On the number of holomorphs of rings of order 8.

by

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Introduction. In a previous paper [1] we have determined the number of holomorphs of rings with additive group of type (p, p) and (p, p^2), (p is a prime number). In this paper *) we want to discuss the rings with additive group of type (2, 2, 2). As an abelian non-cyclic group of order 8 is either the direct sum of a cyclic group of order 2 and a cyclic group of order 4 or the direct sum of three cyclic groups of order 2, we get a survey of the number of holomorphs of all rings of order 8. Together with our previous results of [1] we have determined the number of holomorphs for all finite rings R with order less than 16. Our results for non-zero rings R with additive group R^+ of type (2, 2, 2) can easily be generalized to the case of non-zero rings R with R^+ of type (p, p, p), where p is a prime number. The zero-ring R with additive group R^+ of type (2, 2, 2) has a large number of holomorphs, in fact 367. The question arises now to determine the number of non-isomorphic holomorphs for finite rings with a small number of elements, both for non-zero and zero-rings.

1. The non-zero rings R with additive group R^+ of type (2, 2, 2)

Let R be a ring, whose additive group $R^+=(a_1)\oplus(a_2)\oplus(a_3)$ is the direct sum of three cyclic groups (a_1), (a_2) and (a_3). We assume, that R is not a zero-ring, i.e. the product ab ($a, b\in R$) does not vanish for *all* $a, b\in R$. The annulator n_R of R is the set of all elements $a\in R$, such that $aR=Ra=0$. R^2 is the ideal in R, generated by all products ab ($a, b\in R$). Both n_R and R^2 are characteristic subrings of R, which means that both n_R and R^2 are invariant under all double homothetisms of R (for the definitions and terminology we refer to our paper [1]).

As $O(R)=8$, we have that the orders of n_R and R^2 are divisors of 8, i.e. 1, 2, 4 or 8. If $O(n_R)=1$ or $n_R=(0)$, then R has one holomorph (WEINERT—EILHAUER [3]). If $O(n_R)=8$ or $n_R=R$, then R is a zero-ring, which we have excluded. If $O(R^2)=1$ or $R^2=(0)$, then R is a zero-ring. If $O(R^2)=8$ or $R^2=R$, then R has one holomorph (VAN LEEUWEN [1]). Thus we have to investigate only the cases: a) $O(n_R)=2$, $O(R^2)=4$; b) $O(n_R)=4$, $O(R^2)=2$; c) $O(n_R)=4$, $O(R^2)=4$, and d) $O(n_R)=2$, $O(R^2)=2$.

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Case a) $O(n_R)=2$, $O(R^2)=4$.

a_1) $n_R \cap R^2=(0)$. In this case $R=n_R \oplus R^2$ is the ring-theoretic direct sum of its ideals n_R and R^2 and R has one holomorph (WEINERT—EILHAUER [3], Satz 4).

a_2) $n_R \cap R^2 \neq (0)$. As $n_R \cap R^2 \subseteqq n_R$, $O(n_R)=2$, we must have $O(n_R \cap R^2)=2$. Then $n_R \cap R^2 = n_R$ or $n_R \subseteqq R^2$. Without loss of generality we may suppose that $n_R=\{0, a_1\}$ and $R^2=\{0, a_1, a_2, a_1+a_2\}$. One can construct 6 rings R with $n_R= =\{0, a_1\}$ and $R^2=\{0, a_1, a_2, a_1+a_2\}$ and $R^+=(a_1)\oplus(a_2)\oplus(a_3)$. These rings have the following multiplication tables:

A_1	a_2	a_3;
a_2	0	a_1
a_3	a_1	a_2

A_2	a_2	a_3 ;
a_2	0	a_1
a_3	a_1	a_1+a_2

A_3	a_2	a_3;
a_2	a_2	0
a_3	0	a_1

A_4	a_2	a_3 ;
a_2	a_2	a_2
a_3	a_2	a_1+a_2

A_5	a_2	a_3;
a_2	a_1+a_2	0
a_3	0	a_1

A_6	a_2	a_3 .
a_2	a_1+a_2	a_1+a_2
a_3	a_1+a_2	a_2

Each of the rings A_i $(i=1, \ldots, 6)$ is commutative. By the condition $s_i(ab)= =s_i(a)b$ for all $a, b \in A_i$, those endomorphisms s_i of A_i^+ are selected, which may occur as a first component of a double homothetism of A_i. These endomorphisms s_i form a subring $K_1(A_i)$ of $E(A_i^+)$, the endomorphism ring of A_i^+. Likewise the endomorphisms s_k with $s_k(ab)=a(s_k b)$ for all $a, b \in A_i$ form a subring $K_2(A_i)$ of $E(A_i^+)$. As A_i is commutative, we get $K_1(A_i)=K_2(A_i) \subseteqq E(A_i^+)$. For the uniqueness of the holomorph of a commutative ring A_i the commutativity of the ring $K_1(A_i)= =K_2(A_i) \subseteqq E(A_i^+)$ is necessary and sufficient ([3], Korollar, Satz 1). It is easy to check that for all rings A_i $(i=1, \ldots, 6)$ the ring $K_1(A_i)$ is commutative. Therefore each of the rings A_i has one holomorph. It may be remarked here that the ring A_1, for instance, does not satisfy the conditions of a theorem of POLLÁK [2], which reads: If the ring R has a characteristic subring R', which has one holomorph, and if each homomorphism of R/R' in n_R is the zero-homomorphism, then R has one holomorph. As A_1 has one holomorph, the conditions in this theorem are not necessary for the uniqueness of the holomorph. This is an example of a *finite* ring in which there is no proper characteristic subring R' satisfying Pollák's condition.

