \leqq k \leqq \log_2 x + c\,(\log_2 x)^{1/2}, \tag{1.11}$$

where c is an arbitrary constant.

Using contour-integration and a theorem of Esseen we could improve this lemma (for this see Kubilius's book [4], Ch. 9.).

This lemma suggests me the following theorem concerning the asymptotic behavior of the iteration of the indicator function of the square-free numbers.

Let $\mu_1(n)$ be an arithmetical function defined by

$$\mu_1(n) = \begin{cases} 1, & \text{if } n \text{ is square-free, or } 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $U(n)$ denote as usual the number of different prime divisors of n, i.e. let $U(n)=r$ for $n=p_1^{\alpha_1}\dots p_r^{\alpha_r}$. Let further

$$\mu_2(n)=\mu_1\bigl(U(n)\bigr)\mu_1(n),$$
$$U_k(n)=U\bigl(U_{k-1}(n)\bigr),\ U_1(n)=U(n),$$
$$\mu_k(n)=\mu_1\bigl(U_{k-1}(n)\bigr)\mu_1\bigl(U_{k-2}(n)\bigr)\dots\mu_1(n)=\mu_1\bigl(U_{k-1}(n)\bigr)\mu_{k-1}(n);\quad k=3,4,\dots.$$

In the case $\mu_k(n)=1$ we call n a k-fold square-free number. We call a natural number n total-square free, if it is k-fold square-free for every k satisfying $U_k(n)\geqq 1$. Let $\mu^{(T)}(n)$ denote the indicator of the set of total-square-free numbers, and let

$$M_l(x) = \sum_{n\leqq x}\mu_l(n), \qquad l=1,2,\dots;$$

$$M^{(T)}(x) = \sum_{n\leqq x}\mu^{(T)}(n).$$

* P. Erdős and I proved, that $D_4(x)\sim c_4 x\log_4 x$

From Lemma 1 we can deduce very simply that

$$M_l(x) = \big(1+o(1)\big)\left(\frac{6}{\pi^2}\right)^l x \qquad (x\to\infty) \tag{1.12}$$

in the case $l=2$.

It seems probable, that (1. 12) holds for every l, however we are unable to prove this for $l\geqq 3$.

Let $k(x)$ be an integer valued function defined on the interval $e\leqq x<\infty$ as follows:

$$1\leqq \log_k x<e, \qquad k=k(x).$$

Perhaps the following relation holds:

$$\log\frac{M^T(x)}{x} = \big(1+o(1)\big)\frac{k(x)}{2}\log\frac{6}{\pi^2}.$$

In the following $c, c_1, c_2, \ldots$ denote positive constants, not the same at every places.

2. *The proof of Theorem* 1.

Let $\mathfrak{A}$ denote the set of those natural numbers n, which have every prime divisors at least on the second power, and let

$$\vartheta(x) = \sum_{\substack{n\leqq x\\ n\in\mathfrak{A}}} 1. \tag{2.1}$$

From the Perron-formula we have

$$\vartheta(x) = \frac{1}{2\pi i}\int\limits_{(\frac{1}{2}+\varepsilon)} \frac{x^s}{s}\prod_p\left\{1+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\cdots\right\}ds,$$

whence applying the contour-integral-technique we can deduce the following

Lemma 2.

$$\vartheta(x) = cx^{1/2}+O(x^{\vartheta}), \tag{2.2}$$

where $\vartheta<\frac{1}{2}$ *is a suitable positive constant. Every natural number n can be represented uniquely in the form*

$$n=Km;\quad K\in\mathfrak{A},\quad (m,K)=1,\quad \mu(m)\neq 0. \tag{2.3}$$

Let

$$d(K)=k; \tag{2.4}$$

$$k=2^{\beta}k_1, \tag{2.5}$$

where k_1 is an odd integer. Then we have

$$d\big(d(n)\big) = d\left(\frac{k}{2^{\beta}}\right)\big(\beta+1+U(m)\big) = d(k)+d(k_1)\,U(m). \tag{2.6}$$

Let us now introduce the following notations.

