On the density of certain sequences of integers

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1. Let $p, p_1, p_2, \ldots$ be prime numbers. We use the $\ll$ symbol in Vinogradov's sense. Let $d_k(n)$ denote the number of solutions of $n=x_1, \ldots, x_k$, $d_2(n)=d(n)$, and put $\sigma_a(n)=\sum_{d|n} d^a$. Let ε denote arbitrarily small positive constants, not necessarily the same at every case.

For a general natural number n let B_n denote the set of those integers, all prime factors of which divide n. We call K a square-full number, if all its prime factors occur at least on the second power. In other words the number K having the prime decomposition $K=p_1^{\alpha_1}\ldots p_r^{\alpha_r}$ is square-full, if $\alpha_1\geqq 2, \ldots, \alpha_r\geqq 2$. Let B denote the set of all square-free numbers.

An arbitrary integer n can be written in the form

$$(1.1)\qquad n = a_n\cdot b_n,$$

where $a_n\in B$, and b_n is a square-free number coprime to a_n. This representation is unique. We shall call a_n the quadratic part and b_n the square-free part of n.

Let F denote the set of those arithmetical functions, the values of which depend only on the quadratic part of the number. In other words $f(n)\in F$, if $f(n)=f(a_n)$ for all $n\geqq 1$.

Here we are interested in the local-distribution on some special subsets of integers of the values of functions belonging to F.

The first result of this type is due to A. RÉNYI [1]. He proved the following assertion. If we define $f(n)$ as $f(p_1^{\alpha_1}\ldots p_r^{\alpha_r})=(\alpha_1-1)+\ldots+(\alpha_r-1)$ ($p_1, \ldots, p_r$ are different prime numbers), then $x^{-1}N\{n\leqq x, f(n)=q\}$ tends to a limit d_q for all $q=0, 1, 2, \ldots,$ and $\Sigma d_q=1$. This theorem was improved and generalized by many authors (see [2], [3], [4], [5]).

Suppose that $f(n)\in F$ and let $\lambda_1, \lambda_2, \ldots$ be the different values taken on by $f(n)$. One of us proved the following assertion [6].

Theorem. *$N\{n\leqq x; f(n)=\lambda_i\}=d_i x+O(\sqrt{x}(\log x)\theta_i)$ as $x\to\infty$, where $\Sigma d_i=1$, $0\leqq\theta_i\leqq 1$ and $\Sigma\theta_i\leqq 1$. The constant implied by the O-term is an absolute one.*

2. Let $K_1, K_2\in\mathscr{B}$ and let $B(x; K_1, K_2)$ be the number of $n\leqq x$, for which the quadratic part of n is K_1, and that of $n+1$ is K_2, i.e.

$$(2.1)\qquad B(x; K_1, K_2)=\sum_{n\leqq x} 1 \qquad (a_n=K_1,\ a_{n+1}=K_2).$$

Let

$$(2.2)\qquad \tau(K_1, K_2) = \begin{cases} \prod_{p|K_1K_2}\left(1-\frac{1}{p}\right)\prod_{p\nmid K_1K_2}\left(1-\frac{2}{p^2}\right) & \text{when } (K_1, K_2) = 1, \\ 0 & \text{when } (K_1, K_2) > 1. \end{cases}$$

Let

$$(2.3)\qquad \Delta(x) = \sum_{K_1, K_2 \in \mathscr{B}} \left| B(x; K_1, K_2) - \frac{\tau(K_1, K_2)}{K_1 K_2} x \right|.$$

First we prove the following

Theorem 1.

$$(2.4)\qquad B(x; K_1, K_2) = \frac{\tau(K_1, K_2)}{K_1 K_2} x + O\left(\frac{x^{2/3}(\log x)}{(K_1 K_2)^{1/3}} d(K_1 K_2)\right).$$

Furthermore

$$(2.5)\qquad \Delta(x) \ll x^{\frac{6}{7}+\varepsilon}$$

for all fixed $\varepsilon > 0$.