Case b) $O(n_R)=4$, $O(R^2)=2$.

b_1) $n_R \cap R^2=(0)$. In this case $R=n_R \oplus R^2$ is the direct sum of its ideals n_R and R^2 and as n_R has more than one holomorph (VAN LEEUWEN [1], Satz 3), R has more than one holomorph [1]. Now the subrings n_R and R^2 are characteristic subrings of R. Then the holomorphs of R exist in the form $H=H_1 \oplus H_2$, where H_1 is an arbitrary holomorph of n_R and H_2 is an arbitrary holomorph of R^2, (POLLÁK [2]). The zero-ring n_R with $n_R^+=(a_1)\oplus(a_2)$, $O(a_1)=O(a_2)=2$, has $2^2+2+3 = 9$ holomorphs (VAN LEEUWEN [1], Satz 3). The ring $R^2=(a_3)$ has one holomorph. Therefore $R=n_R \oplus R^2$ has 9 holomorphs. All rings R with $O(n_R)=4$, $O(R^2)=2$, $R^+=(a_1)\oplus(a_2)\oplus(a_3)$, $O(a_1)=O(a_2)=O(a_3)=2$, are isomorphic. Hence all of these rings have 9 holomorphs.

b_2) $n_R \cap R^2 \neq (0)$. As $n_R \cap R^2 \subseteqq R^2$, $O(R^2)=2$, we must have $O(n_R \cap R^2)=2$. Then $n_R \cap R^2 = R^2$ or $R^2 \subseteqq n_R$. Without loss of generality we may suppose that $R^2=\{0, a_1\}$ and $n_R=\{0, a_1, a_2, a_1+a_2\}$. Then $a_1^2=a_1a_2=a_1a_3=a_2a_1=a_2^2=$ $=a_2a_3=a_3a_1=a_3a_2=0$. And $a_3^2=a_1$, as $a_3^2=0$ implies that R is a zero-ring. For this multiplication the ring R is a non-zero ring, which is commutative and has 9 maximal rings of related double homothetisms and therefore 9 holomorphs. Therefore all of the rings in this case b_2) have 9 holomorphs.

Case c) $O(n_R)=4$, $O(R^2)=4$.

Suppose $n_R=\{0, a, b, a+b\}$ and $c \in R$ with $c \notin n_R$. Then $c+a$, $c+b$, $c+a+b$ belong to R, but none of them belongs to n_R and $R=\{0, a, b, c, a+b, a+c, b+c, a+b+c\}$. As $aR=Ra=0$ and $bR=Rb=0$ we get $R^2=\{0, c^2\}$, which is a contradiction to $O(R^2)=4$. Thus there are no rings possible in this case.

Case d) $O(n_R)=2$, $O(R^2)=2$.

d_1) $n_R \cap R^2=(0)$. Suppose $n_R=\{0, a\}$ and $R^2=\{0, b\}$ with $a \neq b$. Again, if $c \neq a$, $c \neq b$ $(c \in R)$, then $R=\{0, a, b, c, a+b, a+c, b+c, a+b+c\}$. If $b^2=0$, then from $bc=b$ we would get $(bc)c=bc=b=b(c^2)$, or $c^2=b$, but then $b^2=b$. Contradiction. Thus $b^2=0$ implies $bc=0$. Similarly $b^2=0$ implies $cb=0$, But now $bR=Rb=0$ or $b \in n_R$, which is impossible. We conclude: $b^2=b$.

If $bc=b$, then $(bc)b=b^2=b=b(cb)$, and $cb=b$. Also $(bc)c=bc=b=b(c^2)$, and $c^2=b$. Then $(b+c)R=R(b+c)=0$, which implies $b+c \in n_R$. Contradiction. If $bc=0$, then $(bc)b=0=b(cb)$, or $cb=0$. Also $(bc)c=0=b(c^2)$ and $c^2=0$. Then $cR=Rc=0$ or $c \in n_R$. Contradiction. Thus there is no ring satisfying the conditions of this case.

d_2) $n_R \cap R^2 \neq (0)$. It follows now that $n_R=R^2$. Without loss of generality we may suppose that $n_R=R^2=\{0, a_1\}$. For each of the elements a_2^2, a_2a_3, a_3a_2 and a_3^2 $(\in R)$ one can choose either 0 or a_1, but not $a_2^2=a_2a_3=a_3a_2=a_3^2=0$, as R is not a zero-ring. One can construct 12 non-zero rings R with $n_R=R^2=\{0, a_1\}$ and $R^+=(a_1)\oplus(a_2)\oplus(a_3)$. These rings have the following multiplication tables:

B_1:

	a_2	a_3
a_2	0	a_1
a_3	a_1	0

B_2:

	a_2	a_3
a_2	0	a_1
a_3	a_1	a_1

B_3:

	a_2	a_3
a_2	a_1	0
a_3	0	a_1

B_4:

	a_2	a_3
a_2	a_1	0
a_3	a_1	a_1

B_5:

	a_2	a_3
a_2	a_1	a_1
a_3	0	a_1

B_6:

	a_2	a_3
a_2	a_1	a_1
a_3	a_1	0

B_7:

	a_2	a_3
a_2	0	0
a_3	a_1	0

B_8:

	a_2	a_3
a_2	0	0
a_3	a_1	a_1

B_9:

	a_2	a_3
a_2	0	a_1
a_3	0	0

B_{10}:

	a_2	a_3
a_2	0	a_1
a_3	0	a_1

B_{11}:

	a_2	a_3
a_2	a_1	0
a_3	a_1	0

B_{12}:

	a_2	a_3
a_2	a_1	a_1
a_3	0	0

As the sets $\{B_2, B_3, B_6\}$, $\{B_4, B_5\}$, $\{B_7, B_8, B_9, B_{10}, B_{11}, B_{12}\}$ are consisting of isomorphic rings each, we need only to consider the rings B_1, B_2, B_4 and B_7. The ring B_1 is commutative. Therefore $K_1(B_1)=K_2(B_1)$ (see case a_2)). As the ring $K_1(B_1)$ is commutative, the ring B_1 has one holomorph ([3], Korollar Satz 1). For the same reason, the ring B_2 has one holomorph.