$$T(y,K)=\sum_{\substack{m\leqq y\\(m,K)=1}}|\mu(m)|\,U(m), \tag{2.7}$$

$$Z(y,K)=\sum_{\substack{m\leqq y\\(m,K)=1}}|\mu(m)|. \tag{2.8}$$

The following inequalities are evident.

$$Z(y,K)\ll y, \tag{2.9}$$

$$T(y,K)\ll y\log_2 y. \tag{2.10}$$

Lemma 3.

$$Z(y,K)=A_K y+O(Ky^{1/2}), \tag{2.11}$$

$$T(y,K)=A_K y\log_2 y+O(y\log_3 y), \tag{2.12}$$

where

$$A_K=\frac{\varphi(K)}{\zeta(2)K}\prod_{p|K}\left(1-\frac{1}{p^2}\right)^{-1}.$$

PROOF. Using that

$$\sum_{\substack{(n,K)=1\\n\leqq z}}1=\frac{\varphi(K)}{K}z+O(K)$$

and

$$\sum_{\substack{\delta^2\leqq y\\(\delta,K)=1}}\frac{\mu(\delta)}{\delta^2}=\frac{1}{\zeta(2)}\prod_{p|K}\left(1-\frac{1}{p^2}\right)^{-1}+O(y^{-1/2}),$$

we have

$$Z(y,K)=\sum_{\substack{\delta^2\leqq y\\(\delta,K)=1}}\mu(\delta)\sum_{\substack{u\leqq y/\delta^2\\(u,K)=1}}1=\sum_{\substack{\delta^2\leqq y\\(\delta,K)=1}}\mu(\delta)\left\{\frac{y}{\delta^2}\frac{\varphi(K)}{K}+O(K)\right\}=A_K y+O(Ky^{1/2}),$$

and hence it follows (2.11).

For the proof of (2.12) let us put

$$T(y,K)=\sum_{\substack{p\leqq y\\(p,K)=1}}\sum_{\substack{np\leqq y\\(n,K)=1}}|\mu(np)|=\sum_{\substack{p\leqq y\\(p,K)=1}}\sum_{\substack{(n,pK)=1\\n\leqq y/p}}|\mu(n)|=\sum_{\substack{(p,K)=1\\p\leqq y}}Z\left(\frac{y}{p},pK\right)=$$

$$=\sum_{p\leqq y^{0,1}}+\sum_{p>y^{0,1}}=\Sigma_1+\Sigma_2.$$

From (2.9),

$$\Sigma_2\ll y.$$

Now using (2.11), we have

$$\Sigma_1=A_K y\sum_{\substack{p<y^{0,1}\\(p,K)=1}}\frac{1}{p}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^2}\right)^{-1}+O(Ky^{0,65}).$$

Further the sum on the right hand side equals to

$$\sum_{\substack{p<y^{0,1}\\(p,K)=1}}\frac{1}{p}+O(1)=\log_2 y-\sum_{\substack{p<y^{0,1}\\p/K}}\frac{1}{p}+O(1).$$

Now we shall prove, that

$$\sum_{p|K}\frac{1}{p}\ll\log_3 K$$

and hence (2. 12) follows.

It is known that

$$\sigma(n)\ll n\log_2 n,$$

where $\sigma(n)$ denotes the sum of the divisors of n. Hence we obtain that

$$\sum_{p|K}\frac{1}{p}\ll\log\prod_{p|K}\left(1+\frac{1}{p}\right)\ll\log\frac{\sigma(K)}{K}\ll\log_3 K.$$

Lemma 4. *For* $y\geqq 1$

(2. 13)
$$\sum_{\substack{K\leqq y\\K\in\mathfrak{A}}}\frac{d(k_1)}{K}\ll y^{-1/2+\varepsilon}$$

holds where $\varepsilon>0$ *is an* $\varepsilon>0$ *arbitrary constant.*

Proof. Using the inequality $dd(n)<d(n)\ll n^{\varepsilon}$ we have

$$\sum_{M\leqq K\leqq 2M}d(d(K))\ll\vartheta(2M)M^{\varepsilon}\ll M^{1/2+\varepsilon}.$$