Let $f_1(n), f_2(n)$ be arbitrary functions belonging to F, with the set of values $\{\lambda_i\}$, $\{\mu_j\}$, respectively. Let

$$E(x) = \sum_{i,j} |N\{n \leqq x;\ f_1(n) = \lambda_i,\ f_2(n+1) = \mu_j\} - d_{i,j} x|,$$

where

$$d_{i,j} = \sum_{K_1, K_2 \in \mathscr{B}} \frac{\tau(K_1, K_2)}{K_1 K_2} \qquad (f_1(K_1) = \lambda_i, f_2(K_2) = \mu_j).$$

From Theorem 1 it follows immediately that $E(x) \leqq \Delta(x) \ll x^{6/7+\varepsilon}$.

3. Let $g(n)$ be an irreducible polynomial over the rational field with integer coefficients. Let $B(x; K) = B(x; K, g)$ denote the number of those $n \leqq x$ for which the quadratic part of $g(n)$ is K. Let $\varrho(m)$ denote the number of solutions of the congruence $g(n) \equiv 0 \pmod{m}$. It is known that $\varrho(p^\alpha) \ll 1$ for $\alpha = 1, 2, \dots$ uniformly in p and that $\varrho(m)$ is a multiplicative function. Let

$$(3.1)\qquad \tau(K) = \prod_{p\nmid K}\left(1-\frac{\varrho(p)}{p}\right)\prod_{p\nmid K}\left(1-\frac{\varrho(p)}{p^2}\right)$$

and

$$(3.2)\qquad P(x) = \sum_{K \in \mathscr{B}} \left| B(x; K) - \frac{\tau(K)}{K} x \right|.$$

It seems likely that

$$(3.3)\qquad x^{-1}P(x) \to 0 \quad \text{as} \quad x \to \infty,$$

for all polynomials. For the moment we can prove this only for polynomials of degree not higher than 3.

Theorem 2. *Let* $g(n) = n^2+1$. *Then we have*

$$B(x;K) = \frac{\tau(K)}{K}x + O\left(\frac{x^{2/3}\log x}{K^{1/3}}d(K)\right) \tag{3. 4}$$

and

$$P(x) \ll x^{\frac{6}{7}+\varepsilon}, \tag{3. 5}$$

where ε *is an arbitrary positive constant.*

We state without proof the

Theorem 3. *Let* $g(n)$ *be an irreducible polynomial of degree* 3. *Then*

$$P(x) = o(x).$$

4. *Proof of Theorem* 1. First we prove (2. 4). Since for $(K_1, K_2)>1$ we have $B(x; K_1, K_2)=0$, thus (2. 4) holds in this case. Assume now that $(K_1, K_2)=1$. Let d_i run over the sets of integers relatively prime to K_i, and δ_i the set $\mathscr{B}_{K_i}$ (resp. for $i=1, 2$). Using the relations

$$\sum_{\delta_1 d_1^2 K_1|n} \mu(\delta_1)\mu(d_1) = \begin{cases} 1, & \text{if } \ a_n = K_1, \\ 0 & \text{otherwise;} \end{cases}$$

$$\sum_{\delta_2 d_2^2 K_2|n+1} \mu(\delta_2)\mu(d_2) = \begin{cases} 1, & \text{if } \ a_{n+1} = K_2, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$B(x; K_1, K_2) = \sum_{n\leqq x}\sum \mu(\delta_1\delta_2 d_1 d_2),$$

where the second Σ means a summation over those $\delta_1, \delta_2, d_1, d_2$ for which $\delta_1 d_1^2 K_1|n$, $\delta_2 d_2^2 K_2|n+1$. By changing the order of the summation we obtain

$$B(x; K_1, K_2) = \sum_{\substack{\delta_1,\delta_2\\ d_1, d_2}} \mu(\delta_1\delta_2 d_1 d_2) S(x; K_1\delta_1 d_1^2, K_2\delta_2 d_2^2), \tag{4. 1}$$