Now we consider the ring B_4. Let s_i be the first component of a double homothetism of B_4. Then $s_i(a_2^2)=s_i(a_1)=s_i(a_2)a_2$, and $s_i(a_1)=0$ or a_1. From $s_i(a_2a_3)=$ $=0=s_i(a_2)a_3$ it follows that $s_i(a_2)=0, a_1, a_2$ or a_1+a_2. From $s_i(a_3a_2)=s_i(a_1)=$ $=s_i(a_3)a_2$ and $s_i(a_3^2)=s_i(a_1)=s_i(a_3)a_3$ we infer that $s_i(a_3)=0, a_1, a_3$ or a_1+a_3. It turns out that $s_i(a_1)=0$ implies $s_i(a_2)=0$ or a_1 and $s_i(a_3)=0$ or a_1 and that $s_i(a_1)=a_1$ implies $s_i(a_2)=a_2$ or a_1+a_2 and $s_i(a_3)=a_3$ or a_1+a_3. Let s_k be the second component of a double homothetism of B_4. Then likewise we find that $s_k(a_1)=0$ implies $s_k(a_2)=0$ or a_1 and $s_k(a_3)=0$ or a_1 and that $s_k(a_1)=a_1$ implies $s_k(a_2)=a_2$ or a_1+a_2 and $s_k(a_3)=a_3$ or a_1+a_3. Thus $K_1(B_4)=K_2(B_4)$. The ring $K_1(B_4)$ consists of the same endomorphisms of $(a_1)\oplus(a_2)\oplus(a_3)$ as the ring $K_1(B_2)$ or $K_1(B_1)$. Therefore $K_1(B_4)$ is commutative. The condition $s_k(s_ia)=s_i(s_ka)$ is satisfied for all $s_i\in K_1(B_4)$, $s_k\in K_2(B_4)$ and for all $a\in B_4$. The condition $(s_ka)b=a(s_ib)$ for all $a,b\in B_4$ with $s_i\in K_1(B_4)$ and $s_k\in K_2(B_4)$ implies that we cannot use all possible pairs (s_i, s_k) $(s_i\in K_1(B_4), s_k\in K_2(B_4))$ as double homothetisms of B_4. But if (s_i, s_k) and (s_i', s_k') are double homothetisms of B_4, then $s_is_k'=s_k's_i$ and $s_ks_i'=s_i's_k$, as $K_1(B_4)=K_2(B_4)$ is commutative. All double homothetisms of B_4 are pairwise related, therefore B_4 has one holomorph.

Finally we investigate the ring B_7. From the conditions $s_i(ab)=s_i(a)b$, $s_k(ab)=$ $=as_k(b)$, $as_i(b)=s_k(a)b$ for s_i, s_k endomorphisms of B_7^+ and all $a, b\in B_7$ we get that $s_i(a_1)=0$ or a_1, $s_i(a_2)=0, a_1, a_2$ or a_1+a_2, $s_i(a_3)=0, a_1, a_3$ or a_1+a_3 and $s_k(a_1)=0$ or a_1, $s_k(a_2)=0, a_1, a_2$ or a_1+a_2, $s_k(a_3)=0, a_1, a_3$ or a_1+a_3. Moreover if $s_i(a_1)=0$ then $s_i(a_3)=0$ or a_1 and if $s_i(a_1)=a_1$ then $s_i(a_3)=a_3$ or a_1+a_3; likewise if $s_k(a_1)=0$ then $s_k(a_2)=0$ or a_1 and if $s_k(a_1)=a_1$ then $s_k(a_2)=a_2$ or a_1+a_2. Also if $s_i(a_2)=0$ or a_1 then $s_k(a_3)=0$ or a_1 and if $s_i(a_2)=a_2$ or a_1+a_2 then $s_k(a_3)=a_3$ or a_1+a_3. A pair of endomorphisms (s_i, s_k) of B_7^+ satisfying the above conditions is a double homothetism of B_7 if $s_k(s_ia)=s_i(s_ka)$ for all $a\in B_7$. It turns out, that the set of all double homothetisms (s_i, s_k) of B_7 consists of 72 elements. The double homothetisms (s_i, s_k) and (s_i', s_k') of B_7 are related if $s_i s_k'=s_k's_i$ and $s_k s_i'=s_i' s_k$ hold Each set of pairwise related double homothetisms of B_7 is contained in a maximal ring D of this kind. The ring B_7 has 4 maximal rings of related double homothetisms and therefore 4 holomorphs.

2. The zero-ring R with additive group R^+ of type (2, 2, 2)

In this section R will be a zero-ring i.e. $ab=0$ for all $a, b\in R$ and $R^+=(a_1)\oplus$ $\oplus(a_2)\oplus(a_3)$ is the direct sum of 3 cyclic groups, each of order 2. An endomorphism s of R^+ is determined by the images $s(a_1)$, $s(a_2)$ and $s(a_3)$. We start with the zero-endomorphism: $s(a_1)=0$, $s(a_2)=0$, $s(a_3)=0$ and call it s_1. Then $s_2(a_1)=0$, $s_2(a_2)=0$, $s_2(a_3)=a_1$; $s_3(a_1)=0$, $s_3(a_2)=0$, $s_3(a_3)=a_2$, etc. The endomorphisms of R^+ are indexed „lexicographically" and are in this order $s_1, \ldots, s_{512}$, where $s_{512}(a_1)=$ $=a_1+a_2+a_3$, $s_{512}(a_2)=a_1+a_2+a_3$, $s_{512}(a_3)=a_1+a_2+a_3$. For the zero-ring R a pair of endomorphisms (s_i, s_k) of R is a double homothetism of R if $s_is_k=s_ks_i$. Two double homothetisms (s_i, s_k) and (s_i', s_k') are related if $s_is_k'=s_k's_i$ and $s_ks_i'=s_i's_k$.

Our problem is to determine the number of holomorphs of R, i.e. the number of maximal rings of related double homothetisms of R. In order to do so, we introduce the following notation for the set of endomorphisms of R^+, consisting of elements which are commuting with a given s: $\{s\}=\{s_i\in E(R^+),\ s_is=ss_i\}$. There are 2 endomorphisms s in $E(R^+)$, such that $\{s\}$ has order 512, namely s_1 and s_{84}: $a_1\to a_1$, $a_2\to a_2$, $a_3\to a_3$. There are 98 endomorphisms s in $E(R^+)$, such that $\{s\}$ has order 32. These 98 endomorphisms can be divided into pairs, such that each pair consists of two endomorphisms s_b, s_a with the same elements in $\{s_a\}$ and $\{s_b\}$, $s_a\neq s_b$. The remaining 412 elements of $E(R^+)$ are each commuting with 8 endomorphisms of $E(R^+)$.