Hence

$$\sum_{K\leqq y}\frac{d(k_1)}{K}=\sum_{v=0}^{\infty}\frac{1}{2^v y}\sum_{2^v y\leqq K\leqq 2^{v+1}y}d(k_1)\ll\sum_{v=0}^{\infty}\frac{1}{2^v y}(2^v y)^{1/2+\varepsilon}\ll y^{-1/2+\varepsilon}.$$

The proof of (1. 7) is straightforward. From (2. 3) it follows, that

$$D_2(x)=\sum_{\substack{K\leqq x\\K\in\mathfrak{A}}}d(k)Z\left(\frac{x}{K},\,K\right)+\sum_{\substack{K\leqq x\\K\in\mathfrak{A}}}d(k_1)T\left(\frac{x}{K},K\right)=\Sigma_3+\Sigma_4.$$

Firstly we have evidently, that

$$\Sigma_3\ll x\sum_{K\in\mathfrak{A}}\frac{d(k)}{K}\ll x.$$

Let now y be any number in the interval $1\leqq y\leqq x$, and let

$$\Sigma_4=\sum_{K\leqq y}+\sum_{K>y}=\Sigma_5+\Sigma_6.$$

Using (2. 10) and Lemma 4 we have

$$\Sigma_6\ll x\log_2 x\sum_{K>y}\frac{d(k_1)}{K}\ll x\log_2 x\cdot y^{-1/2+\varepsilon}\ll x,$$

if

(2. 14) $$y \geqq \log x.$$

Applying now (2. 12) we obtain

$$\Sigma_5 = A_K x \log_2 x \sum_{\substack{K \in \mathfrak{A} \\ K < y}} \frac{d(k_1)}{K} + O(x \log_3 x) = c_1 x \log_2 x + O(x \log_3 x),$$

where

$$c_1 = A_K \sum_{K \in \mathfrak{A}} \frac{d(k_1)}{K}.$$

Hence the relation (1. 7) follows immediately.

For the proof of (1. 8) we need the following Turán type inequality

Lemma 5.

(2.15) $$\sum_{\substack{p \equiv 1 \pmod{Q} \\ p \leqq x}} \left(U\left(\frac{p-1}{Q}\right) - \log_2 x \right)^2 \ll \frac{x}{\varphi(Q) \log x} \log_2 x \cdot \log_3 x,$$

uniformly for $Q \leqq (\log x)^{10}$.

For the proof see e.g. H. HALBERSTAM [5] (p. 24). Using the new result of Bombieri concerning the distribution of prime numbers in arithmetical progressions we could give a better estimation for (2. 15).

Using the notations (2. 3)—(2. 6) we have

$$\bar{D}_2(x) = \sum_{K < x} d(k) \sum_{\substack{p-1 = Km \\ p \leqq x \\ (K,m)=1}} |\mu(m)| + \sum_{K < x} d(k_1) \sum_{\substack{(m,K)=1 \\ p-1 = Km}} U(m)|\mu(m)| = \Sigma_7 + \Sigma_8.$$

Using Lemma 4 and the well known Brun—Titchmarsh inequality stating that

$$\pi(x, K, l) \ll \frac{x}{\varphi(K) \log x}$$

for $K < x^{1/2}$, we have

$$\Sigma_7 \ll \frac{x}{\log x} \sum_{K < x^{1/2}} \frac{d(k)}{\varphi(K)} + x \sum_{x^{1/2} \leqq K < x} \frac{d(k)}{K} \ll \frac{x}{\log x}.$$

Let us now choose $y = (\log x)^{10}$, say, and let

$$\Sigma_8 = \sum_{K \leqq y} + \sum_{K > y} = \Sigma_9 + \Sigma_{10}.$$

By Lemma 4 and the inequality

$$\sum_{m \leqq x} U(m) \ll x \log_2 x$$

we obtain

$$\Sigma_{10} \ll \sum_{K > y} d(k_1) \sum_{m \leqq \frac{x}{K}} U(m) \ll x \log_2 x \sum_{K > y} \frac{d(k')}{K} \ll x/\log^2 x.$$

