On Archimedian ring extensions

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1. Introduction. In literature there are results showing the preservation of Archimedian character of ordering in some ring extensions. For example, the quotient fields of Archimedian rings, the finite extensions of Archimedian fields and semi-simple algebras over Archimedian fields are again Archimedian [1; 127].

In this note we consider the algebraic algebras over Archimedian subrings which are not necessarily fields, and discuss whether they can be Archimedian. This problem is tackled via weak Archimedian rings. It is proved that if a fully ordered (f.o.) ring with identity is algebraic over an Archimedian subfield, then it is only weak Archimedian but not necessarily Archimedian. But if the above ring is an integral domain, then it is certainly an Archimedian ring. The Archimedian nature of f.o. division ring is shown to be completely determined by the algebraic nature of its bounded elements. Furthermore, we shall prove that a commutative integral domain which is an algebraic algebra over Archimedian Noetherian subring is an Archimedian ring.

1. 1 *Definition.* A fully ordered (f.o.) ring R with identity is said to be weak Archimedian if every element in R is bounded. An element x in R is said to be bounded (by the ring of integers I) if $|x|$ is less than some positive integral multiple of the identity.

1. 2 *Definition.* A f.o. ring R with identity is said to be an algebraic algebra over a subring F of R or algebraic over F if every element of R is algebraic over F. An element $x \in R$ is said to be algebraic over F if either $\sum_{i=0}^{n} a_i x^i = 0$ or $\sum_{0}^{n} x^i a_i = 0$ for some $a_i \in F$. If $a_n = 1$, we say that x is integral over F.

1. 3 *Notation.* Throughout this paper all rings are associative rings with identity and Archimedian f.o. rings are defined as in [1].

2. Extensions of archimedian rings

2. 1 Theorem. *Let R be a f.o. ring with identity such that R isalgebraic over a division subring F. Then R is a local ring with the unique maximal right ideal as a nil ideal.*

PROOF. Let R be an integral domain. If $x \neq 0$ and $x \in R$, then $\sum_{r=0}^{n} a_r x^r = 0$, where $a_i \in$ F and n is the minimal degree of the polynomial satisfied by x.

Let $y = a_n x^{n-1} + \ldots + a_1 \neq 0$. Since R is an integral domain, $yx = -a_0 \neq 0$. Hence $(-a_0)^{-1} yx = 1$. Also $x(-a_0)^{-1} y = 1$, since, otherwise $[x(-a_0)^{-1} y - 1]x = 0$, a contradiction. Similarly x has a two-sided inverse if $\sum_{r=0}^{n} x^r a_r = 0$. Thus R is a division ring, a trivial local ring. If R is not an integral domain, then the set N of all nilpotent elements in R is a non-zero convex ideal [1, 130]. So R/N is a f.o. integral domain with identity algebraic over $\bar{F}$, the image of F in R/N. Hence from the above R/N is a division ring. Thus R is a local ring with N as the unique maximal right ideal of R.

2. 2 Lemma. *Let R be a f.o. ring with identity and suppose $x \geqq 0$. Then if $\sum_{r=0}^{n} a_r x^r = 0$ with the coefficients in a subring F and if $a_n \geqq 1$, x is bounded by an element in F.*

PROOF. Let $a_n = 1 + t$, $t \geqq 0$. There exists an $c > 0$ such that $c \in F$ and $c \geqq 1 - a_j$, $j = 0, 1, 2, \ldots n$. The proof is completed by showing $x < c$. Suppose on the contrary $x \geqq c$. Then $a_j \geqq 1 - c \geqq 1 - x$ and hence $a_j x^j \geqq (1-x)x^j$ for $j = 1, 2, \ldots n$. Therefore $0 = a_n x^n + \ldots + a_0 \geqq (1+t)x^{n-} + \ldots + (1-x) = 1 + tx^n > 1$, a contradiction.

2. 3 Theorem. *Let R be a f.o. ring with identity. Then R is weak Archimedian if either* (i) *R is integral over a weak Archimedian subring or* (ii) *R is algebraic over an Archimedian subring F.*

PROOF. The first case is an immediate consequence of the lemma 2. 2. In the second case, let $x \in R \setminus F$. Then $a_n x^n + \ldots + a_0 = 0$, $a_i \in F$. We can always assume that $a_n \geqq 1$. Otherwise, if $a_n < 1$, there exists a positive integer m such that $ma_n > 1$. So $ma_n x^n + \ldots + ma_0 = 0$. Now the result follows from 2. 2.

2. 4 Lemma. *If a division ring D is a weak Archimedian f.o. ring, then D is Archimedian.*

PROOF. Let a and b be any two non-zero elements of D. Suppose $a, b > 0$. By hypothesis, $b < m$ and $a^{-1} < n$ for some positive integers m and n. Then follows $ba^{-1} < N$, N being a positive integer. So $b < Na$ and thus D is Archimedian.

Combining the theorems 2. 1 and 2. 3 we have

2. 5 Corollary. Let R be a f.o. ring with identity and suppose R is algebraic over a weak Archimedian division subring (equivalently Archimedian subfield). Then R is a weak Archimedian local ring with the unique maximal right ideal as a nil ideal.

It is stated in the proposition 3 of [1; 127] that every algebraic algebra over an Archimedian subfield is Archimedian. This need not be true generally unless the algebraic algebra is an integral domain, as can be seen in the following example.