A. First we consider the sets $\{s\}$ with order 8. If $\{s\}$ is a set of order 8, and $s_a, s_b\in\{s\}$, then $s_as_b=s_as_b$. Now we distinguish two cases:

(i) if s_c is an arbitrary element of $\{s\}$, $s_c\neq s_1$, $s_c\neq s_{84}$, then $\{s_c\}=\{s\}$.

(ii) there is at least one element $s_v\in\{s\}$, $s_v\neq s_1, s_v\neq s_{84}$, such that $\{s_v\}\neq\{s\}$. In this case $\{s\}\subset\{s_v\}$ and $\{s_v\}$ has order 32.

Case (i). Let $\{s\}$ be a set of endomorphisms of $E(R^+)$ of order 8. If s_c, s_b are arbitrary in $\{s\}$, then (s_a, s_b) is a double homothetism of R, as $s_as_b=s_bs_a$. It is clear that every pair of double homothetisms (s_a, s_b), (s_a', s_b') is related. The set of all double homothetisms, obtained in this way, is a maximal set and therefore a maximal ring of related double homothetisms of R. For let (s_t, s_u) be related to (s_a, s_b) with $s_a, s_b\in\{s\}$. Then $s_ts_b=s_bs_t$ or $s_t\in\{s_b\}=\{s\}$ and $s_as_u=s_us_a$ or $s_u\in\{s_a\}=\{s\}$. Thus we get 8 maximal rings of related double homothetisms of R and therefore 8 holomorphs of R. These holomorphs are denoted by P_i $(i=1, \ldots, 8)$.

Case (ii). Let s be an endomorphism which commutes with 8 endomorphisms, so that $\{s\}$ has order 8 and let $s_v\in\{s\}$ with $\{s\}\subset\{s_v\}$, where $\{s_v\}$ has order 32. If s_a and s_b are arbitrary in $\{s\}$, then (s_a, s_b) is a double homothetism of R, as $s_as_b= =s_bs_a$. Again we form the set of all pairs (s_a, s_b) with $s_a, s_b\in\{s\}$. Then we obtain a set S of related double homothetisms of R, since, if $s_a, s_b, s_a', s_b'\in\{s\}$, then $s_as_b'=s_b's_a$ and $s_bs_a'=s_a's_b$, whence the double homothetisms (s_a, s_b) and (s_a', s_b') are related. We state that S is a maximal set of related double homothetisms of R. Indeed, let (s_t, s_u) be a double homothetism related to all double homothetisms of S. In particular, (s_t, s_u) is related to (s, s) which means $s_ts=ss_t$ and $s_us=ss_u$ or $s_t\in\{s\}$ and $s_u\in\{s\}$. Therefore $(s_t, s_u)\in S$, hence S is maximal with respect to the property of relatedness. In this way we get 133 maximal rings S_i of related double homothetisms of R and therefore 133 holomorphs of R, which we denote by Q_i $(i=1, \ldots, 133)$.

B. Next we consider the sets $\{s\}$ with order 32. As we have remarked, there are 49 different sets of this kind. Let $\{s_c\}$ and $\{s_d\}$ be two *different* sets, both of order 32, and such that $s_cs_d=s_ds_c$. It may be remarked here, that two arbitrary elements s_a, s_b of a given set $\{s\}$ of order 32 need not be commuting. Now (s_c, s_d) is a double homothetism of R. Then we consider the endomorphisms of $E(R^+)$ which belong to the set $\mathscr{A}=\{s_c\}\cap\{s_d\}$. The endomorphisms of $\mathscr{A}$ are commuting and $\mathscr{A}$ has order 8. There are now two cases: (a) there is at least one endomorphism s in $\mathscr{A}$, such that $\{s\}$ has order 8, then $\mathscr{A}=\{s\}$; (b) for all endomorphisms s in $\mathscr{A}$, except s_1 and s_{84}, the set $\{s\}$ has order 32. In both cases we form the set of all pairs (s_k, s_l) with $s_k, s_l\in\mathscr{A}$. In case (a) we get a set S of pairwise related double homothetisms which were obtained already in case (ii) of **A**. In case (b) the elements of $\mathscr{A}=\{s_c\}\cap\{s_d\}$ are: $s_1, s_{84}, s_c, s_d, s_e$ such that $\{s_e\}=\{s_c\}$, s_f such that $\{s_f\}=$

$=\{s_d\}$, s_g with $\{s_g\}\cap\{s_c\}=\{s_g\}\cap\{s_d\}=\mathscr{A}$ and s_h with $\{s_h\}=\{s_g\}$. The set of all pairs (s_k, s_l) with $s_k, s_l\in\mathscr{A}$ is again a set S' of pairwise related double homothetisms of R. Also S' is a maximal set of related double homothetisms of R. For, let (s_t, s_u) be a double homothetism related to all double homothetisms of S'. Then (s_t, s_u) is related to (s_c, s_c) or $s_t s_c = s_c s_t$ and $s_u s_c = s_c s_u$. Likewise we have $s_t s_d = s_d s_t$ and $s_u s_d = s_d s_u$. Hence $s_t\in\{s_c\}\cap\{s_d\}=\mathscr{A}$, and $s_u\in\mathscr{A}$, whence $(s_t, s_u)\in S'$. Thus we get 42 maximal rings S_i' of related double homothetisms of R and therefore 42 holomorphs of R, which we denote by R_i $(i=1, \ldots, 42)$.