Summarizing our results we have

$$\bar{D}_2(x) = \Sigma_9 + O\left(\frac{x}{\log x}\right).$$

Let now

$$\Sigma_9 = \log_2 x \sum_{K<y} d(k_1) \sum_{\substack{p-1=Km \\ (K,m)=1 \\ m \leq \frac{x}{K}}} |\mu(m)| + \Sigma_{12} = \log_2 x \, \Sigma_{11} + \Sigma_{12},$$

where

$$\Sigma_{12} = \sum_{K<y} d(k_1) \sum_{\substack{p-1=Km \\ (K,m)=1 \\ m \leq \frac{x}{K}}} \left(U(m) - \log_2 x\right) |\mu(m)|.$$

Using Lemma 5 and the Schwarz-inequality we have

$$\Sigma_{12} \ll \sum_{K<y} d(k_1) \pi^{1/2}(x, K, 1) \Big\{ \sum_{\substack{p-1=Km \\ p \leq x}} \left(U(m) - \log_2 x\right)^2 \Big\}^{1/2} \ll \frac{x}{\log x} (\log_2 x \cdot \log_3 x)^{1/2}.$$

Introducing the notation

$$T_K = \sum_{\substack{p-1=Km \\ (K,m)=1 \\ p \leq x}} |\mu(m)|$$

we have

$$\Sigma_9 = \sum_{K<y} d(k_1) T_K.$$

Now we shall give an asymptotic formula for T_K.

$$T_K = \sum_{\substack{p-1=Km \\ (K,m)=1 \\ p \leq x}} \sum_{\delta^2/m} \mu(\delta) = \sum_{\substack{\delta^2 \leq x \\ (\delta,K)=1}} \mu(\delta) \sum_{d|K} \mu(d) \sum_{\substack{p \equiv 1 \pmod{K\delta^2 d} \\ p \leq x}} 1 =$$

$$= \sum_{\substack{\delta^2 \leq x \\ (\delta,K)=1}} \mu(\delta) \sum_{d|K} \mu(d) \pi(x, d\delta^2, 1).$$

Hence one can deduce easily the relation

$$T_K = \prod_{p|K} \left(1 - \frac{1}{p(p-1)}\right) \frac{x}{K \log x} \left(1 + O\left(\frac{1}{\log x}\right)\right),$$

uniformly for $K \leq y$, using the quoted theorem of Brun—Titchmarsh and the theorem of Siegel—Walfisz. Hence

$$\Sigma_9 = c_2 \frac{x \log_2 x}{\log x} + O\left(\frac{x}{\log x} (\log_2 x \cdot \log_3 x)^{1/2}\right)$$

follows, where

$$c_2 = \sum_{\substack{K=1 \\ K \in \mathfrak{A}}}^{\infty} \frac{d(k_1)}{K} \prod_{p|K} \left(1 - \frac{1}{p(p-1)}\right).$$

Therefore

$$D_2(x) = c_2 \frac{x \log_2 x}{\log x}\left(1+O\left(\left(\frac{\log_3 x}{\log_2 x}\right)^{1/2}\right)\right)$$

and so the relation (1. 8) holds.

3. *The proof of Theorem* 2.

Lemma 6. *Let A be a natural number with a canonical representation*

$$A = \Pi p^{l_p}$$

and let

(3. 1) $$\Delta_A(x) = \sum_{n\leqq x}{}' d(An).$$

Then

(3. 2) $$\Delta_A(x) = c_0(A)x\log x + c_1(A)x + O(x^{1/3})$$

uniformly in $A \ll (\log x)^{10}$, *where*

(3. 3) $$c_0(A) = d(A)\prod_{p|A}\left(1-\frac{l_p/(l_p+1)}{p}\right),$$

and

(3. 4) $$c_1(A) = c_0(A)\sum_{p|A}\frac{l_p}{l_p+1}\frac{\log p}{p-\dfrac{l_p}{l_p+1}} + c_1(1)c_0(A).$$

Furthermore we have

(3. 5) $$c_0(A) \ll d(A),$$

(3. 6) $$c_1(A) \ll d(A)\log^2 A.$$

It can be easily verified, that the function

$$f(s) = \sum_{n=1}^{\infty}\frac{d(An)}{n^s}$$

can be written in the form

$$f(s) = \zeta^2(s)\prod_{p|A}\left(l_p+1-\frac{l_p}{p^s}\right),$$

and that

$$f(s) = \frac{c_0(A)}{(s-1)^2}+\frac{c_1(A)}{s-1}+\cdots$$

in some neighborhood of $s=1$.