2. 6 Example. Let $S = \{a + bx, a, b \in R$, the rational number field$\}$. Suppose $x^2 = 0$ and $dx = xd$ for every $d \in R$. Then S is a ring with identity under the usual operations as in polynomials. S is a f.o. ring by setting $a + bx > 0$ if $a > 0$ or $b > 0$ if $a = 0$. Since $(a+bx)^2 - 2a(a+bx) + a^2 = 0$, S is algebraic over R. But S is not Archimedian since it has zero-divisors.

But by combining 2. 4 and 2. 5 we obtain,

2. 7 Corollary. Let R be a f.o. integral domain with identity. Then R is an Archimedian ordered field if R is algebraic over an Archimedian subfield.

It is possible to determine the Archimedian character of a division ring by knowing the algebraic nature of bounded elements. So we begin by proving an interesting result.

2. 8 *Definition.* A ring D with identity is called a valuation ring with respect to a skewfield Q, if Q is a two-sided quotient skewfield of D and if $x \in Q$ then x or $x^{-1} \in D$. This is equivalent to saying that the right ideals and also the left ideals are linearly ordered by set inclusion.

2. 9 *Definition.* A ring Q with identity is called a right quotient ring of a ring R with identity if i) $R \subseteq Q$, ii) every non-zero divisor in R is invertible in Q and iii) every $q(\neq 0) \in Q$ is of the form ab^{-1}, $a, b \in R$ and b is a non-zero divisor in R.

The Archimedian character is always preserved under passage to the (right) quotient fields where as the weak Archimedian character is not all carried to the right quotient skewfields in the non-commutative case, since otherwise, the quotient skewfield becomes Archimedian by lemma 2. 4 and hence the original ring becomes commutative.

2. 10 Proposition. *Let D be the set of all bounded elements in a f.o. division ring Q. Then D is a valuation ring with Q as a two-sided quotient skewfield and D is the maximal weak Archimedian subring of Q.*

Proof. Evidently D is a subring with identity same as that of Q. Let $x > 0$ be in Q and $x \notin D$. Then $x > 1$, which implies $x^{-1} \leqq 1$. Thus $x^{-1} \in D$. If $x \in Q \setminus D$, then $x = x^{-1}[(x^2)^{-1}]^{-1} = [(x^2)^{-1}]^{-1}x^{-1}$, where $x^{-1} \in D$. Also $(x^2)^{-1} \in D$, since, otherwise, $(x^2)^{-1}$ is unbounded. Then x^2 is bounded. So $x = x^2 x^{-1} \in D$. Thus D is a valuation ring with Q as a two-sided quotient skewfield.

2. 11 Theorem. *Let D be the set of all bounded elements in an ordered division ring Q. Then Q is Archimedian if D is algebraic over an Archimedian subring S with identity.*

Proof. If $D = Q$, then by 2. 4 Q is Archimedian. So assume $D \neq Q$. Since S is Archimedian, S is a commutative integral domain with identity and so has a quotient field S^* in Q. Furthermore the quotient field S^* itself can be verified to be Archimedian. Hence $S^* \subseteq D$. Now D is an integral domain with identity which has S^* as a subring and which is algebraic over an Archimedian field S^*, since D is algebraic over S and $S \subseteq S^*$. Hence by 2. 7, D is a field. But Q is the quotientsfield of D by 2. 10. Hence $D = Q$. Thus Q becomes Archimedian.

Now we discuss the Archimedian extensions of Archimedian Notherian rings which need not be necessarily fields.

2. 12 Lemma. *Let R be a commutative integral domain with identity and suppose that S is a Noetherian subring of R with identity. Then* i) *if a and b are in R and algebraic over S, ab is algebraic over S* ii) *if R is a finitely generated S-module, R is algebraic over S.*

PROOF. Since a and b are algebraic over S, the subring $S[a, b]$ generated by S, a and b is a finitely generated S-module. But S is Noetherian and $S[ab]$, a subring generated by S and ab, is a S-submodule of $S[a, b]$. Hence $S[ab]$ is itself finitely generated S-module. Then we can write $S[ab] = Sf_1(ab) + Sf_2(ab) + \dots + Sf_t(ab)$. Choose N greater than the degrees of $\{f(x)\}$. Then $(ab)^N = S_1 f_1(ab) + \dots + S_t f_t(ab)$ where $S_i \in S$. Thus ab is algebraic over S.

To prove (ii), by a similar argument as above, if $a \in R$, there exists N such that $a^N = S_1 f_1(a) + \dots + S_t f_t(a)$ where $S \in S$, since $S[a]$ is a finitely generated S-module. Hence R is algebraic over S.

2. 13 Theorem. *Let R be a commutative f.o. integral domain with identity. Then R is Archimedian if either* i) *R is algebraic over a Noetherian Archimedian subring S with identity or* ii) *R is a finitely generated module over a Noetherian Archimedian subring S with identity.*

PROOF. By lemma 2. 12 (ii)$\Rightarrow$(i). Now R has a quotient field Q and S has also a quotient field S^* and $S^* \subseteq Q$. Since S is Archimeidan, S^* is also Archimedian. Let $x(\neq 0) \in Q$. Then $x = ab^{-1}$, $a, b \in R$. Since b is algebraic over S, b^{-1} is also algebraic over S. Then ab^{-1} is algebraic over S by lemma 2. 12. Consequently Q is algebraic over the Archimedian field S^*. Hence by 2. 7, Q is Archimedian. Thus R is Archimedian.

References

[1] L. FUCHS, Partially Ordered Algebraic Systems *London,* 1963.

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