C. Let $\{s_c\}$ and $\{s_d\}$ be *equal* sets, both of order 32, with $s_c\neq s_d$. As s_c and s_d belong to the same set $\{s_c\}=\{s_d\}$, it is clear that $s_c s_d = s_d s_c$. Now we can form the set S'' consisting of all pairs (s_1, s_t), (s_{84}, s_u), (s_c, s_x), (s_d, s_y), where s_t, s_u, s_x and s_y run through the set $\{s_c\}$. Then S'' consists of related double homothetisms of R. Let (s_e, s_f) be a double homothetism related to all double homothetisms of S''. We are going to prove that $(s_e, s_f)\in S''$. In $\{s_c\}$ we can find two endomorphisms $s_a, s_b \{s_a\}\neq\{s_b\}$ such that both $\{s_a\}$ and $\{s_b\}$ have order 8. We show that $\{s_a\}\cap\{s_b\}= =(s_1, s_{84}, s_c, s_d)$. Indeed, as $s_a\in\{s_c\}=\{s_d\}$ we have that $s_c\in\{s_a\}$ and $s_d\in\{s_a\}$, $s_b\in\{s_c\}=\{s_d\}$ implies that $s_c\in\{s_b\}$ and $s_d\in\{s_b\}$. Hence $(s_1, s_{84}, s_c, s_d)\subseteqq\{s_a\}\cap\{s_b\}$ or the order of $\{s_a\}\cap\{s_b\}$ is $\geqq 4$. But as the orders of $\{s_a\}$ and $\{s_b\}$ are 8, and $\{s_a\}\neq\{s_b\}$, we must have that the order of $\{s_a\}\cap\{s_b\}$ is $\leqq 4$. This proves that $\{s_a\}\cap\{s_b\}=(s_1, s_{84}, s_c, s_d)$. From (s_e, s_f) is related to (s_c, s_a) we infer that $s_e s_a = s_a s_e$ or $s_e\in\{s_a\}$. Likewise $s_e s_b = s_b s_e$ or $s_e\in\{s_b\}$. Hence $s_e\in\{s_a\}\cap\{s_b\}=(s_1, s_{84}, s_c, s_d)$. From (s_e, s_f) is related to (s_c, s_a) we also infer that $s_f s_c = s_c s_f$ of $s_f\in\{s_c\}$. Thus $(s_e, s_f)\in S''$. It follows that S'' is a maximal ring of related double homothetisms of R. As we have 49 different pairs s_c, s_d with $\{s_c\}=\{s_d\}$ and $O(\{s_c\})=32$, we can form 49 maximal rings S_i'' and we get 49 holomorphs of R.

In the same way we can form the set S''' consisting of the pairs (s_t, s_1), (s_u, s_{84}), (s_x, s_c), (s_y, s_d), where s_t, s_u, s_x, s_y run through the set $\{s_c\}=\{s_d\}$ $(s_c\neq s_d,\ O(\{s_c\})= =32)$. As above, we can prove that S''' is a maximal ring of related double homothetisms of R. Thus we get again 49 holomorphs of R, which yields altogether 98 holomorphs of R, denoted by T_i $(i=1, \ldots, 98)$.

D. Let $\{s_c\}$ and $\{s_d\}$ be two *different* sets, both of order 32, and such that $s_c s_d\neq s_d s_c$. Then we consider the endomorphisms of $E(R^+)$, which belong to the set $\mathscr{A}'=\{s_c\}\cap\{s_d\}$. The order of $\mathscr{A}'$ is 4 or 8.

If $\mathscr{A}'$ has order 4, then the elements of $\mathscr{A}'$ are (s_1, s_{84}, s_k, s_k'), where $s_k\in \in\{s_c\}\cap\{s_d\}$ and $\{s_k'\}=\{s_k\}$ with $s_k\neq s_k'$. Now we can form the set of all pairs (s_1, s_t), (s_{84}, s_u), (s_k, s_x), (s_k', s_y), where s_t, s_u, s_x and s_y run through the set $\{s_k\}$; we get a maximal ring of related double homothetisms of R, which was obtained already in case **C**. The set of all pairs (s_t, s_1), (s_u, s_{84}), (s_x, s_k) and (s_y, s_k') is a maximal ring of related double homothetisms of R, if s_t, s_u, s_x and s_y run through the set $\{s_k\}$. This ring, too, is one of the rings of case **C**. If $\mathscr{A}'$ has order 8, then the elements of $\mathscr{A}'$ are $(s_1, s_{84}, s_e, s_e', s_f, s_f', s_g, s_g')$, where each of the last 6 endomorphisms commutes with 32 endomorphisms of $E(R^+)$. The notation s_e, s_e' indicates that the sets $\{s_e\}$ and $\{s_e'\}$ are equal, and likewise for s_f and s_g. In $E(R^+)$ we can find an endomorphism, say, s_h, $s_h\neq s_c$, $s_h\neq s_d$, $s_h\neq s_c'$, $s_h\neq s_d'$, such that $\{s_c\}\cap\{s_d\}= =\{s_h\}\cap\{s_d\}=\{s_c\}\cap\{s_h\}$. Now we form the set $\mathscr{B}'$: $(s_1, s_{84}, s_c, s_c', s_d, s_d', s_h, s_h')$. As $\mathscr{A}'\subset\{s_h\}=\{s_h'\}$, it follows that each element of $\mathscr{B}'$ is commuting with each element of $\mathscr{A}'$. The set (s_k, s_l) of pairs $s_k\in\mathscr{B}'$, $s_l\in\mathscr{A}'$ is a set $S^\wedge$ of related double homothetisms of R. Moreover $S^{\wedge}$ is a maximal set of related double homothetisms of R. For, let (s_t, s_u) be a double homothetism related to all double homothetisms of $S^{\wedge}$. Then $s_u s_c = s_c s_u$ and $s_u s_d = s_d s_u$, hence $s_u \in \{s_c\} \cap \{s_d\} = \mathscr{A}'$. As $s_t s_e = s_e s_t$ and $s_t s_f = s_f s_t$, it follows that $s_t \in \{s_e\} \cap \{s_f\}$. But $\{s_e\} \cap \{s_f\} = \mathscr{B}'$, whence $s_t \in \mathscr{B}'$. Therefore $(s_t, s_u) \in S^{\wedge}$. The set of all pairs (s_l, s_k) with $s_l \in \mathscr{A}'$, $s_k \in \mathscr{B}'$ is likewise a maximal set $S^{\wedge\wedge}$ of related double homothetisms of R. In this way we obtain 42 pairs $S^{\wedge}, S^{\wedge\wedge}$ of maximal sets, hence 84 maximal rings of related double homothetisms of R. The 84 corresponding holomorphs of R are denoted by U_i $(i = 1, \ldots, 84)$.