Using the same analytical method which was elaborated for the Dirichlet divisor problem (see Titchmarsh [6], Ch. XII) we obtain (3. 2).

The relations (3. 5)—(3. 6) from (3. 3)—(3. 4) immediately follow.

Let $P(x, K, r)$ denote the number of integers m for which

$$m \leqq x,\ |\mu(m)| = 1,\ U(m) = r,\ (K,\ m) = 1$$

are satisfied.

Let $V(n)$ denote the total number of the primedivisors of n, i.e. let $V(n) = \alpha_1 + \ldots + \alpha_s$ for $n = p_1^{\alpha_1} \ldots p_s^{\alpha_s}$.

Let further $\lambda(n) = (-1)^{V(n)}$, the so called Liouville-function.

Let $\mathscr{B}_K$ denote the set of all integers n, each of whose primedivisor is a divisor of K, i.e. for $K = p_1^{\alpha_1} \ldots p_s^{\alpha_s}$ let

$$\mathscr{B}_K = \{n;\, n = p_1^{\beta_1} \ldots p_r^{\beta_r},\ \beta_i = 0, 1, \ldots,\ i = 1, \ldots, r\}.$$

For the sake of brevity let us denote

$$P(x, r) = P(x, 1, r).$$

The following relation holds.

Lemma 7.

$$P(x, K, r) = \sum_{\substack{\nu \leqq x \\ \nu \in \mathscr{B}_K}} P\left(\frac{x}{\nu},\ r - V(\nu)\right) \lambda(\nu). \tag{3. 7}$$

For the proof of (3. 7) we start from the identity

$$f_K(s) \overset{\text{def}}{=} \sum_{(m, K)=1} \frac{z^{U(m)} |\mu(m)|}{m^s} = \prod_{p \nmid K} \left(1 + \frac{z}{p^s}\right) = f_1(s) \sum_{\nu \in \mathscr{B}_K} \frac{\lambda(\nu) z^{V(\nu)}}{\nu^s} =$$

$$= \sum \frac{z^{U(m)} |\mu(m)|}{m^s} \sum_{\nu \in \mathscr{B}_K} \frac{\lambda(\nu) z^{V(\nu)}}{\nu^s}.$$

Comparing the coefficients on the left and right hand sides we have

$$\sum_{\substack{U(m)=r \\ (m, K)=1}} \frac{|\mu(m)|}{m^s} = \sum_{\nu \in \mathscr{B}_K} \frac{\lambda(\nu)}{\nu^s} \left\{ \sum_{U(n) = r - V(\nu)} \frac{|\mu(n)|}{n^s} \right\},$$

from which (3. 7) immediately follows.

Lemma 8. *Let $I_x(c)$ denote the interval*

$$\log_2 x - c(\log_2 x)^{1/2} \leqq r \leqq \log_2 x + c(\log_2 x)^{1/2}, \tag{3. 8}$$

where c is an arbitrory positive constant. Then

$$P(x, K, r) = \frac{6}{\pi^2}(1 + o(1)) \prod_{p \mid K} \left(1 + \frac{1}{p}\right)^{-1} \frac{x}{\log x} \frac{(\log_2 x)^{r-1}}{(r-1)!} \tag{3. 9}$$

uniformly for $r \in I_x(c)$ and $K \ll (\log_2 x)^4$.

Hence it follows, that the relation (3. 9) holds uniformly for a suitable sequence of the c_x tending to infinity as $x \to \infty$.