where $S(x; a, b)$ is the number of those $n\leqq x$ for which $n\equiv O \pmod a$ and $n+1\equiv$ $\equiv O \pmod b$ hold. This congruence system is solvable only if $(a, b)=1$, and for $(a, b)=1$

$$S(x; a, b) = \frac{x}{ab} + O(1). \tag{4. 2}$$

Thus we deduce from (4. 1)

$$B(x; K_1, K_2) = \frac{x}{K_1 K_2}\Sigma' + O(\Sigma^{(1)}_{K_1, K_2}) + O(\Sigma^{(2)}_{K_1, K_2}),$$

where

$$\Sigma' = \sum \frac{\mu(\delta_1\delta_2 d_1 d_2)}{\delta_1\delta_2 d_1^2 d_2^2}. \tag{4. 3}$$

The summation in (4. 3) is extended over those $\delta_1, \delta_2, d_1, d_2$ for which $\delta_1\delta_2 d_1^2 d_2^2 \leqq x\delta$ (where β is a constant satisfying the relation $1 \leqq \beta < 2$). $\Sigma^{(1)}_{K_1, K_2}$ is the number of the values $\delta_1, \delta_2, d_1, d_2$ satisfying $\delta_1\delta_2 d_1^2 d_2^2 K_1 K_2 \leqq x^\beta$, and $\Sigma^{(1)}_{K_1, K_2}$ denotes the number of $\delta_1, \delta_2, d_1, d_2$ for which $\delta_1\delta_2 d_1^2 d_2^2 K_1 K_2 > x^\beta$ and $\delta_1 d_1^2 K_1 | n$, $\delta_2 d_2^2 K_2 | n+1$ for one $n \leqq x$ at least.

Taking into account that

$$\sum \frac{\mu(\delta_1 \delta_2 d_1 d_2)}{\delta_1 \delta_2 d_1^2 d_2^2} = \tau(K_1, K_2),$$

we deduce from (4. 3)

$$\Sigma' = \tau(K_1, K_2) + O(\Sigma^{(3)}_{K_1, K_2}),$$

where

$$\Sigma^{(3)}_{K_1, K_2} = \sum \frac{1}{\delta_1 \delta_2 d_1^2 d_2^2} \qquad (\delta_1 \delta_2 d_1^2 d_2^2 K_1 K_2 > x^\beta). \tag{4. 4}$$

Using that $\sum_{v \leqq u} d(v) v^{-2} \ll u^{-1} \log u$ and that $v = d_1 d_2$ has at most $d(v)$ solutions in d_1, d_2, we deduce from (4. 4) that

$$\Sigma^{(3)}_{K_1, K_2} \ll x^{-\beta/2} (\log x) \sqrt{K_1 K_2}\, \sigma_{-1/2}(K_1 K_2). \tag{4. 5}$$

In order to estimate $\Sigma^{(3)}_{K_1, K_2}$, we consider that the number of d_1, d_2 satisfying $\delta_1\delta_2 d_1^2 d_2^2 K_1 K_2 \leqq x^\beta$ is smaller than

$$\sum_{v \leqq N} d(v) \ll N \log x, \quad N = \frac{x^{\beta/2}}{\sqrt{\delta_1 \delta_2 K_1 K_2}},$$

whence after a summation over the δ's we obtain

$$\Sigma^{(1)}_{K_1, K_2} \ll \frac{x^{\beta/2} \log x}{\sqrt{K_1 K_2}} \sigma_{-1/2}(K_1 K_2). \tag{4. 6}$$

For the estimation of $\Sigma^{(2)}_{K_1, K_2}$ we need the following

Lemma 1. *Let a, b arbitrary positive integers. Then the number of solutions $1 \leqq u, v \leqq x$ of the equation*

$$au^2 - bv^2 = 1$$

is at most $O(\log x)$.