E. Finally we consider the two sets $\{s_1\}$ and $\{s_{84}\}$, both consisting of all endomorphisms of R^+. Here we form the set S^{iv} of all pairs (s_1, s_k) and (s_{84}, s_n) where s_k and s_n run through $E(R^+)$. S^{iv} consists of related double homothetisms of R and it is a maximal set of this kind.

Likewise the set S^v, consisting of all pairs (s_k, s_1) and (s_n, s_{84}), where s_k, s_n run through $E(R^+)$, is a maximal ring of related double homothetisms of R. Thus we obtain 2 holomorphs of R, denoted by V_1, V_2. For a zero-ring R, whose additive group $R^+ = (a_1) \oplus (a_2) \oplus (a_3)$ is the direct sum of three cyclic groups each of order 2, we have determined 367 holomorphs. We prove now that these are *all* holomorphs of R.

Let U be an arbitrary maximal ring of related double homothetisms of R.

a. Suppose there is an element $(s_i, s_k) \in U$, $s_i \neq s_1, s_{84}$, $s_k \neq s_1, s_{84}$, such that $\{s_i\}$ has order 8, and if s_c is arbitrary in $\{s_i\}$, then $\{s_c\} = \{s_i\}$ $(s_c \neq s_1, s_{84})$. As we have seen in case (i) of **A**, the set of all pairs (s_a, s_b) with $s_a, s_b \in \{s_i\}$ is a maximal ring M of related double homothetisms of R. It follows that $(s_i, s_k) \in U \cap M$. Now let (s_c, s_d) be an arbitrary element of U $(s_c \neq s_1, s_{84}, s_d \neq s_1, s_{84})$. As $s_c s_k = s_k s_c$ and $s_k \in \{s_i\}$, we get that $\{s_k\} = \{s_i\}$ and $s_c \in \{s_i\}$. From $s_d s_i = s_i s_d$ it follows that $s_d \in \{s_i\}$. Therefore $(s_c, s_d) \in M$. The elements (s_c, s_d) in U with either s_c or s_d or both equal to s_1 or s_{84} belong to M, as $s_1, s_{84} \in \{s_i\}$. Thus $U \subseteqq M$, and as U is maximal, $U = M$. Then the holomorph cooresponding to U is one of the holomorphs P_i $(i = 1, \ldots, 8)$. In the sequel we need the following properties of the ring $E(R^+)$:

Property I. If $\{s_a\}$ has order 8, $\{s\}$ has order 32 and $s_a \in \{s\}$, then $\{s_a\} \subset \{s\}$

Property II. If $s_i, s_k \in \{s_i\} \cap \{s_k\}$ and both $\{s_i\}$ and $\{s_k\}$ have order 8, then $\{s_i\} = \{s_k\}$.

b. Suppose there is an element $(s_i, s_k) \in U$, $s_i \neq s_1, s_{84}$, $s_k \neq s_1, s_{84}$ with $s_i, s_k \in \{s\}$, $\{s\}$ being a set of order 8 of case (ii) of **A**. Moreover we assume that at least one of the two sets $\{s_i\}$ and $\{s_k\}$ is of order 8, say $\{s_i\}$. As $s_i s = s s_i$ and both $\{s\}$ and $\{s_i\}$ have order 8, we get $\{s\} = \{s_i\}$ (property II).

b_1. $\{s_k\}$ has order 8, which implies that $\{s\} = \{s_i\} = \{s_k\}$. Let (s_a, s_b) be an arbitrary element of U. From (s_a, s_b) is related to (s_i, s_k) it follows that $s_a \in \{s_k\} = \{s_i\}$ and $s_b \in \{s_i\}$. This means that $U \subseteqq S_i$, where S_i is a maximal ring of related double homothetisms of R, obtained in case (ii) of **A**. From the maximality of U we infer that $U = S_i$, hence the holomorph corresponding to U is one of the holomorphs Q_i $(i = 1, \ldots, 133)$.

b_2. $\{s_k\}$ has order 32, which implies $\{s_i\} \subset \{s_k\}$ (property I).

b_2(i). All double homothetisms of U have as second component s_1, s_{84}, s_k, or s_k', where s_k' is the uniquely determined endomorphism of R^+ with $\{s_k'\} = \{s_k\}$, $s_k \neq s_k'$. Let (s_a, s_b) be an arbitrary element of U. Then $s_a \in \{s_k\}$, hence $U \subseteqq S_i'''$, where S_i''' is one of the maximal rings of related double homothetisms of R, obtained in **C**.

Again $U=S_i'''$ and the holomorph corresponding to U is one of the holomorphs T_i $(i=1, \ldots, 98)$.

b_2.(ii). There is at least one double homothetism in U, say (s_c, s_d), such that s_d is not one of the endomorphisms s_1, s_{84}, s_k, s_k' of case b_2(i). As (s_c, s_d) is related to (s_i, s_k) it follows that $s_d \in \{s_i\} \subset \{s_k\}$. Let (s_a, s_b) be an arbitrary element of U. Then (s_a, s_b) is related both to (s_i, s_k) and (s_c, s_d). Therefore $s_a \in \{s_k\} \cap \{s_d\}$.

b_2(iia). Suppose $\{s_d\}$ has order 8, then $\{s_d\} \subset \{s_k\}$ (property I) and $s_a \in \{s_d\}$. From $s_d s_i = s_i s_d$ and both $\{s_d\}$ and $\{s_i\}$ have order 8, it follows that $\{s_d\} = \{s_i\}$ (property II). Hence $s_a \in \{s_i\}$. From (s_a, s_b) is related to (s_i, s_k) we infer that $s_b \in \{s_i\}$. Thus in this case $U \subseteqq S_i$, where S_i is one of the maximal rings of related double homothetisms of R, obtained in case (ii) of **A**. Then $U=S_i$ and the holomorph corresponding to U is one of the holomorphs Q_i $(i=1, \ldots, 133)$.