PROOF. We need the following estimation:

$$\sum_{\substack{\nu \in \mathscr{B}_K \\ \nu > \Delta}} \frac{1}{\nu} < \sum_{\nu \in \mathscr{B}_K} \frac{1}{\nu} \left(\frac{\nu}{\Delta}\right)^{1/2} \leqq \frac{1}{\Delta^{1/2}} \prod_{p \mid K} \left(1 - \frac{1}{p^{1/2}}\right)^{-1} \ll \frac{d(K)}{\Delta^{1/2}}.$$

From Lemma 7 we have

$$P(x, K, r) = \sum_{\substack{v \leqq \Delta \\ v \in \mathscr{B}_K}} \lambda(v) P\left(\frac{x}{v},\ r - V(v)\right) + O\left(\sum_{\substack{v > \Delta \\ v \in \mathscr{B}_K}} P\left(\frac{x}{v},\ r - V(v)\right)\right).$$

Choosing $\Delta = (\log_2 x)^6$ the inequality

$$\sum_{\substack{v > \Delta \\ v \in \mathscr{B}_K}} P\left(\frac{x}{v},\ r - V(v)\right) \ll x \sum_{\substack{v > \Delta \\ v \in \mathscr{B}_K}} \frac{1}{v} \ll x \frac{d(K)}{\Delta^{1/2}} \ll \frac{x}{(\log\log x)^2}$$

holds for the remainder term.

Let us suppose, that $r \in I_x(c)$. Then $r - V(v) \in I_x(2c)$, if $v \leqq \Delta$. Applying Lemma 1 we have

$$P(x, K, r) = \frac{6}{\pi^2}\left(1 + O(1)\right) x \sum_{\substack{v < \Delta \\ v \in \mathscr{B}_K}} \frac{\lambda(v)}{v} \frac{\left(\log_2 \frac{x}{v}\right)^{r-1-V(v)}}{\log \frac{x}{v} \cdot \left(r - 1 - V(v)\right)!} + O\left(\frac{x}{(\log_2 x)^2}\right).$$

Using that $V(v) \ll \log \Delta \ll \log_3 x$, we have

$$\frac{\left(\log_2 \frac{x}{v}\right)^{r-1-V(v)}}{\log \frac{x}{v} \cdot \left(r-1-V(v)\right)!} = \left(1 + O(1)\right) \frac{(\log_2 x)^{r-1}}{\log x \cdot (r-1)!}$$

uniformly for $r \in J_x(c)$, $v \leqq \Delta$. Taking into account, that

$$\sum_{\substack{v < \Delta \\ v \in \mathscr{B}_K}} \frac{\lambda(v)}{v} = \prod_{p \mid K} \left(1 + \frac{1}{p}\right)^{-1} + O\left(\frac{d(K)}{\Delta^{1/2}}\right),$$

and that

$$\frac{x}{(\log_2 x)^2} = o\left(\frac{x}{\log x} \frac{(\log_2 x)^{r-1}}{(r-1)!} \prod_{p \mid K} \left(1 + \frac{1}{p}\right)^{-1}\right)$$

for $r \in I_x(c)$, hence (3. 9) follows.

We are now in the position to prove Theorem 2. We have, that

$$D_3(x) = \sum_{\substack{K \leqq x \\ K \in \mathfrak{A}}} \sum_{\substack{n = Km \\ n \leqq x}} d_3(n) = \sum_{K \leqq y} + \sum_{K > y} = \Sigma_1 + \Sigma_2,$$

where $y = (\log_2 x)^4$.

Now we prove that $\Sigma_2 \ll x$. For any K

$$\sum_{\substack{n \leqq x \\ n = Km}} d_3(n) \leqq \sum_{\substack{n \leqq x \\ n = Km}} d_2(n) \leqq \sum_{m \leqq \frac{x}{K}} \{d(k) + d(k_1) U(m)\} \ll \frac{x d(k)}{K} \log_2 x,$$

further

$$\sum_{K > y} \frac{d(k)}{K} \ll (\log_2 x)^{-2},$$

whence $\Sigma_2 \ll x$ follows.