Let

$$n = l_1 \delta_1 K_1 d_1^2, \; n+1 = l_2 \delta_2 K_2 d_2^2 \tag{4. 7}$$

and let $R(\delta_1, \delta_2)$ denote the number of those $n \leqq x$ for which (4. 7) is satisfied with suitable l_1, l_2, d_1, d_2 satisfying the inequality $\delta_1\delta_2 K_1 K_2 d_1^2 d_2^2 > x^\beta$. For fixed $l_1, l_2, \delta_1, \delta_2$ the number of the n's in (4. 7) is at most $O(\log x)$ by Lemma 1. Hence by $l_1 l_2$ $\left(\leqq 2x(\delta_1\delta_2 K_1 K_2 d_1^2 d_2^2)^{-1}\right) \leqq 2x^{2-\beta}$ we have

$$R(\delta_1, \delta_2) \ll (\log x) \sum_{l_1 l_2 \leqq 2x^{2-\beta}} 1 \ll x^{2-\beta} \log^2 x$$

Summing over the δ's we obtain

$$\Sigma^{(2)}_{K_1, K_2} \ll x^{2-\beta} \log^2 x d(K_1 K_2). \tag{4. 8}$$

Now we choose β as follows. If $K_1K_2=x^\gamma$, then $\beta=\frac{\gamma+4}{3}$. With this β we have

$$\frac{x^{\beta/2}}{\sqrt{K_1K_2}}=x^{2-\beta}=\frac{x^{2/3}}{(K_1K_2)^{1/3}}.$$

Thus we obtain (2. 4) immediately by combining our inequalities (4. 8), (4. 6), (4. 4). In order to prove (2. 5) we remark that $\Delta(x)\leqq O(\Sigma^{(1)})+O(\Sigma^{(2)})+O(\Sigma^{(3)})$, where

$$\Sigma^{(1)}=\sum_{K_1,K_2\in\mathscr{B}}\Sigma^{(1)}_{K_1,K_2},\quad \Sigma^{(2)}=\sum_{K_1,K_2\in\mathscr{B}}\Sigma^{(2)}_{K_1,K_2},$$

$$\Sigma^{(3)}=\sum_{K_1,K_2\in\mathscr{I}}\frac{x}{K_1K_2}\Sigma^{(3)}_{K_1,K_2}.$$

Using $\sum_{K\leqq x}\frac{\sigma_{-1/2}(K)}{\sqrt{K}}\ll\log x$ we have from (4. 6) and (4. 4)

$$\Sigma^{(1)}\ll x^{1-\beta/2}(\log x)^3,\quad \Sigma^{(3)}\ll x^{\beta/2}(\log x)^3. \tag{4. 9}$$

Now we consider $\Sigma^{(2)}$. Set $N_1=\delta_1d_1^2K_1$, $N_2=\delta_2d_2^2K_2$. In our case $N_1N_2>x^\beta$ and N_i has at most $d_3(N_i)$ representations as product of δ_i, d_i^2, K_i. Let $N_i=u_i^2v_i$, where u_i^2 is the greatest quadratic divisor of N_i. Using the fact that N_i is a squarefull number, we have $v_i\leqq N_i^{1/3}$. Taking $n=l_1N_1=l_1v_1u_1^2$, $n+1=l_2v_2u_2^2$ we have that $\Sigma^{(2)}\ll x^\varepsilon R$, where R denotes the number of solutions of the equation

$$l_2v_2u_2^2-l_1v_1u_1^2=1 \tag{4. 10}$$

for those l_1,l_2,v_1,v_2,u_1,u_2 which satisfy the inequality

$$l_1v_1l_2v_2\left(\leqq 2\frac{x^2}{u_1^2u_2^2}\leqq 2\frac{x^2}{(N_1N_2)^{2/3}}\right)\leqq 2x^{2-\frac{2}{3}\beta},\qquad u_1\leqq x,\quad u_2\leqq x.$$

By Lemma 1 (4. 10) has at most $O(\log x)$ solutions in u_1,u_2 for fixed l_1,l_2,v_1,v_2. Thus

$$R\ll(\log x)\sum_{l_1l_2v_1v_2\leqq 2x^{2-\frac{2}{3}\beta}}1\ll x^{2-\frac{2}{3}\beta+\varepsilon},$$

and consequently

$$\Sigma^{(2)}\ll x^{2-\frac{2}{3}\beta+\varepsilon}. \tag{4. 11}$$

Now we choose $\beta=\frac{12}{7}$. Taking into account (4. 11), (4. 9) we deduce (2. 5.)