b_2(iib). Suppose $\{s_d\}$ has order 32. Now $s_k s_d = s_d s_k$, both $\{s_k\}$ and $\{s_d\}$ have order 32 and $\{s_k\} \neq \{s_d\}$. From (s_a, s_b) is related to (s_i, s_k) it follows that $s_a \in \{s_d\} \cap \{s_k\}$. Also $s_d \in \{s_i\}$ and, as $\{s_i\}$ has order 8, one gets $s_d s_b = s_b s_d$ or $s_b \in \{s_d\}$. Hence $s_b \in \{s_k\} \cap \{s_d\}$. Now $U \subseteqq \mathscr{A}_i$, where $\mathscr{A}_i$ is one of the maximal rings of related double homothetisms of R of case **B**. Then $U=\mathscr{A}_i$ and the holomorph corresponding to U is one of the holomorphs R_i $(i=1, \ldots, 42)$.

Remark. The case: "there exists an element $(s_i, s_k) \in U$, $s_i \neq s_1, s_{84}$, $s_k \neq s_1, s_{84}$ with $s_i, s_k \in \{s\}$, $\{s\}$ being a set of order 8 of case (ii) of **A** and $\{s_k\}$ having order 8" is quite similar to the case b just finished.

c. Suppose there is an element $(s_i, s_k) \in U(s_i \neq s_1, s_{84}; s_k \neq s_1, s_{84})$ with $s_i, s_k \in \{s\}$, where $\{s\}$ is a set of order 8 of case (ii) of **A**. Now we assume that both $\{s_i\}$ and $\{s_k\}$ have order 32.

First we remark that we may assume that for every element $(s_a, s_b) \in U(s_a \neq s_1, s_{84}; s_b \neq s_1, s_{84})$ both $\{s_a\}$ and $\{s_b\}$ have order 32. For let $(s_c, s_d) \in U$, where, say s_c, has order 8. As $s_d \in \{s_c\}$, we have that $s_c, s_d \in \{s_c\}$, $\{s_c\}$ being a set of order 8. Then we get a ring U, already considered in case a or case b. Secondly we state that if $(s_c, s_t) \in U$, $s_c \neq s_1, s_{84}$, then $(s_c', s_t) \in U$, where s_c' is the uniquely determined element of $E(R^+)$ with $\{s_c\} = \{s_c'\}$, $s_c \neq s_c'$. For, if (s_a, s_b) is an arbitrary element of U, then $s_c' s_b = s_b s_c'$, as $s_c s_b = s_b s_c$ and $s_a s_t = s_t s_a$. Likewise from $(s_c, s_t) \in U$ it follows that $(s_c, s_t') \in U$. In the sequel the notation s_a, s_a' means that the sets $\{s_a\}$ and $\{s_a'\}$ are equal, both of order 32, and $s_a \neq s_a'$.

c_1. All double homothetisms of U have as a first component s_1, s_{84}, s_i or s_i'. Let (s_a, s_b) be an arbitrary element of U. Then $s_b \in \{s_i\}$ and $U \subseteqq S_i''$, where S_i'' is one of the maximal rings of related double homothetisms of R of case **C**. The maximality of U implies $U=S_i''$. The holomorph corresponding to U is one of the holomorphs T_i $(i=1, \ldots, 98)$.

c_2. There is at least one double homothetism of U, say (s_c, s_d), such that $s_c \notin (s_1, s_{84}, s_i, s_i')$. Now the endomorphisms $s_1, s_{84}, s_i, s_i', s_c, s_c'$ occur as first components of double homothetisms of U.

c_2(i). First we suppose that $s_i s_c = s_c s_i$. Let (s_a, s_b) be an arbitrary element of U, then $s_b \in \{s_c\} \cap \{s_i\}$. Therefore the set K of all second components of double homothetisms of U is a subset of a set $\mathscr{A}_i = \{s_c\} \cap \{s_i\}$ of case **B**. Now $s_k, s_k' \in K$, $s_k \neq s_1, s_{84}$, hence the order of $K \geqq 4$. If the order of K is 4, then all double homothetisms of U have as second component s_1, s_{84}, s_k or s_k'. This case has been considered in c_1, hence $O(K) > 4$. Then the endomorphisms $s_1, s_{84}, s_k, s_k', s_l, s_l'$ occur in K, where $s_l \neq s_1, s_{84}, s_k, s_k'$ and $s_l \in \{s_c\} \cap \{s_i\}$. As s_a commutes with all second components, it follows that $s_a \in \{s_k\} \cap \{s_l\} = \mathscr{A}_i$. Thus in this case $U \subseteqq S_i'$, where S_i' is a maximal ring of related double homothetisms of R of case **B**. As U is also a maximal ring, we get that $U = S_i'$. The holomorph corresponding to U is one of the holomorphs R_i $(i = 1, \ldots, 42)$.