For the sake of brevity let $k_2 = d(k_1)$. The sum Σ_1 can be written as follows:

$$\Sigma_1 = \sum_{K \leqq y} \sum_{\substack{(m, K)=1 \\ m \leqq \frac{x}{K}}} |\mu(m)|\, d\big(d(k) + k_2 U(m)\big) = \sum_{K \leqq y} \sum_{r=1}^{\infty} d\big(k_2(\beta+1+r)\big) P\left(\frac{x}{K}, K, r\right) =$$

$$= \sum_{K \leqq y} \sum_{r \in L_x} + \sum_{K \leqq y} \sum_{r \notin L_x} = \Sigma_3 + \Sigma_4,$$

where L_x denotes the interval

$$L_x = \left[\frac{1}{2}\log_2 x, \frac{3}{2}\log_2 x\right].$$

Now we prove that $\Sigma_4 \ll x$. Really

$$\sum_{U(m) \notin L_x} d_3(n) \ll \sum_{K<y} d(k) \sum_{\substack{m \leqq \frac{x}{K} \\ U(m) \notin L_x}} U(m) \ll (\log_2 x)^{-1} \sum_{K \leqq y} d(k) \sum_{m \leqq \frac{x}{K}} \big(U(m) - \log_2 x\big)^2 \ll$$

$$\ll x \sum_{K \leqq y} \frac{d(k)}{K} \ll x.$$

Here we have used the inequality of TURÁN [7] stating that

$$\sum_{m \leqq x} \big(U(m) - \log_2 x\big)^2 \ll x \log_2 x.$$

Let now

$$L_x = T_x(c) + I_x(c) + R_x(c),$$

where

$$T_x(c) = [\tfrac{1}{2}\log_2 x, \log_2 x - c(\log_2 x)^{1/2}]$$

$$R_x(c) = [\log_2 x + c(\log_2 x)^{1/2}, \tfrac{3}{2}\log_2 x]$$

and correspondingly let

$$\Sigma_3 = \sum_{K \leqq y} \sum_{r \in T_x} + \sum_{K \leqq y} \sum_{r \in I_x} + \sum_{K \leqq y} \sum_{r \in R_x} = \Sigma_T + \Sigma_I + \Sigma_R.$$

Let further

$$t_l = \frac{1}{2}\log_2 x + l(\log_2 x)^{1/2}, \quad r_l = \frac{3}{2}\log_2 x - l(\log_2 x)^{1/2},$$

for $l = 1, 2, \ldots, l_0$, where $l_0 = [(\log_2 x)^{1/2} - c]$, and

$$T_{x,l} = [t_{l-1}, t_l),\ R_{x,l} = [r_{l-1}, r_l).$$

We shall prove, that the sums Σ_T and Σ_R are $o(x \log_3 x)$, whenever $c = c_x$ tends to infinity as $x \to \infty$. For this we need the Hardy—Ramanujan inequality stating that

$$P(x, K, r) \leqq P(x, r) \ll \frac{x}{\log x} \frac{(\log_2 x + c)^{r-1}}{(r-1)!}$$

for $r \in L_x$.

For any $K \leqq y$ and $l \leqq l_0$

$$\sum_{r \in T_{x,l}} d\big(k_2(\beta+1+r)\big) P\left(\frac{x}{K},\, K,\, r\right) \ll \frac{xk_2}{K \log x} \sum_{r \in T_{x,l}} d(\beta+1+r) \frac{(\log_2 x + c)^{r-1}}{(r-1)!} \ll$$

$$\ll \frac{xk_2}{K \log x} \frac{(\log_2 x + c)^{t_l - 1}}{(t_l - 1)!} \sum_{r \in T_{x,l}} d(\beta+1+r).$$

Using Lemma 6 with $A=1$ we have

$$\sum_{r \in T_{x,l}} d(\beta+1+r) = \Delta(t_l+\beta+1) - \Delta(t_{l-1}+\beta+1) \ll (\log_2 x)^{1/2} (\log_3 x).$$

Using the monotonity of $\dfrac{(\log_2 x + c)^{t-1}}{(t-1)!}$ we have

$$\Sigma_T \ll x(\log_2 x)^{1/2} \log_3 x \sum_{K \leqq y} \frac{k_2}{K} \frac{1}{(\log_2 x)^{1/2}} \sum_{r \in T_x} \frac{(\log_2 x + c)^{r-1}}{(r-1)!} \ll$$

$$\ll x \log_3 x \sum_{r \in T_x} \frac{(\log_2 x + c)^{r-1}}{(r-1)!} = o(x \log_3 x),$$

when $c_x \to \infty$ as $x \to \infty$.