5. Proof of Theorem 2. This is a similar one to that of Theorem 1, therefore we give only a scetch for it. It is known that $\varrho(p^\alpha)=2$ or 0, according to $p\equiv 1$ or $\equiv -1 \pmod 4$. Furthermore $\varrho(2)=1$ and $\varrho(2^\alpha)=0$ for $\alpha\geqq 2$.

Let δ run over the divisors of K, and let d denote the numbers coprime to K. Then

$$B(x;K)=\sum_{\delta,d}\mu(\delta d)\varrho_x(\delta d^2K), \tag{5. 1}$$

where $\varrho_x(m)$ denotes in general those number of $n \leqq x$ for which m divides n^2+1. Then we have

$$\varrho_x(m) = \frac{\varrho(m)x}{m} + O(\varrho(m)). \tag{5. 2}$$

From (5. 1) and (5. 2) we have

$$B(x; K) = \frac{\tau(K)}{K}x + O\left(\frac{x}{K}\Sigma_K^{(3)}\right) + O(\Sigma_K^{(1)}) + O(\Sigma_K^{(2)}), \tag{5. 3}$$

where

$$\Sigma_K^{(1)} = \sum_{\delta, d} \varrho(\delta K d^2) \qquad (\delta d^2 K \leqq x^\beta) \tag{5. 4}$$

$$\Sigma_K^{(2)} = \sum_{\delta, d} \varrho_x(\delta K d^2) \qquad (\delta d^2 K > x^\beta) \tag{5. 5}$$

$$\Sigma_K^{(3)} = \sum_{\delta, d} \frac{\varrho(\delta K d^2)}{\delta K d^2} \qquad (\delta d^2 K \leqq x^\beta). \tag{5. 6}$$

Using that $\varrho(\delta K) = \varrho(K)$, $\varrho(d^2) = \varrho(d)$ and that $\sum_{m \leqq y} \varrho(m) \ll y$ we have without any difficulty, that

$$\Sigma_K^{(1)} \ll \frac{x^{\beta/2}}{\sqrt{K}} \varrho(K) \sigma_{-1/2}(K), \tag{5. 7}$$

$$\Sigma_K^{(3)} \ll x^{-\beta/2} \sqrt{K} \sigma_{-1/2}(K). \tag{5. 8}$$

For the estimation of $\Sigma_K^{(2)}$ we use Lemma 1. $\Sigma_K^{(2)}$ is not greater than the number of the solutions of

$$n^2 - l\delta K d^2 = 1 \qquad (n \leqq x) \tag{5. 9}$$

for those d, n, l, δ which satisfy the inequality $l \ll x^{2-\beta}$. For fixed l, δ (5. 9) has at most $O(\log x)$ solutions. Consequently

$$\Sigma_K^{(2)} \ll x^{2-\beta} (\log x) d(K). \tag{5. 10}$$

Choosing $\beta = \frac{\gamma+4}{3}$, where $K = x^\gamma$, and taking into account (5. 3), (5. 7), (5. 8), (5. 10) we obtain (3. 4).

The estimation of $P(x)$ goes in a similar manner as that of $\Delta(x)$ and thus we drop it.

6. The proof of Theorem 3 goes in a similar way as that of Theorem 2 applying the following lemma due to C. Hooley [8].

Lemma 2. *If $g(n)$ is an irreducible polynomial of degree 3, then the number of $n \leqq x$, for which there exists a p^2 divisor of $g(n)$, $> \log x$ is at most*

$$O\left(x (\log x)^{-A/\log\log\log x}\right),$$

$A > 0$ is a constant.

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(Received September 12, 1966.)