c_2(ii). Secondly we suppose that $s_c s_i \neq s_i s_c$. Again, if (s_a, s_b) is an arbitrary element of U, $s_b \in \{s_c\} \cap \{s_i\}$. Therefore the set L of all second components of double homothetisms of U is a subset of a set $\mathscr{A}_i' = \{s_c\} \cap \{s_i\}$ of case **D**. As $s_k, s_k' \in L$, $s_k \neq \neq s_1, s_{84}$, we have that $O(L) \geqq 4$. If $O(L) = 4$, then all double homothetisms of U have as second components s_1, s_{84}, s_k or s_k', which is again case c_1, hence $O(L) \geqq 4$. Then the endomorphisms $s_1, s_{84}, s_k, s_k', s_l, s_l'$ occur in L, where $s_l \neq s_1, s_{84}, s_k, s_k'$ and $s_l \in \{s_c\} \cap \{s_i\}$. An element s_a that commutes with all elements of $\mathscr{A}_i'$ belongs to $\mathscr{B}_i'$, where $\mathscr{B}_i' = \{s_k\} \cap \{s_l\}$. Hence $U \subseteqq S_i^{\wedge}$, where $S_i^{\wedge}$ is a maximal ring of related double homothetisms of R of case **D**. It follows that $U = S_i^{\wedge}$. The holomorph corresponding to U is one of the holomorphs U_i $(i = 1, \ldots, 84)$. In the preceding cases a, b and $c$we have assumed the existence of an element $(s_i, s_k) \in U$ with $s_i \neq s_1, s_{84}$ and $s_k \neq s_1, s_{84}$. Now every endomorphism $s_n \in E(R^+)(s_n \neq s_1, s_{84})$ belongs either to a set $\{s\}$ of case (i) of **A** or to at least one set $\{s'\}$ of case (ii) of **A**. If s_i and s_k belong to the *same* set $\{s\}$ of case (i) of **A**, we get case a. The case that s_i and s_k belong to *different* sets of case (i) of **A** is impossible, as $s_i s_k = s_k s_i$. It is a consequence of properties I and II that $s_i \in \{s\}$ of case (i) of **A** and $s_k \in \{s'\}$ of case (ii) of **A** or conversely is also impossible. There remains the case that s_i and s_k both belong to sets of case (ii) of **A**. If s_i and s_k belong to the *same* set $\{s'\}$ of case (ii) of **A**, we get cases b and c. Suppose now that $s_i \in \{s\}$ and $s_k \in \{s'\}$, where both $\{s\}$ and $\{s'\}$ are sets of case (ii) of **A**, such that $\{s\} \neq \{s'\}$. Then $O(\{s_i\}) = O(\{s_k\}) = 8$ is impossible, since this would imply $\{s\} = \{s'\}$ by property II. If at least one of the sets $\{s_i\}$ and $\{s_k\}$ has order 32, we can proceed as in the cases b_2 or c, since in b_2 and in c the assumption that s_i and s_k belong to the *same* set of case (ii) of **A** is not essential.

d. Suppose there is no element $(s_i, s_k) \in U$ with $s_i \neq s_1, s_{84}$ and $s_k \neq s_1, s_{84}$. It is clear that, if $(s_1, s_c) \in U$, $(s_{84}, s_c) \in U$ and, if $(s_c, s_1) \in U$, $(s_c, s_{84}) \in U$. Now suppose (s_1, s_t) and (s_u, s_1) belong to U, where $s_t \neq s_1, s_{84}$ and $s_u \neq s_1, s_{84}$. If (s_a, s_b) is an arbitrary element of U, then $s_a \in \{s_t\}$ and $s_b \in \{s_u\}$. If $\{s_t\}$ and $\{s_u\}$ both have order 8, then $\{s_t\} = \{s_u\}$ (property II) and this means that U is a *proper* subset of a maximal ring of **A** or **B**, which contradicts the maximality of U. Also if either $\{s_t\}$ or $\{s_u\}$ or both have order 32, we can prove that U is a *proper* subring of a maximal ring of related double homothetisms of R of one of the cases **A**, **B**, **C** or **D** by the same method as in case c with only a few modifications. As U is maximal this gives a contradiction at any case. Hence $(s_1, s_t), (s_u, s_1) \in U$ with $s_t \neq s_1, s_{84}$ and $s_u \neq s_1, s_{84}$ is impossible. Thus, if $(s_1, s_t) \in U$ with $s_t \neq s_1, s_{84}$, then for every element $(s_a, s_b) \in U$ one gets $s_a = s_1$ or $s_a = s_{84}$. But then $U \subseteqq S^{iv}$, and as U is maximal, $U = S^{iv}$. If there is no element $(s_1, s_t) \in U$ with $s_t \neq s_1, s_{84}$, then for every element $(s_a, s_b) \in U$ one has $s_b = s_1$ or $s_b = s_{84}$. In that case $U \subseteqq S^v$, and as U is maximal, $U = S^v$. The only holomorphs corresponding to rings U of d are V_1 and V_2. This completes the proof that the zero-ring R with additive group of type $(2, 2, 2)$ has 367 holomorphs.

In [1] we have determined the number of holomorphs of all rings R with additive group R^+ of type (p, p) and (p, p^2), where p is a prime number. Together with our previous results for rings R with R^+ of type $(2, 2, 2)$ we can give a survey of the number of holomorphs of all rings R with $O(R) < 16$. If R has 2, 3, 5, 7, 11 or 13 elements, then the additive group R^+ of R is a cyclic one, and as $E(R^+)$ is commutative, R has one holomorph [1]. If R has 6, 10, 14 or 15 elements, then, as is well known, the additive group R^+ of R is again cyclic and R has one holomorph. In the sequel we leave the rings R with cyclic additive group R^+ out of consideration. If R has 4 resp. 9 elements, then the additive group R^+ of R is of type (p, p) with $p=2$ resp. 3. In this case, if R is a non-zero ring, R has one holomorph (Satz 2, [1]). A zero-ring R with R^+ of type (2, 2) has $2^2+2+3 = 9$ holomorphs. A zero-ring R with R^+ of type (3, 3) has $3^2+3+3 = 15$ holomorphs. If R has 12 elements and R^+ is not a cyclic group, then R^+ is the direct sum of a four-group of Klein and a group of order 3. Therefore R is the ring-theoretical direct sum of its subrings R_1 resp. R_2, consisting resp. of the 4 elements of order 2 and the 3 elements of order 3 in R. As both R_1 and R_2 are characteristic subrings in R all holomorphs of R have the form $H=H_1\oplus H_2$, where H_i is an arbitrary holomorph of R_i $(i=1, 2)$ [2]. As R_2 has one holomorph it follows that R has one holomorph if R_1 has one holomorph. If R_1 is a non-zero ring it has one holomorph. If R_1 is a zero-ring it has 9 holomorphs as we have seen. Consequently, a ring R with 12 elements has one holomorph, except when the subring of elements of order 2 in R is a zero-ring. In the last case R has 9 holomorphs.

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