It can be seen similarly, that

$$\Sigma_R = o(x \log_3 x)$$

when $c_x \to \infty$ as $x \to \infty$.

Now we start to investigate the sum Σ_I. It follows from Lemma 8, that

$$\Sigma_I = \big(1+o(1)\big) \frac{6}{\pi^2} \frac{x}{\log x} \sum_{K \leqq y} \frac{\prod_{p|K} \left(1+\frac{1}{p}\right)^{-1}}{K} \sum_{r \in I_x(c)} d\big(k_2(\beta+1+r)\big) \frac{(\log_2 x)^{r-1}}{(r-1)!}.$$

Put

$$\Sigma_K \overset{\text{def}}{=} \sum_{r \in I_x(c)} d\big(k_2(\beta+1+r)\big) \frac{(\log_2 x)^{r-1}}{(r-1)!}$$

and let

$$V = (\log_2 x)^{0,4}.$$

It is easy to show that

$$\frac{(\log_2 x)^{r-1}}{(r-1)!} = \big(1+o(1)\big) \frac{1}{V} \sum_{\nu=0}^{V-1} \frac{(\log_2 x)^{r+\nu-1}}{(r+\nu-1)!}$$

uniformly for every $r \in I_x(c_x)$, whenever c_x tends suitable slowly to infinity.

Using partial summation we have

$$V\Sigma_K = \big(1+o(1)\big) \sum_{r \in I_x(c)} \frac{(\log_2 x)^{r-1}}{(r-1)!} \sum_{\nu=r-V}^{r} d\big(k_2(\beta+1+\nu)\big) =$$

$$= \big(1+o(1)\big) \frac{6}{\pi^2} \sum_{r \in I_x(c)} \frac{(\log_2 x)^{r-1}}{(r-1)!} \{\Delta_{k_2}(\beta+1+r) - \Delta_{k_2}(\beta+1+r-V)\}.$$

Let us now assume, that

$$k_2 = d(k_1) \ll (\log_3 x)^4 \tag{S}$$

Then using Lemma 7 with $A = k_2$ we obtain that

$$\begin{aligned}\Delta_{k_2}(\beta+1+r) - \Delta_{k_2}(\beta+1+r-V) &= c_0(k_2)\,V\log r + O\big(c_1(k_2)V\big) + O(V) = \\ &= c_0(k_2)V\log_3 x + O\big(d(k_2)(\log k_2)^2 + d(k_2)c_x(\log_2 x)^{-1/2}\big)\cdot V = \\ &= c_0(k_2)\,V\log_3 x + O\big(d(k_2)(\log_4 x)^2 V\big) = c_0(k_2)\big(1+o(1)\big)\,V\log_3 x,\end{aligned}$$

whenever $c_x = O(\log_3 x)$, say.
Using the relation

$$\sum_{r \in I_x(c_x)} \frac{(\log_2 x)^{r-1}}{(r-1)!} = \big(1+o(1)\big)\log x \quad \text{for} \quad c_x \to \infty,$$

we have

$$\Sigma_K = c_0(k_2)\big(1+o(1)\big)\frac{6}{\pi^2}\log x \cdot \log_3 x \quad \text{for} \quad c_x \to \infty.$$

Further for every K by $d(mn) \leqq d(m)d(n)$

$$\begin{aligned}\Delta_{k_2}(\beta+1+r) - \Delta_{k_2}(\beta+1+r-V) \leqq d(k_2)\big(\Delta_1(\beta+1+r) - \Delta_1(\beta+1+r-V)\big) \ll \\ \ll d(k_2)V\log_3 x\end{aligned}$$

follows. Therefore for the K-s not satisfying the condition (S) we have

$$\Sigma_K \ll \frac{x k_2 d(k_2)}{(\log_3 x)^3}.$$

From these inequalities it follows rapidly that

$$\Sigma_I = \big(1+o(1)\big)c_3\, x\log_3 x,$$

where

$$c_3 = \frac{6}{\pi^2}\sum_{K \in \mathfrak{A}} \frac{c_0(k_2)\prod_{p|K}\left(1+\frac{1}{p}\right)^{-1}}{K}.$$

Combining our results the assertion of Theorem 2 immediately follows.

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